# A Flea on Schrödinger's Cat

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## Abstract

We propose a technical reformulation of the measurement problem of quantum mechanics, which is based on the postulate that the final state of a measurement is classical; this accords with experimental practice as well as with Bohr's views. Unlike the usual formulation (in which the post-measurement state is a a unit vector in Hilbert space, such as a wave-function), our version actually admits a purely technical solution within the confines of conventional quantum theory (as opposed to solutions that either modify this theory, or introduce unusual and controversial interpretative rules and/or ontologies).

To that effect, we recall a remarkable phenomenon in the theory of Schrödinger operators (discovered in 1981 by Jona-Lasinio, Martinelli, and Scoppola), according to which the ground state of a symmetric double-well Hamiltonian (which is paradigmatically of Schrödinger's Cat type) becomes exponentially sensitive to tiny perturbations of the potential as $\hbar \to 0$. We show that this instability emerges also from the textbook WKB approximation, extend it to time-dependent perturbations, and study the dynamical transition from the ground state of the double well to the perturbed ground state (in which the cat is typically either dead or alive, depending on the details of the perturbation). Numerical simulations show that, in an *individual* experiment, certain (especially adiabatically rising) perturbations may (quite literally) *cause* the collapse of the wavefunction in the classical limit. Thus we combine the technical and conceptual virtues of dynamical collapse models à la GRW (which do solve the measurement problem) with those of decoherence (in that our perturbations come from the environment) without sharing their drawbacks: although single measurement outcomes are obtained (instead of merely diagonal reduced density matrices), no modification of quantum mechanics is needed.


#### Motto

> 'Another secondary readership is made up of those philosophers and physicists who—again like myself—are puzzled by so-called foundational issues: what the strange quantum formalism implies about the nature of the world it so accurately describes. (...) My presentation is suffused with a perspective on the quantum theory that is very close to the venerable but recently much reviled Copenhagen interpretation. Those with a taste for such things may be startled to see how well quantum computation resonates with the Copenhagen point of view. Indeed, it had been my plan to call this book *Copenhagen Computation* until the excellent people at Cambridge University Press and my computer-scientist friends pursuaded me that virtually no members of my primary readership would then have any idea what it was about.'
>
> David Mermin, *Quantum Computer Science: An Introduction* (Preface)


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# 1 Introduction

Citizens of most democratic countries know the phenomenon of a "hung parliament", in which two major political parties have a large numer of seats each, but are short of a majority separately and mutually exclude each other as potential coalition partners. In that case, tiny parties with just a few seats can tip the balance to the left or to the right and hence, quite undemocratically, acquire an importance far exceeding their relative size.

An analogous phenomenon in quantum mechanics was discovered in 1981 by Jona-Lasinio, Martinelli, and Scoppola [57, 58] (see also Section 3 below). Here, the ground state of (say) a symmetric double-well Hamiltonian becomes exponentially sensitive (in $1/\hbar$) to tiny perturbations of the potential as $\hbar \to 0$. In particular, whereas the ground state of the unperturbed Hamiltonian has two pronounced and well-separated peaks, the ground state of the perturbed Hamiltonian typically features one of those peaks only and hence may be said to have "collapsed". We will henceforth denote such a perturbation as a "flea" (residing, of course, on Schrödinger's Cat rather than on Simon's elephant [95]).

As we will explain and make precise in Section 2, this phenomenon acquires acute relevance for the measurement problem as soon as one accepts just two postulates:[1]

1. Bohr's dogma—coinciding with experimental practice—to the effect that a measurement be a classical snapshot (or "readout") of a quantum state;[2]

2. The fundamental nature of quantum theory, which implies that measurement devices (like anything else) are ultimately quantum mechanical in nature.[3]

On the first clause of this conjunction, the post-measurement state of the pertinent apparatus should be a classical state, whilst on the second, it has to be the classical limit of some quantum state. After coupling to some microscopic object, the latter state might evolve into a superposition à la Schrödinger's Cat, and this is what causes the measurement problem. But it is exactly in the classical limit that the sensitivity of the wave-function to the flea arises! Thus the correct formulation of the measurement problem as a problem concerning classical limits of quantum states already contains the seed of its solution.

In Section 3 we review *static* aspects of "flea" instability (using the two-level system as a pedagogical example), in that the perturbations are taken to be time-independent, and the perturbed ground state is just studied for its own sake. This review is backed up by a new, more technical analysis based on the familiar WKB approximation from the textbooks,[4] which we delegate to the appendix in order not to interrupt our story. In order to trace the fate of Schrödinger's Cat as a dynamical process, we need to take up the *dynamical* study of the instability. In other words, the perturbation should be made time-dependent, and in addition the transition from the unperturbed "Schrödinger Cat" ground state to the perturbed ground state (in which the cat is either dead or alive) should be followed in time. This will be done (largely numerically) in in Section 4, providing a "proof of concept" that the solution to the measurement problem suggested here might actually work (at least in a toy example), including an easy derivation of the Born rule.

In the final Discussion section, we explore connections between our approach and symmetry breaking and phase transitions, quantum metastability, and determinism versus Free Will type Theorems (at least one of whose assumptions our flea violates, of course).

[1] We regard these as Wittgensteinian "hinge propositions" [97], on which contemporary physics is based.

[2] This is sometimes called Bohr's *doctrine of classical concepts* [87]. See also [66] for a detailed analysis.

[3] For our work it is a moot point whether Bohr endorsed this second point as well; it is hard to say.

[4] The original expositions [25, 46, 53, 57, 58, 95] might be hard to follow for non-mathematicians (see, however, [21]). Applications in chemistry and solid-state physics may be found in [22, 40, 59, 74].

## 2 Rethinking the measurement problem

### 2.1 Historical overview

Roughly speaking, the measurement problem consists in the fact that the Schrödinger equation of quantum mechanics generically fails to predict that measurements have outcomes. Instead, it apparently predicts (empirically) unacceptable "superpositions" thereof. Slightly more technically, the problem is usually formulated in approximately the following way (see e.g. the excellent presentations in [2, 17, 18] for more detail). Suppose that one measures some observable $O$ pertaining to a microscopic system $S$, in such a way that if $S$ is in an eigenstate $\Phi_i$ of $O$, then the associated macroscopic apparatus $A$ is in state $\Psi_i$. It is important to note that in this description (pure) states are seen as unit vectors in some Hilbert space, as usual in quantum mechanics. Now, although there is hardly any problem with the existence of a *microscopic* superposition $\Phi = \sum_i c_i\Phi_i$ of $S$ (where $\sum_i |c_i|^2 = 1$), the linearity of the Schrödinger equation implies that it would bring $A$ in a similar superposition $\Psi = \sum_i c_i\Psi_i$, which, *if macroscopic*,[5] is never seen in nature. Instead, as a matter of fact one always observes one of the states $\Psi_i$ (or so it is claimed).

This problem was immediately recognized by the founders of quantum theory. In response, in 1926, Born (generalizing earlier ideas of Bohr and Einstein on light emission by atoms) stated that quantum theory indeed did not predict individual outcomes, but merely computed their probabilities (according to the formula now named after him) [14, 69]. In 1927, Heisenberg (again in the wake of Bohr's electronic "quantum jumps") proposed the "collapse of the wavepacket" [51], which (in the above language) implies that during the course of the measurement, the apparatus state $\Psi$ miraculously "jumps" to one of the states $\Psi$. Bohr immediately endorsed this idea, ordered that such quantum jumps ought not to be analyzed any further, and claimed that they were the source of irreducible randomness in physics. During the period 1927–1935 (with an aftermath running until 1949), Bohr famously defended these ideas against a highly critical Einstein [10], as did Born [13]. Adding considerable conceptual and mathematical precision, von Neumann gave an account of the measurement problem in his book from 1932 [78], which has formed the basis for most discussions of the issue ever since. The early period was closed in 1935, the year in which Schrödinger (following a correspondence with Einstein [39]) published a penetrating analysis, including the metaphor around the cat later named after him [89].

The introduction of the collapse process (and the associated Born probabilities) has turned out to be an incredibly successful move, on which practically all empirical successes of quantum theory are based. This may well be the reason why few physicists are bothered by the measurement problem: the underlying slick manoeuvre of ad hoc collapse seems an acceptable price for these successes, unprecedented in science as they are.[6] But for those in the foundations of physics, there is no doubt that this is a pseudo-solution. Consequently, despite arguments claiming to prove their non-existence [16, 18, 38], many solutions to the measurement problem have been proposed. Among those, it is fair to say that at least the quasi-philosophical solutions have failed to convince the scientific community at large.[7]

[5] This notion needs to be quantified, of course [2, 18].

[6] So-called "Bayesians" even deny that there is a measurement problem [20].

[7] This is true in particular for the Many-Worlds (aka "Everett") Interpretation [86], or the Modal Interpretation of quantum mechanics [32], in which radical changes are proposed in the ontology and/or the usual interpretative rules of the theory, without clarifying in any way what is really going on during measurements (a question one indeed is not supposed to ask, according to received wisdom). Bohmian mechanics (as a modern incarnation of de Broglie's pilot-wave theory) does a better job here [29, 34, 35],

Within the realm of technical approaches to the measurement problem, one may distinguish between those proposals that do and those that do not modify quantum theory. Among the latter, the main effort so far has been towards attempts to eliminate interference terms (i.e., between the states $\Psi_i$ of the previous section), sometimes accompanied by the (implicit or explicit) suggestion that their removal would actually solve the problem.[8] Such attempts come in (at least) two kinds. In the wake of the Swiss school [36, 54], mathematical physicists typically worked in the formalism of *superselection rules* (see [63, 64] for reviews and [93] for recent work in this direction), whereas theoretical physicists tend to exploit *decoherence* (see [60, 88] for recent reviews). There seems to be a general consensus, though, that neither of these solves the problem (at least in the form stated above); they rather reconfirm it. Indeed, granted that measurements yield classical data, since classical physics by definition does not have quantum-mechanical interference terms, their disappearance in appropriate limiting situations (like the ones described by decoherence and/or superselection theory) is just a necessary condition that (in part) *defines* measurement (which after all is supposed to produce some classical state as its outcome).[9] Consequently, in our opinion the measurement problem is *posed*, rather than *solved* by proving that such interference terms vanish under particular (limiting) conditions.[10]

In contrast, the *dynamical collapse models* of Pearle, Ghirardi–Rimini–Weber, and others (cf. [15] for a comprehensive survey) *do* solve the measurement problem. But they do so at a heavy prize: the Schrödinger equation is modified by adding a novel and universal stochastic process that even makes the equation nonlinear, and which, like the solution of Heisenberg and von Neumann, is completely ad hoc except for its goal of causing collapse.

The approach to the measurement problem we are going to propose below uses key ideas from both dynamical collapse models and decoherence (and could not have been conceived without the inspiration from these earlier approaches), but in such a way that we avoid their main drawbacks (though probably at the cost of others!):

1. Dynamical collapse is obtained without modifying quantum theory.
2. While decoherence preserves all peaks (i.e., potential measurement outcomes) in the density matrix, and hence subsequently needs e.g. some kind of a Many Worlds Interpretation [56, 88], our mechanism, if correct, leads to just one outcome.

Our approach starts with a technical reformulation of the measurement problem, which relies on a specific mathematical formalism for dealing with classical states, including their role as potential limits of quantum states. For completeness's sake, we explain this first.

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but its narrow applicability (at least in its current form), focusing as it does on position as the only physical observable, makes it unattractive to many (including the authors).

[8]This applies, for example, to the famous paper by Danieri, Loinger, and Prosperi [30], to early papers on decoherence [100], and to much of the mathematical physics literature on the measurement problem, including the work of the senior author [36, 54, 63, 64, 93]. We now regard such papers as mathematically interesting but conceptually misguided, at least on this point. In any case, it is to the credit of especially the Swiss school that it drew attention to the idea that measurement involves limiting procedures, so that solutions of the measurement problem should at least *incorporate* the appropriate limits.

[9]In this sense even von Neumann's book [78] is misleading, as he suggested that the act of observation may be identified with a cut in the chain now named after him. What is right about this idea is that observation is linked to a voluntary loss of information, but it would have been preferable to point out—with Bohr— that such a loss, in so far as it defines measurement, should be a loss of *quantum* information.

[10]Furthermore, despite its outspoken ambition to derive classical physics from quantum theory [60, 88, 100], decoherence hardly (if at all) invokes limits like Planck's constant going to zero, which are needed, for one thing, to derive the correct classical equations of motion (cf. [66, 67]).

## 2.2 Intermezzo: classical states

In order to describe classical states as limits of quantum states, we describe states algebraically. Although this formalism implicitly uses the language of C*-algebras, very little of that theory will be needed here [48, 65]. In our case, the C*-algebra of observables will simply be $A = K(L^2(\mathbb{R}))$ (i.e., the compact operators on the Hilbert space $L^2(\mathbb{R})$ of square-integrable wave-functions) in the quantum case, and $A_0 = C_0(\mathbb{R}^2)$ in the classical case (that is, the continuous functions on the phase space $\mathbb{R}^2$ that vanish at infinity). Note that the classical algebra is commutative. A *state*, then, is a positive linear functional of norm one on the C*-algebra of observables. For the above algebras, this means that a classical state $\mu$ (i.e., a state on $A_0$) is the same thing as a probability measure $\hat{\mu}$ on phase space, seen as a map $f \mapsto \mu(f) = \int_{\mathbb{R}^2} d\hat{\mu}\, f$, $f \in C_0(\mathbb{R}^2)$. Such a probability *measure* may or may not be given by a probability *density*, i.e., a positive $L^1$-function $\chi$ on $\mathbb{R}^2$ integrating to unity with respect to the Liouville measure $dpdq/2\pi$ on $\mathbb{R}^2$, such that

$$\int_{\mathbb{R}^2} d\hat{\mu}\, f = \int_{\mathbb{R}^{2n}} \frac{dpdq}{2\pi}\, \chi(p,q) f(p,q). \tag{2.1}$$

On the other hand, a quantum state, i.e., a state $\rho$ on $K(L^2(\mathbb{R}))$, is essentially the same as a density matrix $\hat{\rho}$ on $L^2(\mathbb{R})$, seen as a map $a \mapsto \rho(a) = \mathrm{Tr}\,(\hat{\rho}a)$, where $a \in K(L^2(\mathbb{R}))$.

A *pure* state $\omega$ has no nontrivial convex decomposition, i.e., if $\omega = p\omega_1 + (1-p)\omega_2$ for some $p \in (0,1)$ and certain states $\omega_1$ and $\omega_2$, then $\omega_1 = \omega_2 = \omega$. Pure states on $C_0(\mathbb{R}^2)$ are probability measures of the Dirac form $\delta_z$, $z \in \mathbb{R}^2$ (i.e., $\delta_z(f) = f(z)$ for $f \in C_0(\mathbb{R}^2)$), and hence bijectively correspond to points of $\mathbb{R}^2$. Equally familiar, pure states $\psi$ on $K(L^2(\mathbb{R}))$ are just unit vectors $\Psi$ (up to a phase), with associated density matrices $\hat{\rho} = |\Psi\rangle\langle\Psi|$ given by the (orthogonal) projection on $\mathbb{C}\Psi$; i.e., a unit vector $\Psi$ defines an algebraic state $\psi$ by

$$\psi(a) = \langle\Psi|a|\Psi\rangle \equiv \langle\Psi, a\Psi\rangle. \tag{2.2}$$

The following notion of convergence of quantum states to classical ones is standard (cf. [24, 65, 83, 85] and many other sources),[11] and has been used especially in quantum chaology [80]. We first recall the *coherent states*, labeled by $z = (p,q) \in \mathbb{R}^2$,

$$\Phi_\hbar^{(p,q)}(x) = (\pi\hbar)^{-1/4} e^{-ipq/2\hbar} e^{ipx/\hbar} e^{-(x-q)^2/2\hbar}, \tag{2.3}$$

with associated *Berezin quantization* map $f \mapsto Q_\hbar(f)$, $f \in C_0(\mathbb{R}^2)$, $Q_\hbar(f) \in K(L^2(\mathbb{R}))$,

$$Q_\hbar(f) = \int_{\mathbb{R}^{2n}} \frac{dpdq}{2\pi\hbar}\, f(p,q) |\Phi_\hbar^{(p,q)}\rangle\langle\Phi_\hbar^{(p,q)}|. \tag{2.4}$$

Now let $(\rho_\hbar)$ be a family of quantum states, indexed by $\hbar$ (say $\hbar \in (0,1]$), with associated density matrices $(\hat{\rho}_\hbar)$, and let $\rho_0$ be a state on $C_0(\mathbb{R}^2)$, with associated probability measure $\hat{\rho}_0$ on $\mathbb{R}^2$. The quantum states $(\rho_\hbar)$ *converge* to the classical state $\rho_0$, $\lim_{\hbar\to 0} \rho_\hbar = \rho_0$, if

$$\lim_{\hbar\to 0} \rho_\hbar(Q_\hbar(f)) = \rho_0(f), \text{ for all } f \in C_0(\mathbb{R}^2). \tag{2.5}$$

[11] Often Weyl quantization $Q_\hbar^W$ is used instead of Berezin quantization $Q_\hbar$, as in [9], but for Schwartz functions $f$ on phase space these have the same asymptotic properties as $\hbar \to 0$ [65]. The advantage of Berezin quantization is that it is well defined also for continuous functions (vanishing at infinity), in that for any unit vector $\Psi \in L^2(\mathbb{R})$ the map $f \mapsto \langle\Psi|Q_\hbar(f)|\Psi\rangle$ defines a probability measure on phase space. In contrast, the Wigner function defined by $f \mapsto \langle\Psi|Q_\hbar^W(f)|\Psi\rangle$ may fail to be positive, as is well known.

This condition more explicitly reads

$$\lim_{\hbar\to 0} \mathrm{Tr}\,(\hat{\rho}_\hbar Q_\hbar(f)) = \int_{\mathbb{R}^2} d\hat{\rho}_0\, f. \tag{2.6}$$

If $\rho_\hbar = |\Psi_\hbar\rangle\langle\Psi_\hbar|$, then obviously

$$\langle\Psi_\hbar|Q_\hbar(f)|\Psi_\hbar\rangle = \int_{\mathbb{R}^{2n}} \frac{dpdq}{2\pi\hbar}\,\chi_{\Psi_\hbar}(p,q)f(p,q), \tag{2.7}$$

where the probability density $\chi_{\Psi_\hbar}$, called the *Husumi function* of $\Psi_\hbar$, is given by

$$\chi_{\Psi_\hbar}(p,q) = |\langle\Phi_\hbar^{(p,q)}|\Psi_\hbar\rangle|^2, \tag{2.8}$$

in which the inner product is in $L^2(\mathbb{R})$. Consequently, if the limit in (2.5) exists for specific $\rho_\hbar = |\Psi_\hbar\rangle\langle\Psi_\hbar|$, then the limit measure $\hat{\rho}_0$ is the weak (or pointwise) limit of the probability measures $\mu_{\Psi_\hbar}$ that are defined by the probability densities $\chi_{\Psi_\hbar}$ according to (2.1).

Let us illustrate this formalism for the ground state of the one-dimensional harmonic oscillator. Taking $V(x) = \frac{1}{2}\omega^2x^2$ in the usual quantum Hamiltonian (with mass $m = 1/2$),

$$H = -\hbar^2\frac{d^2}{dx^2} + V(x), \tag{2.9}$$

it is well known that the ground state is unique and that its wave-function

$$\Psi_\hbar^{(0)}(x) = \left(\frac{\omega}{2\pi\hbar}\right)^{1/4} e^{-\omega x^2/4\hbar} \tag{2.10}$$

is a Gaussian, peaked above $x = 0$. As $\hbar \to 0$, this ground state converges to the ground state $\rho_0^{(0)} = (0,0) \in \mathbb{R}^2$ (i.e., $(p = 0, q = 0)$) of the corresponding classical system. Slightly less familiar, the same is true for the *an*harmonic oscillator (with small $\lambda > 0$), i.e.,

$$V(x) = \tfrac{1}{2}\omega^2x^2 + \tfrac{1}{4}\lambda x^4, \tag{2.11}$$

the peak, of course, now being only approximately Gaussian. But it is a deep and counterintuitive feature of quantum theory that even the symmetric double-well potential

$$V(x) = -\tfrac{1}{2}\omega^2x^2 + \tfrac{1}{4}\lambda x^4 + \tfrac{1}{4}\omega^4/\lambda = \tfrac{1}{4}\lambda(x^2 - a^2)^2, \tag{2.12}$$

where $a = \omega/\sqrt{\lambda} > 0$ (assuming $\omega > 0$ as well as $\lambda > 0$), has a unique quantum-mechanical ground state [55, 84], despite the fact that the corresponding classical system has two degenerate ground states, given by the phase space points $\rho_0^\pm \in \mathbb{R}^2$ defined by

$$\rho_0^\pm = (0, \pm a). \tag{2.13}$$

The wave-function $\Psi_\hbar^{(0)}$ remains real and positive definite, but this time it has *two* peaks, above $x = \pm a$, with exponential decay $|\Psi_\hbar^{(0)}(x)| \sim \exp(-1/\hbar)$ in the classically forbidden region $x \notin \{-a, a\}$ [55, 84]. As a quantum-mechanical shadow of the classical degeneracy, energy eigenfunctions (and the associated eigenvalues) come in pairs. In what follows, we will be especially interested in the first excited state $\Psi_\hbar^{(1)}$, which like $\Psi_\hbar^{(0)}$ is real, but it has one peak *above* (say) $x = -a$ and another peak *below* $x = a$. See Figure 1.

As $\hbar \to 0$, the eigenvalue splitting $E_1 - E_0$ vanishes exponentially in $-1/\hbar$ like

$$\Delta \equiv E_1 - E_0 \sim \frac{\hbar\omega}{\sqrt{\frac{1}{2}e\pi}} e^{-d_V/\hbar} \ (\hbar \to 0), \tag{2.14}$$

where the typical WKB-factor is given by

$$d_V = \int_{-a}^{a} dx\, \sqrt{V(x)}; \tag{2.15}$$

see [43, 62] (heuristic), or [52, 55, 95] (rigorous) for details. In particular, the probability densities defined by the wave-functions $\Psi_\hbar^{(0)}$ and $\Psi_\hbar^{(1)}$ approach $\delta$-function peaks above the classical minima $\pm a$. See Figure 2, displayed just for $\Psi_\hbar^{(0)}$, the other one being analogous.

We can make the correspondence between the (nondegenerate) pair $\Psi_\hbar^{(0)}$ and $\Psi_\hbar^{(1)}$ of low-lying quantum-mechanical wave-functions and the pair $(\rho_0^+, \rho_0^-)$ of degenerate classical ground states much more precise and impressive by invoking the notion of a classical limit of states explained above. Indeed, in terms of the algebraic states $\psi_\hbar^{(0)}$ and $\psi_\hbar^{(1)}$ one has

$$\lim_{\hbar\to 0} \psi_\hbar^{(0)} = \lim_{\hbar\to 0} \psi_\hbar^{(1)} = \rho_0^{(0)}, \tag{2.16}$$

$$\rho_0^{(0)} := \tfrac{1}{2}(\rho_0^+ + \rho_0^-), \tag{2.17}$$

where $\rho_0^\pm$ are the pure classical ground states (2.13) of the double-well Hamiltonian.[12] To see this, one may either consider numerically computed Husumi functions, as shown in Figure 3 (just for $\Psi_\hbar^{(0)}$, as before), or one may proceed analytically, combining the relevant estimates in [50] or in [95] with the computations in §II.2.3 of [65]. Either way, it is clear that the *pure* (algebraic) quantum ground state $\psi_\hbar^{(0)}$ converges to the *mixed* classical state (2.17). On the other hand, the localized (but now time-dependent) wave-functions

$$\Psi_\hbar^\pm = \frac{\Psi_\hbar^{(0)} \pm \Psi_\hbar^{(1)}}{\sqrt{2}}, \tag{2.18}$$

which of course define pure (algebraic) states as well, converge to *pure* classical states, i.e.,

$$\lim_{\hbar\to 0} \psi_\hbar^\pm = \rho_0^\pm. \tag{2.19}$$

On the one hand this is not surprising, because $\Psi_\pm$ has a single peak above $\pm a$, but on the other hand it is, since neither $\Psi_\hbar^+$ nor $\Psi_\hbar^-$ is an energy eigenstate (whereas their limits $\rho_0^+$ and $\rho_0^-$ are, in the classical sense of being fixed points for the Hamiltonian flow). The explanation is that the energy difference (2.14) vanishes exponentially as $\hbar \to 0$, so that in the classical limit $\Psi_\hbar^+$ and $\Psi_\hbar^-$ approximately do become energy eigenstates. In similar vein, because of (2.14) the tunneling time $\tau = 2\pi\hbar/\Delta$ of the oscillation between $\Psi_\hbar^+$ and $\Psi_\hbar^-$ becomes exponentially large in $1/\hbar$ as $\hbar \to 0$.

In the above examples (and many others) time evolution of states is defined both classically (by the Liouville equation for measures, which is equivalent to Hamilton's equations) and quantum mechanically (by the von Neumann equation for density matrices, which is equivalent to Schrödinger's equation), and provided that $\lim_{\hbar\to 0} \rho_\hbar = \rho_0$ as in (2.5), for each fixed time $t \in \mathbb{R}$, one has Egorov's Theorem in the form [65, Thm. II.2.7.2], [85]

$$\lim_{\hbar\to 0} (\rho_\hbar(t)) = \rho_0(t). \tag{2.20}$$

[12] In (2.17) we regard classical states as probability measures on phase space; hence the addition on the right-hand side has nothing to do with addition in the particular phase space $\mathbb{R}^2$ (whose linear structure is accidental and irrelevant), but is a convex sum of measures.

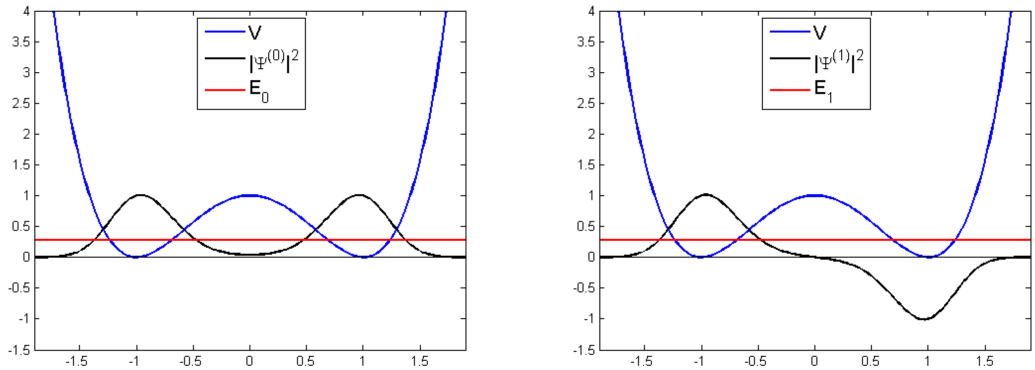


Figure 1: Double well potential with ground state $\Psi^{(0)}_{\hbar=0.5}$ and first excited state $\Psi^{(1)}_{\hbar=0.5}$.

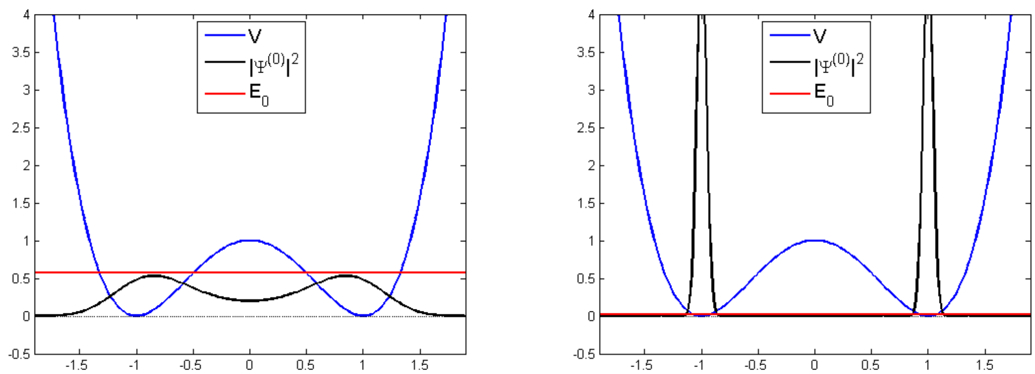


Figure 2: Probability densities for $\Psi^{(0)}_{\hbar=0.5}$ (left) and $\Psi^{(0)}_{\hbar=0.01}$ (right).

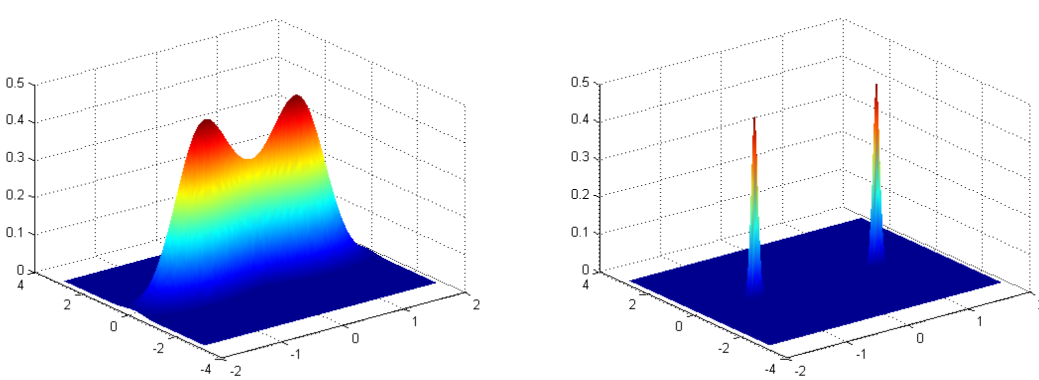

Figure 3: Husumi functions for $\Psi^{(0)}_{\hbar=0.5}$ (left) and $\Psi^{(0)}_{\hbar=0.01}$ (right).

### 2.3 Reformulation of the measurement problem

We return to the measurement problem. If measurement is merely seen as the establishment of certain correlations between two quantum systems, then the problem does not arise, since *a priori* nothing is wrong with the existence of superpositions of such correlated quantum states. What *is* wrong is that at first sight such superpositions seem to survive the classical limit, as shown above by the ground state of the double-well potential. More generally, the measurement problem arises whenever in the classical limit a *pure* quantum state converges to a *mixed* classical state, since in that case quantum theory fails to predict a single measurement outcome. Rather, it suggests there are many outcomes, *not* just because the wave-function has several peaks *per se*, but because *in addition* in the classical limit each of these peaks converges to a different classical state.

Consequently, the measurement problem is by no means solved by proving that such interference terms vanish under certain (limiting) conditions. Instead, the real problem is to show that under realistic measurement conditions pure quantum states actually have *pure* classical limits. Indeed, Schrödinger's Cat is exactly of this nature [66]:

- If one were to study the cat as a quantum system, nothing would be wrong with the famous superposition it resides in. However, such a study is practically impossible.[13]
- The "paradox" arises if one only uses macroscopic variables in order to give a classical description of the cat, so that notions like (being) "alive" or "dead" make sense. In that case, the naive classical state of the cat is of the kind $\rho_0^{(0)} = \frac{1}{2}(\rho_0^+ + \rho_0^-)$, cf. (2.17), where (say) $\rho_0^+$ stands for being alive and $\rho_0^-$ is the (classical) state of death. A classical state like $\rho_0^{(0)}$ is indeed intolerable, but since our flea destabilizes it, it fortunately enough cannot arise in practice (in theory, such a state could be created in a totally isolated system, in which case its paradoxical features disappear).

Having wholeheartedly endorsed the Bohrian (or rather: practical) view of what a measurement is, we emphatically reject the (typically) accompanying claims that the measurement process itself cannot be analyzed or described in principle, and that its outcome is irreducibly random (except for special initial states).[14] But if measurement *by definition* produces some classical state from a quantum state, and quantum (field) theory is agreed to be fundamental and hence classical physics is some limit of it [65, 66], then it would seem almost perverse not to describe the pertinent limiting procedure explicitly. Take our example of the classical limit of the double well, which we regard as a model of a measurement apparatus; the ground state $\Psi_\hbar^{(0)}$ models the state the apparatus has assumed after coupling to some microscopic 'object' system, prepared in a superposition $(O^- + O^+)/\sqrt{2}$, where the object state $O_\pm$ correlates with the localized state $\Psi_\hbar^\pm$, cf. (2.18). Subsequently, the object system is dropped from consideration, as it does not take part in the classical description of the apparatus, and its microscopic superposition does not pose any problem.

[13]This appeal to "practice" does not mean that we are resigned to FAPP (i.e., "for all practical purposes") solutions to the measurement problem. As in [66], we remain convinced that the classical description of a measurement apparatus is a purely epistemic move, relative to which outcomes are *defined*. So even if it were possible to study a cat as a quantum system, there would be no measurement problem, since in that case there would be innumerable superpositions but not a single (undesirable) mixture of classical states.

[14]We share this rejection with the Bohmians [28]. The folk wisdom (shared by the Founding Fathers) that the Copenhagen Interpretation has no measurement problem relies on these secondary Copenhagenian claims, which indeed sweep the problem under the rug. Incidentally, these claims seem much more popular than Bohr's doctrine of classical concepts, which is generally not well understood, and/or mistaken for the idea that the goal of physics is to explain experiments, or that reality does not exist, et cetera.

We then interpret the double limit $\hbar \to 0$, $t \to \infty$ (in an appropriate order to be discussed below) as the "unfolding" of the measurement, in that the apparatus is described increasingly classically.[15]

- According to the Copenhagen Interpretation, by some inexplicable mystery, at some stage of the classical description the wave-function suddenly collapses.
- According to our analysis, the collapse is not inexplicable at all: it is caused by a perturbation, and in principle it can be exactly described and followed in time.

Here it is important to note that the values $\hbar = 0$ and $t = \infty$ are never actually reached—we are talking about *limits*! In particular, the instability of the ground state described in the next section already arises for *very small* (as opposed to zero or 'infinitesimal') effective values of $\hbar$. And this is how it should be: truly classical states (like strictly infinite systems) do not exist in nature, but you should be able to make the difference between the quantum-mechanical approximation to such a state and the actual limit state as small as you like, for sufficiently small $\hbar$ and large $t$. Indeed, the whole point of the reformulation of the measurement problem proposed here is that the usual (superposition) state $\Psi_\hbar^{(0)}$ of Schrödinger's Cat does *not* have this feature: the classical states that (almost) occur in nature are $\rho_0^+$ (alive) and $\rho_0^-$ (dead), and for any $\hbar > 0$, the state $\psi_\hbar^{(0)}$ defined by the wave-function $\Psi_\hbar^{(0)}$ dramatically fails to approximate either of these,[16] although it perfectly well approximates the unphysical mixture $\frac{1}{2}(\rho_0^+ + \rho_0^-)$. In other words, returning to the original mechanical meaning of the double-well system, quantum mechanics is apparently unable to predict that a classical ball lies at the bottom of either the right or the left well.

Fortunately, this inability is only apparent: depending on the sign and localization of the perturbation $\delta V$ of the double well (cf. the next section), the collapsed states $\psi_\hbar^{(\delta)}$ induced by the "flea on the cat" *do* approximate either $\rho_0^+$ or $\rho_0^-$ as $\hbar \to 0$.

Unfortunately, this insight concerning perturbed ground states and their associated localized wave-functions is only the first, *static* part of the solution of the measurement problem. The *dynamical* part of the solution would be to find an appropriate time-dependent way for the flea to jump onto Schrödinger's Cat (in its superposition state), and either kill it, or let it live. That is, one needs to find a suitable perturbed (but nonetheless unitary!) quantum time-evolution operator $U_\hbar^{(\delta)}(t)$ such that the (algebraic) state $\psi_\hbar^{(0)}(t)$ defined by the wave-function $U_\hbar^{(\delta)}(t)\Psi_\hbar^{(0)}$ converges to either $\rho_0^+$ or $\rho_0^-$ as $t \to \infty$ and $\hbar \to 0$. Moreover, a completely satisfactory solution of the measurement problem (or at least of its Schrödinger Cat instance) would have the additional property that measurement results that are already pre-classical, which in this case means that they are either $\psi_\hbar^+$ or $\psi_\hbar^-$, be stable under perturbations. This leads to the following conditions:

[15] The analogy with the thermodynamic limit $V \to \infty$ will be discusses at the end of the paper. As to $\hbar \to 0$, we repeat [66, pp. 471–472] that although $\hbar$ is a dimensionful *constant*, in practice one studies the (semi)classical regime of a given quantum theory by forming a dimensionless combination of $\hbar$ and other parameters; this combination then re-enters the theory as if it were a dimensionless version of $\hbar$ that can indeed be varied. The oldest example is Planck's radiation formula, with the associated limit $\hbar\nu/kT \to 0$, and another example is the Schrödinger operator (2.9), with mass reinserted, where one may pass to a dimensionless parameter $\hbar/\lambda\sqrt{2m\epsilon}$, where $\lambda$ and $\epsilon$ are typical length and energy scales, respectively.

[16] Paraphrasing Bell [6]: the difference between $\rho_0^\pm$ and $\psi_\hbar^{(0)}$ can be made 'as big as you do *not* like.'.

$$\lim_{\hbar\to 0, t\to\infty} \psi_{\hbar}^{(0)}(t) = \rho_0^{\pm}, \tag{2.21}$$
$$\lim_{\hbar\to 0, t\to\infty} \psi_{\hbar}^{+}(t) = \rho_0^{+}; \tag{2.22}$$
$$\lim_{\hbar\to 0, t\to\infty} \psi_{\hbar}^{-}(t) = \rho_0^{-}. \tag{2.23}$$

As in (2.16) - (2.19), these conditions do not contradict each other, since in passing from unit vectors $\Psi$ to algebraic states $\psi$, *linearity is lost.* Similarly, (2.16) - (2.17) would be impossible for unit vectors (noting that $\Psi_{\hbar}^{(0)}$ and $\Psi_{\hbar}^{(1)}$ are orthogonal), but they are perfectly alright for states in the algebraic sense. Indeed, families of unit vectors like $\Psi_{\hbar}^{(i)}$, where $i = 0, 1, +, -$, do not have a limit as unit vectors (or even as density matrices, including one-dimensional projections). This marks a decisive difference with standard approaches to the measurement problem [2, 17, 18], which are victim to "insolubility" theorems of the kind proved by von Neumann, Fine, and others [16, 18, 38, 78]. Such theorems assume that the post-measurement state (if pure) is a unit vector in Hilbert space (or a density matrix otherwise), and totally rely on the linearity of the Schrödinger equation. In contrast, we take the post-measurement state *by definition* to be classical.

In the above setting, determining the correct way to take the double limit $\hbar \to 0, t \to \infty$ is a highly nontrivial problem. In the theory of semiclassical asymptotics [85] (with quantum chaology as an important subfield [12, 37, 90, 98]), the goal of this limit is to find the long-time behaviour of some quantum system by first describing the underlying classical system (especially if it is chaotic), and subsequently using suitable classical expressions to approximate the corresponding quantum formulae. For example, suppose one wants to find the time evolution of a wave-packet that initially is strongly (micro) localized, i.e., is a coherent state for some small (effective) value of $\hbar$. For fixed time $t$, one has Egorov's Theorem (2.20), which, supplemented by exponentially small error terms, shows that for any finite $t$ the limit $\hbar \to 0$ delivers the above goal. For large times, however, there is a competition between the limit $\hbar \to 0$ making it more localized (and hence more classical), and the limit $t \to \infty$ making it less so (and hence more wave-like or quantum-mechanical). Intuitively, spreading is enhanced if the classical dynamics is chaotic, and suppressed if it is integrable. In the chaotic case, it turns out that (micro) localization defeats the spread in time as long as $t \leq C \ln(1/\hbar)$, with $C$ of order one, so that one may take the double limit in the order $\hbar \to 0, t \to C \ln(1/\hbar)$ [1, 15, 26]. If the system is integrable, on the other hand, one expects to push this to much larger times $t \sim \hbar^{-k}$, for some $k \in \mathbb{N}$.

Our situation is more complicated than that. First, in $d = 1$, time-dependent perturbations of the "flea" type render the double-well potential no longer integrable, without the perturbed dynamics becoming really chaotic either. Second, the initial state being the ground state of the (unperturbed) double well, it is not even localized to begin with.[17] Thirdly, we will actually invoke another limit, namely the adiabatic one. As we will explain in the Discussion, combining these features poses a new problem in the practically unexplored territory of quantum metastability, whose solution will not only involve new mathematical results in semiclassical asymptotics, but also calls for genuinely new physical understanding. For now, our goal is just to explain our program and provide a "proof of concept" that it might work. Thus at the present stage we merely present some numerical results, showing that for fixed small $\hbar$, localization takes place for sufficiently large $t$.

[17] As explained above, the nonlinearity inherent in the limit $\hbar \to 0$ makes it impossible to find the limit of this ground state $\Psi_{\hbar}^{(0)}$ by just adding the results for two localized wave-functions like $\Psi_{\hbar}^{+}$ and $\Psi_{\hbar}^{-}$

# 3 A collapse process within quantum mechanics

## 3.1 The "flea" perturbation of the double-well potential

Regarding the doubly-peaked ground state $\Psi_\hbar^{(0)}$ of the symmetric double well as the quantum-mechanical counterpart of a hung parliament, the analogue of a small party that decides which coalition is formed is a tiny *asymmetric* perturbation $\delta V$ of the potential. Indeed, the following spectacular phenomenon in the theory of Schrödinger operators was discovered in 1981 by Jona-Lasinio, Martinelli and Scoppola [57, 58], using stochastic techniques. Using more conventional methods, it was subsequently reconfirmed and analyzed further by mathematical physicists [25, 46, 52, 53, 95].[18] In view of this extensive mathematical literature, we hardly see a need for yet another rigorous treatment, but rather take it as our goal to explain the main idea to physicists and philosophers. This section just gives the key results; a more detailed treatment using the well-known WKB approximation from the textbooks may be found in the appendix.

Replace $V$ in (2.9) by $V + \delta V$, where $\delta V$ (i.e., the "flea") is assumed to:

1. be real-valued with fixed sign, and $C_c^\infty$ (hence bounded) with connected support not including the minima $x = a$ or $x = -a$;[19]
2. satisfy $|\delta V| >> e^{-d_V/\hbar}$ for sufficiently small $\hbar$ (e.g., by being independent of $\hbar$);
3. be localized not too far from at least one the minima, in the following sense.

   First, for $y, z \in \mathbb{R}$ and $A \subset \mathbb{R}$, we extend the notation (2.15) to

$$d_V(y,z) = \left| \int_y^z dx\, \sqrt{V(x)} \right|; \tag{3.24}$$
$$d_V(y,A) = \inf\{d_V(y,z), z \in A\}. \tag{3.25}$$

   Second, we introduce the symbols

$$d'_V = 2 \cdot \min\{d_V(-a, \text{supp}\ \delta V), d_V(a, \text{supp}\ \delta V)\}; \tag{3.26}$$
$$d''_V = 2 \cdot \max\{d_V(-a, \text{supp}\ \delta V), d_V(a, \text{supp}\ \delta V)\}. \tag{3.27}$$

The localization assumption on $\delta V$, then, is that one of the following conditions holds:

$$d'_V < d_V < d''_V; \tag{3.28}$$
$$d'_V < d''_V < d_V. \tag{3.29}$$

In the first case, the perturbation is typically localized either on the left or one the right edge of the double well, whereas in the second it resides somewhere on the middle bump. Note that symmetric perturbations are excluded by 3., as these would satisfy $d'_V = d''_V$.

Under these assumptions, the ground state wave-function $\Psi_\hbar^{(\delta)}$ of the perturbed Hamiltonian (which had two peaks for $\delta V = 0$!) localizes as $\hbar \to 0$, in a direction which *given that localization happens* may be understood from energetic considerations. For example, if $\delta V$ is positive and is localized to the right, then the relative energy in the left-hand part of the double well is lowered, so that localization will be to the left. See Figures 4 - 6.

[18] The "Flea on the Elephant" terminology used in [95] for the phenomenon in question evidently motivated the title of the present paper, which has identified the proper host animal at last!

[19] Some of the details in this section depend on the latter assumption, but our overall scenario in section 4 does not. For example, if the value and/or the curvature of one of the minima is decreased, then the ground state wave-function will localize above that minimum, as follows from standard minimax techniques taking single harmonic eigenfunctions as trial states [46, 84]. So collapse is actually easier in that case.

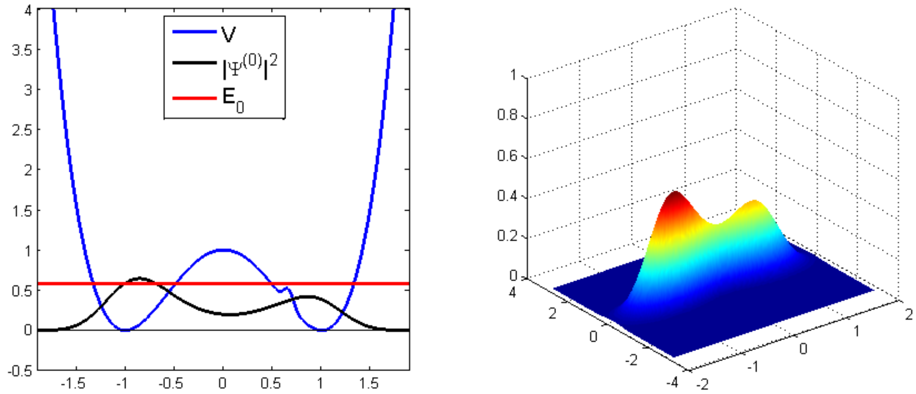


Figure 4: Flea perturbation of ground state $\Psi^{(\delta)}_{\hbar=0.5}$ with corresponding Husumi function. For such relative large values of $\hbar$, little (but some) localization takes place.

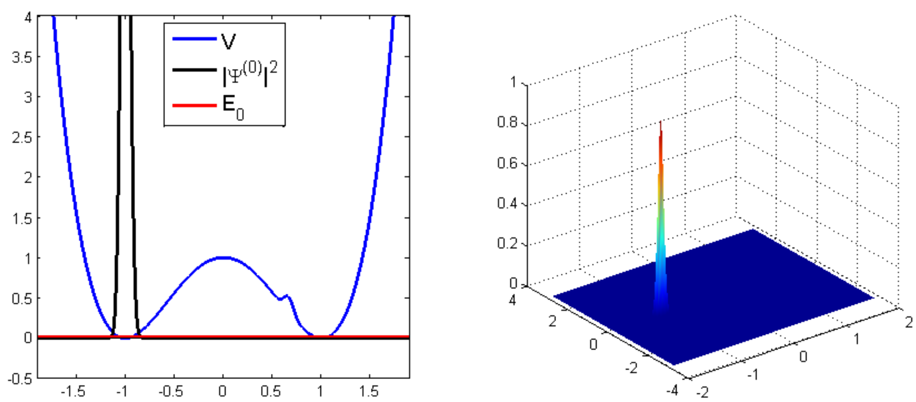


Figure 5: Same at $\hbar = 0.01$. For such small values of $\hbar$, localization is almost total.

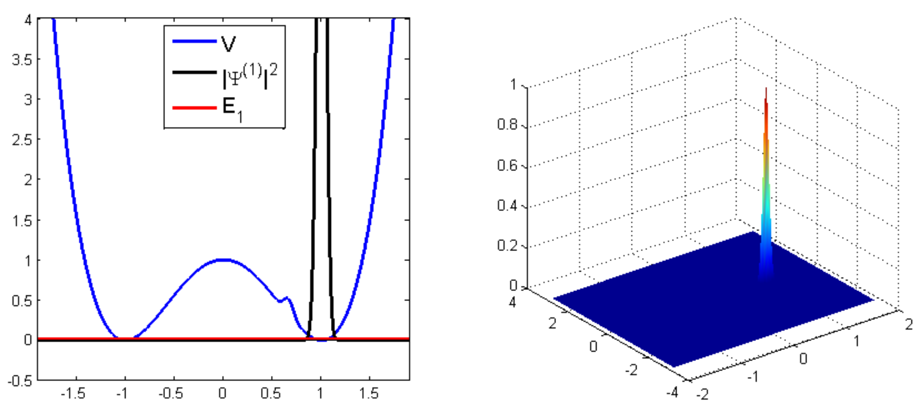


Figure 6: First excited state for $\hbar = 0.01$. Note the opposite localization area.

In more detail, for the perturbed ground state we have (subject to assumptions 1–4):

$$\frac{\Psi_\hbar^{(\delta)}(a)}{\Psi_\hbar^{(\delta)}(-a)} \sim e^{\mp d_V/\hbar} \quad (\pm\delta V > 0,\ \mathrm{supp}(V) \subset \mathbb{R}^+); \tag{3.30}$$

$$\frac{\Psi_\hbar^{(\delta)}(a)}{\Psi_\hbar^{(\delta)}(-a)} \sim e^{\pm d_V/\hbar} \quad (\pm\delta V > 0,\ \mathrm{supp}(V) \subset \mathbb{R}^-), \tag{3.31}$$

with the opposite localization for the perturbed first excited state (so as to remain orthogonal to the ground state).[20] A more precise version of the energetics used above is then as follows. The ground state tries to minimize its energy according to the rules:[21]

- The cost of localization (if $\delta V = 0$) is $\mathcal{O}(e^{-d_V/\hbar})$.
- The cost of turning on $\delta V$ is $\mathcal{O}(e^{-d'_V/\hbar})$ when the wave-function is delocalized.
- The cost of turning on $\delta V$ is $\mathcal{O}(e^{-d''_V/\hbar})$ when the wave-function is localized in the well around $x_0 = \pm a$ for which $d_V(x_0, \mathrm{supp}\ \delta V) = d''_V$.

In any case, these results only depend on the support of $\delta V$, but not on its size: this means that the tiniest of perturbations may cause collapse in the classical limit.

Although the collapse of the perturbed ground state for small $\hbar$ is a mathematical theorem, supported (or rather illustrated) both by our numerical simulations and by the WKB analysis in the appendix, it remains an enigmatic phenomenon of a purely quantum-mechanical nature. Indeed, despite the fact that in quantum theory the localizing effect of the flea is enhanced for small $\hbar$, the corresponding classical system has no analogue of it. Trivially, a classical particle residing at one of the two minima of the double well at zero (or small) velocity, i.e., in one of its degenerate ground states, will not even notice the flea; the ground states are unchanged. But even under a stochastic perturbation, which leads to a nonzero probability for the particle to be driven from one ground state to the other in finite time (as some form of classical "tunneling", where in this case the necessary fluctuations come from Brownian motion), the flea plays a negligible role. For example, in the case at hand the famous Eyring–Kramers formula for the mean transition time reads

$$\langle\tau\rangle \cong \frac{2\pi}{\sqrt{V''(a)V''(0)}} e^{V(0)/\epsilon}, \tag{3.32}$$

where $\epsilon$ is the parameter in the pertinent Langevin equation $dx_t = -\nabla V(x_t)dt + \sqrt{2\epsilon}dW_t$, in which $W_t$ is standard Brownian motion.[22] Clearly, this expression only contains the height of the potential at its maximum and its curvature at its critical points; most perturbations satisfying assumptions 1–4 above do not affect these quantities.

[20] If $\delta V$ has support on both sides of the real axis (which is possible in the case (3.29)), a more detailed analysis of its shape is necessary in order to predict the direction of collapse.

[21] Compare [84, 95, p. 35] for such arguments. Nonetheless, the effect of the flea is counterintuitive even from the point of view of quantum-mechanical tunneling: for example, with a perturbation of the kind displayed in Figures 4 - 6, which falls under case (3.29), one would expect tunneling from the right into the left-handed well to be discouraged, even increasingly so as $\hbar \to 0$, because the potential barrier through which to tunnel has been heightened, but in fact the right-handed peak of the unperturbed ground state tunnels to the left so as to localize the ground state wave-function. See §5.2 for further discussion.

[22] Cf. [7] (for mathematicians) or [49] (for physicists), and references therein.

### 3.2 Two-level approximation

The instability of the ground state of the double-well potential under "flea" perturbations as $\hbar \to 0$ is easy to understand (at least heuristically) if one truncates the infinite-dimensional Hilbert space $L^2(\mathbb{R})$ to a two-level system living in $\mathbb{C}^2$ [52, 95].[23] This simplification is accomplished by keeping only the lowest energy states $\Psi_\hbar^{(0)}$ and $\Psi_\hbar^{(1)}$, in which case the full Hamiltonian (2.9) with (2.12) is reduced to the $2 \times 2$ matrix

$$H_0 = \tfrac{1}{2} \begin{pmatrix} 0 & -\Delta \\ -\Delta & 0 \end{pmatrix}, \tag{3.33}$$

with $\Delta > 0$ given by (2.14). We drop the label $\hbar$. The eigenstates of $H_0$ are given by

$$\Phi_0^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \Phi_0^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{3.34}$$

with energies $E_0 = 0$ and $E_1 = \Delta$, respectively; in particular, $E_1 - E_0 = \Delta$. If

$$\Phi_0^{\pm} = \frac{\Phi_0^{(0)} \pm \Phi_0^{(1)}}{\sqrt{2}}, \tag{3.35}$$

as in (2.18), then

$$\Phi_0^{+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi_0^{-} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.36}$$

Hence in this approximation $\Phi_0^+$ and $\Phi_0^-$ play the role of wave-functions (2.18) localized above the classical minima $x = +a$ and $x = -a$, respectively, with classical limits $\rho_0^{\pm}$. The "flea", then, is introduced as follows: if its support is in $\mathbb{R}^+$, then we put

$$\delta_+ V = \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix}, \tag{3.37}$$

where $\delta \in \mathbb{R}$ is a constant, whereas a perturbation with support in $\mathbb{R}^-$ is approximated by

$$\delta_- V = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.38}$$

Without loss of generality, let us take the latter (a change of sign of $\delta$ leads to the former). The eigenvalues of $H^{(\delta)} = H_0 + \delta_- V$ are $E_0 = E_-$ and $E_1 = E_+$, with energies

$$E_{\pm} = \tfrac{1}{2}(\delta \pm \sqrt{\delta^2 + \Delta^2}), \tag{3.39}$$

and normalized eigenvectors

$$\Phi_\delta^{(0)} = \frac{1}{\sqrt{2}} \left(\delta^2 + \Delta^2 + \delta\sqrt{\delta^2 + \Delta^2}\right)^{-1/2} \begin{pmatrix} \Delta \\ \delta + \sqrt{\delta^2 + \Delta^2} \end{pmatrix}; \tag{3.40}$$

$$\Phi_\delta^{(1)} = \frac{1}{\sqrt{2}} \left(\delta^2 + \Delta^2 - \delta\sqrt{\delta^2 + \Delta^2}\right)^{-1/2} \begin{pmatrix} \Delta \\ \delta - \sqrt{\delta^2 + \Delta^2} \end{pmatrix}. \tag{3.41}$$

Note that $\lim_{\delta \to 0} \Phi_\delta^{(i)} = \Phi_0^{(i)}$ for $i = 0, 1$. Now, if $\hbar \to 0$, then $|\delta| >> \Delta$, in which case $\Phi_\delta^{(0)} \to \Phi_0^{\pm}$ for $\pm\delta > 0$ (and if we had started from (3.37) instead of (3.38), one would have had the opposite case, i.e., $\Phi_\delta^{(0)} \to \Phi_0^{\mp}$ for $\pm\delta > 0$). Thus the ground state localizes as $\hbar \to 0$, which resembles the situation (3.30) - (3.31) for the full double-well problem.

[23] This approximation is extremely well known also in physics [70], but has hardly been studied in the present context. It is too simple to display the behaviour (2.21) - (2.23), though.

# 4 Time-dependent collapse

As already remarked in Subsection 2.3, for a solution of the measurement problem it is not enough to just note that under a typical "flea" type perturbation (as defined in Subsection 3.1) the ground state $\Psi_\hbar^{(\delta)}$ of the perturbed Hamiltonian is localized. In addition, the archetypical Schrödinger Cat state $\Psi_\hbar^{(0)}$, which results from some measurement, needs to evolve into $\Psi_\hbar^{(\delta)}$ under the influence of this perturbation. This is a very complicated problem in quantum metastability, about which little is known. But we can say something.

As a first orientation, we continue our discussion of the two-level system. The simplest idea would be to launch the flea as a so-called "quench", which means that for times $t < 0$ the dynamics is given by $H_0$, upon which for $t \geq 0$ the Hamiltonian is $H_0 + \delta\_V$. Hence

$$H(t) = \begin{pmatrix} \delta(t) & -\frac{1}{2}\Delta \\ -\frac{1}{2}\Delta & 0 \end{pmatrix}, \tag{4.42}$$

where $\delta(t) = 0$ for $t < 0$ and $\delta(t) = \delta$ for $t \geq 0$. Writing $\Phi^{(0)}(t)$ for the solution of the corresponding time-dependent Schrödinger equation with initial condition $\Phi^{(0)}(0) = \Phi_0^{(0)}$, see (3.34), for the localization probability "on the left", i.e., above $x = -a$, we find

$$P_L(t) \equiv |\langle \Phi_0^-, \Phi^{(0)}(t)\rangle|^2 = P_L(0) + \tfrac{1}{2}\frac{\delta\Delta}{\delta^2+\Delta^2} \cdot \left[\cos\left(\frac{it}{\hbar}\sqrt{\delta^2+\Delta^2}\right) - 1\right], \tag{4.43}$$

where $\Phi_0^-$ is given in (3.36). Since $\delta\Delta/(\delta^2 + \Delta^2) \to 0$ as $\hbar \to 0$, we see from this and similar calculations for other initial states that for any $t$ (including $t \to \infty$ in whatever, even $\hbar$-dependent, way), in the classical limit the initial state freezes rather than collapses.

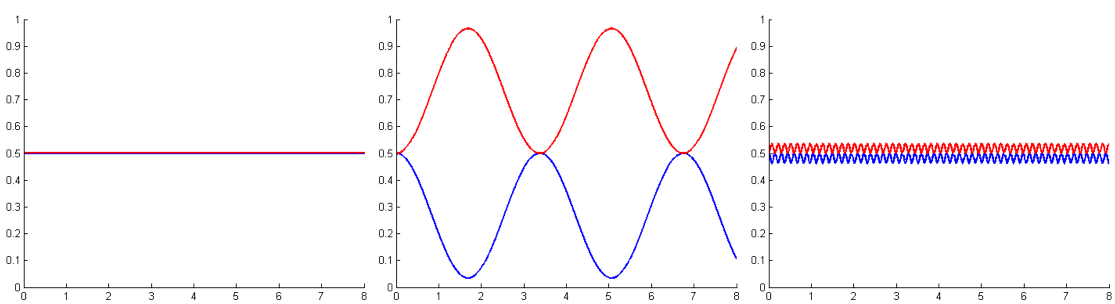

Figure 7: Time evolution of the probabilities $P_L(t)$ and $P_R(t) = 1 - P_L(t)$. The left image has $\delta = 0$, the middle has moderate $\delta$, and the third (displaying "freezing") has large $\delta$.

Towards less naive time-dependent models for the flea perturbation, we also investigated adding white noise or Poisson noise to the time-dependent Schrödinger equation. In the two-level case, the pertinent stochastic differential equations are

$$d\Phi = -(\tfrac{1}{2}i\Delta\sigma_x dt + i\delta\sigma_z dB_t + \tfrac{1}{2}\delta^2 dt)\Phi; \tag{4.44}$$
$$d\Phi = -(\tfrac{1}{2}i\Delta\sigma_x dt + (\sigma_z - \mathbb{I}_2)dN_t)\Phi, \tag{4.45}$$

respectively, where $(\sigma_k)$ are the Pauli matrices, $B_t$ is Brownian motion, and $N_t$ is a Poisson process, both with tunable parameters. However, neither of these leads to dynamical collapse in the classical limit: this is equivalent to strong noise, in which case a quantum Zeno-like effect seems to dominate any desire of the system to localize. See also [8, 11, 73].

Similar (negative) conclusions follow for the full double well (at least, numerically). As far as we have been able to determine, the most effective way to produce dynamical collapse is to let the flea jump on the cat adiabatically. This is easily shown for the two-level system, but we might as well return to the full double-well problem here. We perturb this potential $V$ with a flea with center $b$, width $2c$, and height $d$, as follows:

$$\delta V_{b,c,d}(x) = \begin{cases} d \cdot e^{\frac{1}{c^2} - \frac{1}{c^2-(x-b)^2}} & \text{if } |x-b| < c \\ 0 & \text{if } |x-b| > c \end{cases} . \tag{4.46}$$

This perturbation rises adiabatically according to

$$V(x,t) = \begin{cases} V(x) & \text{if } t \leq 0; \\ V(x) + \delta V_{b,c,d}(x) \sin\left(\frac{\pi t}{2T}\right) & \text{if } 0 \leq t \leq T; \\ V(x) + \delta V_{b,c,d}(x) & \text{if } t > T. \end{cases} \tag{4.47}$$

The corresponding time-dependent Schrödinger equation can be solved numerically with the ground state $\Psi_\hbar^{(0)}$ as the initial condition at $t = 0$. This yields the following pictures, in which dynamical localization is clearly visible, in agreement with the adiabatic theorem.[24]

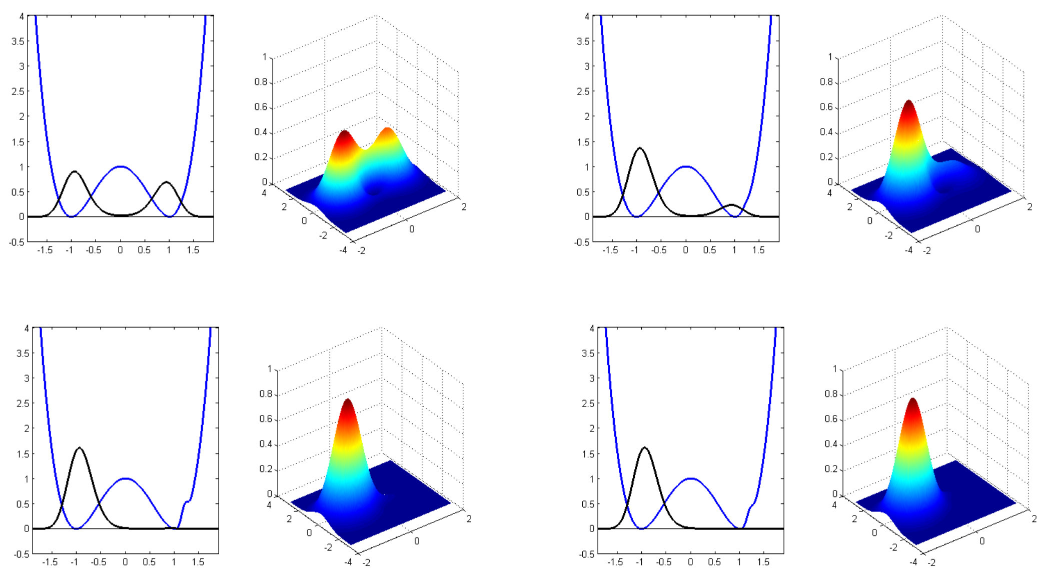

Figure 8: Plots of both $|\Psi(t)|^2$ and the corresponding Husumi function for the solution $\Psi(t)$ of the time-dependent Schrödinger equation defined by the potential (4.47), with $b = 7.5$, $c = 0.5$, $d = 0.3$, $\hbar = 0.3$, and $T = 800$. Starting in the upper left corner and proceeding clockwise, the pictures correspond to $t = 50$, $t = 100$, $t = 400$, $t = 800$.

Of course, the symmetry of the situation implies that the Born rule holds if one averages over all perturbations (which corresponds to averaging over a series of experiments) in any reasonable way, i.e., any way in which $\delta V_{b,c,d}$, $\delta V_{-b,c,d}$, $\delta V_{b,c,-d}$, and $\delta V_{-b,c,-d}$ have equal probability. For in that case, according to the rules in Subsection 3.1 any collapse to the left (in some experiment in a long run) will be accompanied by a collapse to the right (in another experiment of the same run) if one of the signs changes. Hence the probabilities for collapse to the left and to the right will both equal 1/2, in agreement with (2.17).

[24] A corresponding movie may be found on `www.math.ru.nl/~landsman/flea.avi`.

# 5 Discussion

To put the above result in perspective: we claim neither that the double-well system realistically models Schrödinger's Cat, nor that, even if it does, a typical perturbation that causes the collapse of the wave-function should rise adiabatically.[25] But we do believe that the instability that works in this particular example is an example of the general mechanisms that solves the measurement problem (i.e., in our reformulated version), and we do claim to have given a "proof of concept" that our envisaged solution to the measurement problem is viable in principle. But many issues remain to be resolved.

Leaving details to future papers, we briefly discuss three topics. The first indicates that the measurement problem (and hence the mechanism we propose to resolve it) has a far wider scope than is usually imagined, the second points at the unexplored territory of quantum metastability, whilst the third touches on the very issue of determinism.

## 5.1 Symmetry breaking and phase transitions

The symmetric double-well potential provides one of the simplest models of spontaneous symmetry breaking (SSB). Both the classical Hamiltonian and its quantization have a $\mathbb{Z}_2$ symmetry, given by reflection in the origin of the $x$-axis. As we have seen, a remarkable difference between classical and quantum mechanics arises: the classical ground state is degenerate and breaks the symmetry, whereas the quantum ground state $\Psi_\hbar^{(0)}$ is unique and hence symmetric. If we see the splitting of the ground state as a phase transition, then evidently the quantum system has no phase transition, whereas its classical counterpart does. At first sight this appears to be quite paradoxical, since the presence or absence of symmetry breaking is a major *qualitative* difference between the system described either classically or quantum-mechanically, while at the same time we *quantitatively* expect the classical theory to be a limiting case of the quantum theory. Indeed, this is nothing but the measurement problem in disguise: if, for any $\hbar > 0$, the (delocalized) quantum ground state prevails, then the classical ground states $\rho_0^{\pm}$ totally fail to be approximated by it.

We resolved this problem by the "flea" instability. Similarly, the ground state of a large but finite quantum systems ($V < \infty$) is typically unique and hence symmetric. But at $V = \infty$, for suitable Hamiltonians SSB occurs, in that the ground state (or thermal equilibrium state at low temperature) fails to be symmetric. Thus the limit $V$ *to* infinity does not approximate the phenomenon of SSB when $V$ *equals* infinity.[26] Based on [61], we expect to find some analogue of the "flea" perturbation and expect it to be especially effective for large $V$. This should destabilize the ground state so as to break the symmetry already in large *but finite* volume. An instability like this *must* underly the Higgs mechanism, if this is to be phenomenologically relevant (as it indeed seems to be since July 4, 2012).

[25] If it happens to be true that measurement outcomes emerge adiabatically, it would be a marked break with tradition, starting with von Neumann's model, in which both the measurement interaction and the alleged collapse take place instantly [78]. Of course, the question arises relative to which time scale the flea would enter adiabatically, if it does.

[26] Like the measurement problem, this seemingly paradoxical situation does not seem to bother physicists very much, although their Higgs mechanism relies on a resolution of it: apparently, in any finite volume the system refuses to choose a ground state (or vacuum), although all perturbative calculations underlying the successful Standard Model of elementary particle physics rely on such a choice. But it has been the subject of recent discussions in the philosophy of science [4, 19, 71, 75, 81], in which some claim that this "discontinuity" in passing from $V < \infty$ to $V = \infty$ is crucial for the possibility of emergence ('More is Different'), whilst others try to find arguments for continuity and hence defend some form of reductionism.

## 5.2 Quantum metastability

According to our analysis, the measurement problem is really a problem in *quantum metstability*. Metastability is well understood if it is thermally driven, both in classical and in quantum theory [49, 82, 91, 92], but in our approach the driving force is not a persistent heat bath or Brownian motion but a minute single perturbation that (for small $\hbar$) dramatically changes the ground state of a quantum system. In that case, it is quantum fluctuations that are supposed to drive the old (unperturbed) ground state to the new (perturbed) one. Little is known about this situation even heuristically [79, 94], let alone analytically (hence our recourse to a numerical approach). What we can say, on the basis of both these simulations and physical intuition (for what it is worth), is that the limit $\hbar \to 0$ is a double-edged sword: on the one hand, it enhances the instability of the original (Schrödinger Cat) ground state, and hence favours static collapse (in that the perturbed ground state is localized, unlike the original one), but on the other hand, it suppresses tunneling and hence acts against dynamical collapse (in the sense that the unperturbed ground state evolves into the perturbed one under the influence of the perturbed dynamics). This explains the exceptionally long time it takes for localization to happen in our numerical simulations, well beyond the (already long) time scales typical for the thermal case. We expect the same phenomenon in (quantum) phase transitions, as above.

## 5.3 Determinism and Bell/Free Will type Theorems

In so far as determinism is concerned, there are two ways to look at our proposal.

First, the "flea" perturbation might itself be a genuine random process, perhaps ultimately being of quantum-mechanical origin. In that case, its own intrinsic randomness is simply transferred to the set of possible measurement outcomes. Although the flea may still be said to "cause" one particular outcome (of some experiment), and as such solves the measurement problem, it fails to restore determinism. Rather, the experiment amplifies the randomness that was already inherent in the flea.[27]

Second, the flea might be deterministic (but is just modeled stochastically for pragmatic reasons). This opens the door to a complete restoration of determinism. For now the flea transfers its determinism to the experiment (rather than its randomness, as in the previous scenario). The mistaken impression that quantum theory implies the irreducible randomness of nature then arises because measurement outcomes are merely unpredictable "for all practical purposes", though in a way that (because of the exponential sensitivity to the flea in $1/\hbar$) dwarfs even the unpredictability of classical chaotic systems.

In both cases, one has to deal with the Bell inequalities [5, 17] or the Free Will Theorem (FWT) [27]. In this respect the situation is the same as for dynamical collapse models à la GRW [3]: such models are necessarily nonlocal, but they do satisfy a no-signalling theorem.[28] To see which assumption of e.g. the FWT is violated,[29] note that the flea entails local contextuality, which by the TWIN assumption of Conway and Kochen induces a violation of FIN (in the original FWT) or MIN (in the Strong Free Will Theorem).[30]

[27] See [23] for a recent discussion of randomness amplification, which focuses on the way experiments may be construed to amplify the randomness inherent in the (alleged) "free" choice of an experimentalist.

[28] A similar analysis holds also for Bohmian solutions to the measurement problem.

[29] In our opinion, the Free Will Theorem is sharper than any kind of constraint derived from Bell-type inequalities, whose derivation relies on tacit assumptions like the use of the Kolmogorov formalism for measure theory in averaging over hidden states. This formalism lacks a sound conceptual foundation.

[30] In current parlance surrounding the Bell inequalities [17], *parameter independence* is violated.

# 6 Appendix: the flea from WKB

In this appendix,[31] we study the "flea" type perturbation from the point of view of the WKB method of the physics textbooks (like [47, 62]).[32] As explained in [41, 42], the connection formulae stated in such books are actually correct only for simple potentials like a single well, but with due modifications (see below), the formalism will reproduce both the rigorous and the numerical results described in the main body of this paper.

## 6.1 Quantization condition for an asymmetric double well

We start by recalling some standard WKB formulas. The WKB wave-function in the classically allowed region without turning points ($E > V(x)$) can be written as

$$\Psi(x) \cong \frac{1}{\sqrt{p(x)}}\left[Ae^{\frac{i}{\hbar}\int^x p(y)dy} + Be^{-\frac{i}{\hbar}\int^x p(y)dy}\right], \tag{6.48}$$

where

$$p(x) = \begin{cases} \sqrt{[E-V(x)]} & \text{if } E \geq V(x) \\ \pm i\sqrt{[V(x)-E]} & \text{if } E < V(x) \end{cases}. \tag{6.49}$$

A similar formula holds for the classically forbidden region ($E < V(x)$), namely

$$\Psi(x) \cong \frac{1}{\sqrt{|p(x)|}}\left[Ce^{-\frac{1}{\hbar}\int^x |p(y)|dy} + De^{\frac{1}{\hbar}\int^x |p(y)|dy}\right]. \tag{6.50}$$

These wave-functions can be connected across turning points via so-called connection formulas, stated in books like [47]. First, we need to distinguish between two kinds of turning points in the usual way: we use the coefficients $A_l$, $B_l$, $C_l$ and $D_l$ for a left-hand turning point and $A_r$, $B_r$, $C_r$ and $D_r$ for a right-hand one. The lower limit of the integrals in the above equations is always the coordinate of the turning point. The connection formulas for a left-hand turning point are given by

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \overbrace{e^{i\pi/4}\begin{pmatrix} \frac{1}{2} & -i \\ -\frac{i}{2} & 1 \end{pmatrix}}^{M_{C_l/D_l\to A_l/B_l}} \begin{pmatrix} C_l \\ D_l \end{pmatrix} \text{ or } \begin{pmatrix} C_l \\ D_l \end{pmatrix} = \overbrace{e^{-i\pi/4}\begin{pmatrix} 1 & i \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix}}^{M_{A_l/B_l\to C_l/D_l}} \begin{pmatrix} A_l \\ B_l \end{pmatrix}, \tag{6.51}$$

whilst those for a right-hand turning point are given by

$$\begin{pmatrix} A_r \\ B_r \end{pmatrix} = \overbrace{e^{i\pi/4}\begin{pmatrix} 1 & -\frac{i}{2} \\ -i & \frac{1}{2} \end{pmatrix}}^{M_{C_r/D_r\to A_r/B_r}} \begin{pmatrix} C_r \\ D_r \end{pmatrix} \text{ or } \begin{pmatrix} C_r \\ D_r \end{pmatrix} = \overbrace{e^{-i\pi/4}\begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ i & 1 \end{pmatrix}}^{M_{A_r/B_r\to C_r/D_r}} \begin{pmatrix} A_r \\ B_r \end{pmatrix}. \tag{6.52}$$

Now consider a general asymmetric double well, as shown in Figure 9. This figure also introduces part of the notation used.

[31] The authors are indebted to Koen Reijnders (Radboud University) for help with this appendix.

[32] As opposed to the extremely sophisticated and mathematically rigorous methods of Helffer and Sjöstrand [33, 52, 53], who somewhat confusingly suggest they use the ordinary WKB method.

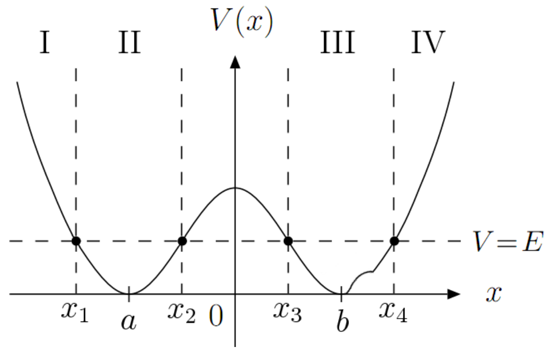


Figure 9: An asymmetric double-well potential $V$. The minima are $a$ and $b$. We assume that the particle has energy $E$. This provides us with turning points $x_1$, $x_2$, $x_3$ and $x_4$ and hence with five distinct regions. Four of these regions are named with Roman numerals.

We need some more notation for the WKB coefficients used in our calculation. As in (6.48) and (6.50), $A, B$ and $C, D$ denote the coefficients of the WKB wave-function in the classically allowed region and the classically forbidden region, respectively. The number attached to a letter shows to which turning point it belongs, e.g. $A_1$ and $B_1$ are the coefficients of the WKB wave-function in region II with respect to $x_1$ (i.e. $x_1$ is the lower boundary of the integral in (6.48)). We also need the following three quantities:

$$\theta_1 = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)dx, \qquad \theta_2 = \frac{1}{\hbar}\int_{x_3}^{x_4} p(x)dx, \qquad K = \frac{1}{\hbar}\int_{x_2}^{x_3} |p(x)|dx. \tag{6.53}$$

A final quantity we need is

$$\tilde{\phi} = \arg\left[\Gamma\left(\tfrac{1}{2} + i\frac{K}{\pi}\right)\right] + \frac{K}{\pi} - \frac{K}{\pi}\ln\left(\frac{K}{\pi}\right). \tag{6.54}$$

We are interested in the limit $K \to \infty$, since this implies that the barrier is very high and broad, which corresponds to the classical limit $\hbar \to 0$. Note that $\tilde{\phi} \to 0$ as $K \to \infty$. Our goal is the following quantization condition for the general double well in Figure 9:

$$\left(1 + e^{-2K}\right)^{1/2} = \frac{\cos(\theta_1 - \theta_2)}{\cos(\theta_1 + \theta_2 - \pi + \tilde{\phi})}\ . \tag{6.55}$$

This condition can be derived in the following way:

1. We start out in region I (coefficients $C_1$ and $D_1$). The wave-function needs to be square integrable, so we immediately see that $C_1 = 0$.

2. Using the left connection matrix from (6.51), we move to region II (coefficients $A_1$ and $B_1$). We can then write the WKB wave-function with respect to $x_2$ by using

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \tag{6.56}$$

which can be proved by changing the lower boundary of the integrals in the WKB wave-function (6.48). The result is

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} -ie^{i\theta_1} \\ e^{-i\theta_1} \end{pmatrix} D_1. \tag{6.57}$$

3. In a similar way, we start in region IV (coefficients $C_4$ and $D_4$), and see that $D_4 = 0$. After moving to region III with a connection matrix and rewriting the wave-function with respect to $x_3$, we find

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} e^{-i\theta_2} \\ -ie^{i\theta_2} \end{pmatrix} C_4 \ . \tag{6.58}$$

4. We now use a result derived in [41] to jump over the barrier and connect the WKB wave-functions in region II and III, viz.[33]

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \left(1+e^{2K}\right)^{1/2} e^{-i\tilde{\phi}} & ie^{K} \\ -ie^{K} & \left(1+e^{2K}\right)^{1/2} e^{i\tilde{\phi}} \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}. \tag{6.59}$$

5. Combining the above results (i.e. inserting (6.57) and (6.58) in (6.59)), we find

$$\frac{D_1}{C_4} = i\left[\left(1+e^{2K}\right)^{1/2} e^{-i(\theta_1+\theta_2+\tilde{\phi})} + e^{K} e^{-i(\theta_1-\theta_2)}\right], \tag{6.60}$$

$$\frac{D_1}{C_4} = -i\left[\left(1+e^{2K}\right)^{1/2} e^{i(\theta_1+\theta_2+\tilde{\phi})} + e^{K} e^{i(\theta_1-\theta_2)}\right]. \tag{6.61}$$

6. The equality of the above two equations leads to the quantization condition (6.55).

As will be discussed in the next two subsections, eqs. (6.55) and (6.60) have implications for the energy levels and the wave-functions in an asymmetric double well.

### 6.2 Energy splitting in an asymmetric double-well potential

Assume that for a certain (unperturbed) symmetric double well and given energy $E$, the constants $\theta_1$ and $\theta_2$ equal some value $\theta$. As in Figure 9, we introduce a perturbation in the right-hand well. For example, by (6.54), this means that $\theta = \theta_1 > \theta_2$ for a positive perturbation. We therefore write $\theta_1 = \theta$, $\theta_2 = \theta - \delta$ with $\delta \in \mathbb{R}$ (e.g. $\delta > 0$ in Figure 9). The quantization condition (6.55) then becomes

$$\left(1+e^{-2K}\right)^{1/2} = \frac{\cos(\delta)}{\cos(2\theta - \delta - \pi + \tilde{\phi})} \ . \tag{6.62}$$

We can solve for $\theta$, yielding two solutions

$$\theta_{\pm} = (n+\tfrac{1}{2})\pi + \tfrac{1}{2}\delta - \tfrac{1}{2}\tilde{\phi} \pm \tfrac{1}{2}\arccos\left[\frac{\cos(\delta)}{(1+e^{-2K})^{1/2}}\right]. \tag{6.63}$$

This resembles the original quantization condition $\theta = (n+\frac{1}{2})\pi$ for a single well, which is derived using connection formulas in [47]. Here, the energy levels have split up in pairs around the original ones (where the minus sign in (6.63) corresponds to the lower energy by (6.54)). To see what this means, we will examine this equation for two special cases. We first set $\delta = 0$ and check if this reproduces known results for a symmetric double well:

$$\theta_{\pm} = (n+\tfrac{1}{2})\pi - \tfrac{1}{2}\tilde{\phi} \pm \tfrac{1}{2}\arccos\left[\frac{1}{(1+e^{-2K})^{1/2}}\right]. \tag{6.64}$$

[33] This result can also be found by applying the method of comparison equations, which is explained in [99]. Further references are [72] and [77].

Supposing that $K$ is large, this means that

$$\theta_\pm \approx (n+\tfrac{1}{2})\pi \pm \tfrac{1}{2}e^{-K}, \tag{6.65}$$

since for $K$ large, $\tilde{\phi} \approx 0$ and $\arccos\left(\frac{1}{\sqrt{1+x^2}}\right) = \arctan x \approx x$ for small $x$. We once again find that the energy levels of the single well have split into two. As discussed in [42], this leads exactly to the familiar energy splitting for a symmetric double-well potential stated in texts like [62]. That means that our method for general double wells reproduces known results for a symmetric one. Now that this has been confirmed, let us look at (6.63) in the classical limit $K \to \infty$. Solving (6.63) for $K \to \infty$ (and so $\tilde{\phi} \to 0$) gives

$$\theta_- = (n+\tfrac{1}{2})\pi \text{ (lower energy) }; \tag{6.66}$$
$$\theta_+ = \delta + (n+\tfrac{1}{2})\pi \text{ (higher energy)}. \tag{6.67}$$

This differs from the symmetric well, which for $K \to \infty$ gives a twofold degeneracy for each energy level labeled by $n$. Equation (6.67) can be understood in the following way: in the classical limit, tunneling is suppressed. Therefore, the particle is localized in one of the wells, where it obeys the familiar quantization condition for a single well. If it is in the left well, then $\theta_1 = (n+\frac{1}{2})\pi = \theta_-$, but if it is in the right well, we have $\theta_2 = (n+\frac{1}{2})\pi = \theta_+ - \delta$.

## 6.3 Localization in an asymmetric double-well potential

Now that we have analyzed the behaviour of the energy splitting, we turn to the WKB wave-function. With the notation used in the previous section, (6.60) leads to

$$\frac{D_1}{C_4} = i\left[\left(1+e^{2K}\right)^{1/2} e^{-i(2\theta_\pm - \delta + \tilde{\phi})} + e^K e^{-i\delta}\right]. \tag{6.68}$$

Inserting (6.63), the reader can check that for $\delta \in [-\pi, \pi]$ one has

$$\frac{D_1}{C_4} = \sin(\delta)e^K \mp \sqrt{\sin^2(\delta)e^{2K}+1}\ . \tag{6.69}$$

This allows us to derive localization of the WKB wave-function in the classical limit $K \to \infty$. As can be seen from (6.57), $D_1$ is a measure of the amplitude of the WKB wave-function in regions I and II in Figure 9. In a similar way, (6.58) shows that $C_4$ is a measure of the amplitude of the WKB wave-function in regions III and IV. Therefore, the fraction $D_1/C_4$ indicates whether the wave-function is localized, and if so, where. Doing the same calculation again for $\delta \in [\pi, 3\pi]$ gives the above result multiplied by $-1$. Of course, this can be generalized: for $n \in \mathbb{Z}$ and $\delta \in [(2n-1)\pi, (2n+1)\pi]$, the result (6.69) is correct for $n$ even and should be multiplied by $-1$ for $n$ odd. This will not affect our conclusions, as we will see. We consider some cases and check what (6.69) tells us:

- For $\delta = 0$ (no perturbation), we find that $\frac{D_1}{C_4} = \mp 1$. The general double well has pairs of energy levels (labeled by $n$). Such a pair consists of a lower and higher lying level, corresponding to $\theta_-$ and $\theta_+$ in (6.63), respectively. Here, we see that for the lower level $D_1 = C_4$, i.e. the WKB wave-function is even. However, for the higher level we find $D_1 = -C_4$, which means the WKB wave-function is odd. This is a well-known fact and it is nice to see our method reproducing it. Note that this conclusion is not only independent of $n$, but also of $K$, as expected.

- For $\delta > 0, \delta \notin \{k\pi | k \in \mathbb{Z}\}$ (which corresponds to a positive perturbation in the right well, e.g. the potential in Figure 9), we find, in the limit $K \to \infty$, that:

$$\frac{D_1}{C_4} \longrightarrow \begin{cases} \infty & \text{for } \theta_- \text{ in (6.63) (lower energy)} \\ 0 & \text{for } \theta_+ \text{ in (6.63) (higher energy)} \end{cases} .$$

  Hence for low (high) energy, the WKB wave-function is localized on the left (right).

- For $\delta < 0,\ \delta \notin \{k\pi | k \in \mathbb{Z}\}$, i.e., a negative perturbation in the right well, we find

$$\frac{D_1}{C_4} \longrightarrow \begin{cases} 0 & \text{for } \theta_- \text{ in (6.63) (lower energy)} \\ \infty & \text{for } \theta_+ \text{ in (6.63) (higher energy)} \end{cases} .$$

  For the lower (higher) energy, the WKB wave-function is localized on the right (left).

- For $\delta \in \{k\pi | k \in \mathbb{Z} \backslash \{0\}\}$, something peculiar happens, in that either $\frac{D_1}{C_4} = \pm 1$ or $\frac{D_1}{C_4} = \mp 1$. This implies that no localization takes place.[34]

- So far, we have interpreted $\delta$ as the result of a perturbation in the right well. However, our approach allows us to interpret a positive perturbation in the right-hand well as a negative one in the left-hand well, and vice versa. Therefore, the above results change places if we put the perturbation in the left-hand well.

Our method produces the results we would expect. However, to be precise, the above reasoning needs to be amended as follows. We have treated $\delta$ as a constant, but in reality it depends on $K$. The reason for this is that $K$ affects $\theta_1$ and $\theta_2$, and therefore $\delta = \theta_1 - \theta_2$, via the quantization condition. Now consider a fixed energy level (i.e. fixed $n$ and fixed sign $\pm$ in (6.63)) in a given double-well potential that has a perturbation in one of the wells. In the limit of completely decoupled wells ($K \to \infty$), we know this energy level has some fixed limit higher than the minimum of the potential. As long as the perturbation is below this energy level, we know that $\theta_1 - \theta_2 \neq 0$ by (6.54). This means that there exists some $K_0$ such that $|\theta_1 - \theta_2| \neq 0$ for any $K > K_0$. We may then apply the above reasoning to verify that our conclusions about localization are still correct.[35]

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[34] This can be explained by level crossing, i.e. certain energy levels of the two individual wells coincide.

[35] To keep the discussion straightforward, we ignored the 'special' case $\delta \in \{k\pi | k \in \mathbb{Z} \backslash \{0\}\}$ here.

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