Anti-Realism and Anti-Revisionism in Wittgenstein's Philosophy of Mathematics

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Abstract
Since the publication of the *Remarks on the Foundations of Mathematics*, Wittgenstein's interpreters have endeavored to reconcile his general constructivist/anti-realist attitude towards mathematics with his confessed anti-revisionary philosophy. In this article, I revisit the issue and present a solution. The basic idea consists in exploring the fact that the so-called "non-constructive results" could be interpreted so that they do not appear non-constructive at all. I substantiate this solution by showing how the translation of mathematical results, given by the possibility of translation between logics, can be seen as a tool for partially implementing the solution Wittgenstein had in mind.

Keywords: Philosophy of mathematics; Wittgenstein; Anti-realism; Anti-revisionism;

1 Overview. Since the publication of the *Remarks on the Foundations of Mathematics* in 1956, Wittgenstein's interpreters have endeavored to reconcile his general anti-realist attitude towards mathematics (particularly his insistence on the idea that the mathematician is not a discoverer, but an inventor) with his confessed anti-revisionary philosophy. This topic has been approached by many authors, e.g., Wright (1980), Wrigley (1980), Redecker (2006) and Marion and Okada (2012). In this article, I revisit the issue and present a solution.

The basic idea, which I develop thoroughly later on, is the following: in his middle period, Wittgenstein thought that the mathematical proofs normally called "non-constructive" by mathematicians can and should be interpreted in constructive terms. Therefore, it is the interpretation of the proof that is revised, not the proof itself, which is preserved in its entirety. It is the standard "prose" of traditional mathematical results that suggests a non-constructive reading, while their correct analysis shows the proofs are in the end constructive. The temptation to call a mathematical result "non-constructive" arises from the fact that mathematical prose amalgamates different proofs by calling them proofs of the same mathematical proposition. It is said, for instance, that there are constructive and non-constructive proofs for the fact that there are infinitely many prime numbers. I.e., these two kinds of proofs are taken as proofs of the same end-result, namely, that the mathematical concept "prime number" has infinitely many instances. However, if we, as a result of an analysis of what has actually been done in the proofs, start to think that the proofs actually prove distinct mathematical propositions, we can resist this temptation and correct misunderstandings that arise from it. In other words, if we conceive of the proof as giving the identity criterion of a mathematical proposition, we are able to correct wrong perspectives resulting from the mathematical "prose".

It is true that, in his later writings, Wittgenstein came to see the idea that the proof is the identity criterion of a mathematical proposition as too simple. While he insisted in his middle period that to know what has been proved one must look at the proof, in his later writings he recognized that this was only a half-truth.[1] This change does not, however, mean that the old solution to the dilemma was entirely wrong. As far as I can see, the solution is only refined by alluding to a more differentiated identity criterion for mathematical propositions (a criterion that also considers, for example, the possible applications of the proposition).[2] Before considering the solution, however, I shall begin by presenting the issue.

2 Presenting the issue. The dilemma posed by Wittgenstein's anti-realism and his anti-revisionism in philosophy of mathematics is usually presented as a dilemma posed by constructivism and anti-revisionism. This is comprehensible, since the identification of anti-realism and constructivism is commonplace. The rationale behind this equation is that, contrary to the realist conception of the mathematician as an explorer of an independently given world of mathematical entities and its properties, the anti-realist holds that these entities and their properties are better called constructs (hence the term "constructivism"). I will present the dilemma in this form as well, but it is important to keep in mind that "anti-realism" and "constructivism" in philosophy of mathematics are different. Strictly speaking, the *constructivist* in philosophy of mathematics can be understood as the one who argues that the "path" to get to a particular mathematical object, or to show that this object has a certain property, or that some mathematical objects are in some relation to one another, etc. is not a mere expedient, but is essential to the object, property or relation (because the path "constructs", so to speak, the object, property or relation[3]). Because of this, the constructivist says that proofs in mathematics are not merely vehicles for obtaining mathematical truths, but are in some sense constitutive of these truths.

Now, according to the commonly accepted opinion, constructivism has revisionary consequences for mathematical practice, such as the rejection of reasonings derived from the law of excluded middle, impredicative definitions, and so on. Revisionism is thus an inevitable consequence of constructivism:

> [...] an extreme constructivist philosophy of mathematics involves drastic revisions of mathematics and by no means leaves it as it is. (Wrigley 1977, 51)
>
> Surely, one cannot deny the law of excluded middle or rule out non-constructive existence proofs and at the same time leave "mathematics as it is" (Maddy 1993, 55)

The anti-revisionist slogan of "leaving mathematics as it is" is taken from Wittgenstein's writings in which he discusses the question of the proper method for philosophy. I refer in particular to the chapters on philosophy in the *Big Typescript* and to paragraphs 89–133 of the *Philosophical Investigations* (i.e. to writings belonging to different phases of his intellectual development yet remarkably similar in content). The passage that includes the slogan is the following:

1 See, in particular, Wittgenstein (1976, 39); also (1978, 282).
2 On this point, see Wittgenstein (1978, 366).
3 I am using these terms in a rather loose and neutral way. I do not presuppose, for instance, that these objects are mental. More importantly, the jargon of "objects" is not to be associated with a descriptive conception of mathematical statements (they may just be elements belonging to grammatical statements).

> Philosophy must not interfere in any way with the actual use of language, so it can in the end only describe it.
>
> For it cannot justify it either.
>
> It leaves everything as it is.
>
> It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other. (Wittgenstein 2009, 124)

However, some commentators have pointed out that this requirement of non-revisionism conflicts with the numerous remarks on the axiom of choice, existence proofs, the law of excluded middle, etc., which have a constructivist flavour. It may seem, then, that Wittgenstein is like the man of Carnap's metaphor, who on weekdays does with qualms many things which are not in accord with the high moral principles he professes on Sundays. Based on this dilemma, some interpreters have argued that Wittgenstein is not an anti-realist or constructivist, for he is *ex confesso* not a revisionist, while others have responded that he is, despite his own words, a revisionist of mathematical practice, for "he legislates what is proper mathematics and what is not proper mathematics (e.g., what is mathematically meaningful, and what is not)." (Rodych 1997, 214)

3 Was Wittgenstein an anti-realist? In the revised edition of *Insight and Illusion*, Hacker (1986) devotes Chapter XI to explaining why he was formerly wrong in supposing that in later writings Wittgenstein defended a form of anti-realism. His diagnosis was, roughly, that Wittgenstein was not committed to any kind of theory of meaning in the sense of Dummett, let alone an anti-realist theory of meaning. In the same vein, other authors have argued that Wittgenstein's critiques of mathematical realism do not make him into an advocate of a *positive doctrine* such as mathematical anti-realism. For, were this true, it would imply that the philosophical question of realism in mathematics would be, for him, a truly philosophical question, worthy of a genuine answer, and this is not compatible with his philosophical method of not answering a problem, but putting "a pseudo-problem to rest." (Maddy 1993, 58) It seems to me, however, that the label "anti-realism" is not to be used to describe a positive doctrine, nor to entail commitment to any theory of meaning whatsoever. As Dummett makes clear, the term "anti-realism" denotes "not a specific philosophical doctrine but the rejection of a doctrine." (Dummett 1991, 4) Accordingly, the *anti-realist* in philosophy of mathematics is best understood, not as someone who has a positive doctrine about what mathematical reality is, but as someone who opposes the conception that mathematical truths are discoveries of what is "already there" waiting for someone to be discovered. In this sense, Wittgenstein is undoubtedly an anti-realist in philosophy of mathematics.[4] He repeatedly denies that there is a realm of mathematical entities explored by the mathematician. And when he insists that the mathematician is not a discoverer, but an inventor, he is surely denying philosophical imagery associated with mathematical realism. Furthermore, when he associates proofs in mathematics with concept-formation, he constantly emphasizes that the proof does not establish that the connections between concepts involved in the proof are there beforehand (see *e.g.* Wittgenstein 1978, 166). One may want to argue that the idea of the mathematician as an inventor forging connections between concepts is a positive idea. The point Wittgenstein wants to make in saying such things, however, does not concern mathematical reality, but how

4 In saying that Wittgenstein is an anti-realist, I do not want to project an extraneous point of view into Wittgenstein's own way of thinking and correcting wrong perspectives in philosophy of mathematics. I only mean to reinforce the well-known fact that he opposed the realist imagery prevalent in mathematics, which is precisely the element which causes a potential conflict with his non-revisionist attitude.

we describe mathematical activity. The mathematician is an inventor because, as Wittgenstein says elsewhere, what is normally called a mathematical discovery had much better be called a mathematical invention (Wittgenstein 1976, 22). He is merely pointing to the use of the words “discovery” and “invention” when used to describe what mathematicians do, and saying that the word “invention” fits better than the word “discovery”. No positive doctrine is to be found here, only an attempt to describe mathematical practice less misleadingly.

4 Was Wittgenstein a constructivist? The force of the dilemma presented in §2 derives from the fact that Wittgenstein's writings on mathematics embody a great deal of argument and reasoning that is strongly constructivist or, at least, acceptable to the constructivist’s way of thinking. The discussion about whether Wittgenstein is or is not a constructivist in philosophy of mathematics is very delicate and controversial, and I will not discuss the topic in detail. Instead, I will merely assume for the sake of the argument that Wittgenstein was, at least in his middle period, a constructivist. All the same, it is worth looking closely at two passages that seem clearly constructivist:

> Russell's "multiplicative axiom" has its origin in the fact that he speaks of constructed classes of arithmetic as extensions of real concepts. The existence of a construction can never be doubtful.
> The confusing (because misleading) character of the usual mode of expression in set theory here shows itself clearly. If I always seem to describe a construction instead of giving it, then doubts can arise whether there is a construction that satisfies a certain description. (Wittgenstein 1994a, 62, my translation)

> What does a construction like that for √2 show? Does it show how there is yet room for this point in between all the rational points? It shows that the point *yielded* by the construction, yielded by *this* construction, is *not rational*. (Wittgenstein 1974, 460)

The first passage is a critique of the genesis of Russell’s “multiplicative axiom” (equivalent to the axiom of choice in set theory). It is possible to read this critique as a critique of the way the principle of comprehension is normally understood within set theory, namely as a principle according to which a class can be defined by a description (that is, a description satisfied by all elements of the class). In this case, if there are classes that can be given by means of a description (and not by means of a construction of their elements), then it may make sense to postulate the existence of a class for which a construction cannot be given (as is the case with the axiom of choice). Wittgenstein says that this mode of expression is “confusing”, “misleading”, for the “existence of a construction can never be dubious.” This is because, if in mathematics the object is the *construct*, it makes no sense to ask for the existence of a construction for an object given independently of any construction.

The second passage also clearly manifests a constructivist way of thinking. The geometric construction of the point corresponding to √2 is not the discovery of a new way of presenting a point that “was already there” in a given *continuum* of points pre-existent to our constructions. The first emphasis of the passage makes it clear that the point is essentially the result of the construction (i.e. if we abstract from the construction the point does not correspond to √2, or rather, there is no way of “picking out” a point corresponding to √2 without the construction).

These passages and many others could be cited to show that Wittgenstein defended a strong form of constructivism in his writings from the middle period. At the same time, however, it can be shown that he always defended the anti-revisionary idea that philosophy leaves everything as it is, that it is "purely descriptive." (Wittgenstein 1979, 106) This is a constant in his philosophy since his early writings. In the next section, I will show how in his middle period he thought his constructivism and his anti-revisionism to be compatible.

5 Wittgenstein's strategy: interpretation and analysis. Assume that some constructivists reach the conclusion that indirect proofs are illegitimate in mathematics (the chain of reasons that lead them to this conclusion is irrelevant here). Now, if they are presented with a proof that is ordinarily called indirect, there are two attitudes they may have. Either they may take the revisionist route and consider the proof illegitimate, or they may take the anti-revisionist route and say that the proof, albeit called indirect, is not really indirect. That is, they may reject the standard interpretation of the proof (as an indirect proof) and argue that there is another, "proper", interpretation of it as a direct proof. I will call this second attitude the *interpretation-strategy*.

I maintain that Wittgenstein thought that every seemingly illegitimate proof can be reinterpreted as a legitimate proof, i.e., he thought that the interpretation-strategy works in every case.[5] This is his solution to the dilemma in his middle period. On the one hand, the strategy preserves the status of the proof as a piece of mathematics: the proof would not be lost due to constructivist criticisms. On the other hand, its proper interpretation would satisfy the constructivist's requirements.

Wittgenstein's commitment to this strategy can be seen by paying attention to the consequences of adopting the strategy and comparing them to his remarks about the meaning of mathematical statements. One of the consequences of the interpretation-strategy is that, when a mathematician says that he or she proved some proposition, p, we do not really know what he or she really proved until we see the proof, because he or she will normally describe p using the standard interpretation, not the proper interpretation. Another consequence is that p is *ambiguous* when, according to the standard interpretation, there are two proofs of p, but, according to the proper interpretation, these proofs are proofs of q and r. These are basically the reasons why Wittgenstein said that the result, once it is read following the standard interpretation (or, in his jargon, once it is expressed in "prose") is sometimes misleading. It is only by going through the proof that we are able to recognize what has been proved:

> If you want to know what the expression 'continuity of a function' means, look at the proof of continuity; that will show what it proves. Don't look at the result as it is expressed in prose, or in the Russellian notation, which is simply a translation of the prose expression; but fix your attention on the calculation actually going on in the proof. The verbal expression of the allegedly proved proposition is in most cases misleading, because it conceals the real purport of the proof, which can be seen with full clarity in the proof itself. (Wittgenstein 1974, 369–370)

The distinction Wittgenstein draws between proof and prose can thus be seen as a tool used to implement the interpretation-strategy and to avoid choosing sides between constructivism and anti-revisionism. Non-constructivist mathematicians do not need this distinction, because they can take mathematical practice at its face value: prose is, for them, a kind of symbolic thinking in natural language, and there is no reason (or so they suppose) to

5 Lampert (2018) distinguishes between algorithmic proofs and meta-mathematical proofs involving meta-mathematical interpretations and argues that while Wittgenstein's non-revisionist understanding of mathematics applies to the former, it does not apply to the latter. As far as I can see, however, there is no reason not to include these proofs under the interpretation-strategy, i.e. they can also be seen as legitimate proofs.

call into question this kind of thinking. Revisionists, on the other hand, even when they have reasons to criticize an alleged proof, can simply say that it is not mathematics proper since it does not follow the requirements for a legitimate proof.

Note that, in the last passage quoted, the adjective "prose" is applied only to the result, i.e., to the proved statement, not the proof itself. However, this qualification is applied by Wittgenstein to elements or parts of the proof or, at least, to the "standard reading" of the proof. This is not surprising, since each step in a proof can be seen as a proof in its own right (albeit in most cases an uninteresting one). Because there can be "prose" within the standard reading of the proof as well, the "real purpose" of the proof may not be immediately visible. To see it, Wittgenstein says, we must "fix our attention on the calculation actually going on in the proof."

In Wittgenstein's manuscripts there is a beautiful example of the interpretation-strategy being put into practice, and the example is actually one involving a seemingly indirect proof. He applies the strategy by focusing his attacks not on the mathematical *proof* that is said to be indirect, but on the *interpretation* of the proof as being indirect, i.e., as supposing the opposite of what it wants to prove. The passage in question is a commentary on a proof that dates back to Saccheri's work *Euclid Vindicated from Every Blemish* (1733), the burden of which is that "two lines cannot have a segment in common". Saccheri's proof is very long, and obscure at some points; I shall not discuss it in detail here.[6] The two all-important points regarding the proof for the present discussion are that: (i) it is presented as an indirect proof; (ii) in the course of the proof an iterative procedure is described which approaches a limit. More precisely, the proof starts from the supposition that the lines AXB and AXC, which are supposed to be straight, share a segment AX and that, therefore, BC describes the arc of a circle with center X and radius AX. At some point, it is proved that, with the assumption made, there will be another two points M and F having the same properties as B and C but with the arc MF contained inside the arc BC (see Figure 1). This situation gives rise to an iterative procedure because two other points within the arc MF may be likewise designated and so on *ad infinitum*. Moreover, this procedure converges to the limit where both points collapse into a single point.

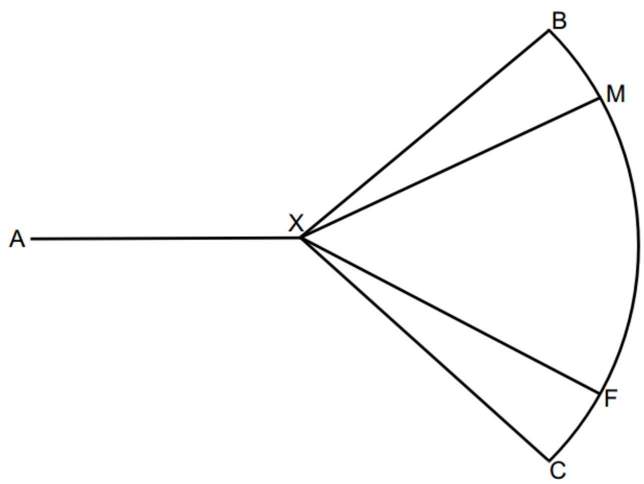
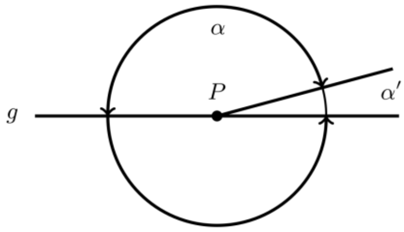


Figure 1: Diagram for Saccheri's proof

Commenting on this (or rather on an analogous) proof, Wittgenstein remarks:

6 See Saccheri (2014, 158–165) for the full presentation of the proof.

How does the indirect proof work, for instance in geometry? What is weirdest about it is that one sometimes tries to draw an ungeometric figure (the exact analogue to an illogical proposition). But this, of course, only comes from an erroneous interpretation of the proof. It is weird, for instance, to say "suppose that the straight-line g has two continuations from point P". But there is really no need to assume such a thing. Proofs in geometry, in mathematics, cannot be indirect in the real sense of the word, because one cannot suppose the opposite of a geometrical proposition as long as one sticks to a specific geometry. (That proof simply shows that the arcs α and α + α' approximate each other all the more and without limit, the more α' approximates 0.) (Wittgenstein 1994b, 146, my translation)



Here Wittgenstein suggests that a proof that seemed to be indirect can be interpreted as a direct proof of a slightly different theorem. Note that Wittgenstein does not say that the proof provided is illegitimate, but that what the proof shows must be interpreted in a certain way to avoid an *erroneous interpretation of the proof*. I take it that this "erroneous interpretation" is precisely the interpretation (which I called *standard*), according to which the proof in question is an indirect one where one supposes the opposite of what one wants to prove.

Another example of a seemingly indirect proof that Wittgenstein discusses at length is the following given by Hardy of the irrationality of the square root of 2:

The following alternative proof that $\sqrt{2}$ cannot be rational is interesting. Suppose, if possible, that p/q is a positive fraction, in its lowest terms, such that $(p/q)^2$ or $p^2 = 2q^2$. It is easy to see that then we must have $(2q - p)^2 = 2(p - q)^2$, and so $(2q - p)/(p - q)$ is another fraction having the same property. But clearly $q < p < 2q$, and so $p - q < q$. Hence we obtain another fraction equal to p/q and having a *smaller* denominator, which contradicts the assumption that p/q is in its lowest terms. (Hardy 1908, 6)

Wittgenstein begins the discussion of this proof in MS 126 by declaring that what disturbs him in a presentation like that of Hardy is the "apparently senseless variety of proofs of the same sentence" (MS 126, 122) (notice that the proof is presented by Hardy as an *alternative* proof). After recasting the proof in his own terms, Wittgenstein remarks that, although $(p/q)^2$ does not equal $((2q - p)/(p - q))^2$, both fractions can be brought arbitrarily near when $(p/q)^2$ approaches 2. However, $((2q - p)/(p - q))^2$ will always be a worse approximation of 2 in comparison with $(p/q)^2$. Therefore, Hardy's proof immediately gives us an iterative procedure to produce, from a given fraction p/q, ever worse approximations of $\sqrt{2}$, namely, $(2q - p)/(p - q)$, $(3p - 4q)/(3q - 2p)$, $(10q - 7p)/(5p - 7)$, and so on.[7] By inverting the procedure, we obviously get an iterative procedure to produce, from a given fraction p/q, ever better approximations of $\sqrt{2}$, namely, $(p + 2q)/(p + q)$, $(3p + 4q)/(2p + 3q)$, $(7p + 10q)/(5p + 7q)$, and so on.[8]

The erroneous interpretation of the proof in this case is the one that says that it is an (alternative) indirect proof of the irrationality of the square root of 2, while the proper interpretation is the one that says that it is a proof exhibiting a certain procedure that approaches a limit. The proper interpretation, then, gives the proof a special status insofar as

7 The rule for the k+1-th fraction of this series is given by $p_{k+1}/q_{k+1} = (-p_k + 2q_k)/(-q_k + p_k)$.
8 The rule for the k+1-th fraction of this series is now given by $p_{k+1}/q_{k+1} = (p_k + 2q_k)/(q_k + p_k)$.

it takes what is proved to be capable of being proved only by the proof in question. The proof is not regarded as a path to a place that could be reached by another path, but as the only path to the place. The right interpretation of mathematical proofs is thus capable of *disambiguating* mathematical results.

It is true that it remains unclear how general Wittgenstein's suggestion of re-interpreting proofs is, and how the right interpretation of any proof is obtained merely by "fixing our attention on the calculation actually going on in the proof". I shall not consider this unclarity as an insurmountable difficulty for Wittgenstein's view, but I stress that it needs to be addressed once the strategy is adopted.

In the next two sections I will explore this strategy a bit further, and in a way independent of Wittgenstein's texts. I will first show how this strategy can be used to understand intuitionistic logic as a tool for analysis of classical arguments, in the sense that intuitionistic logic disambiguates classical results. Then, exploiting work of Griss and some of his followers, I will indicate how this reasoning could be extended to cover indirect proofs.[9] The main motivation for these two sections is that the exploration of this topic enhances our understanding of the solution to the dilemma I am attributing to Wittgenstein.

6 Seeing intuitionistic logic as a tool for analysis. The strategy described in the last section also opens the possibility of seeing the philosophical meaning of intuitionistic logic and mathematics in a new light. Instead of seeing it as an attempt to criticize the grounds on which the classical mathematical edifice rests and erect a new edifice on other grounds, it can be regarded as a tool for the *analysis* of classical mathematics. In fact, after Gödel's and Gentzen's translations of classical arithmetic into intuitionist arithmetic, we know that under this translation no theorem of classical Peano Arithmetic (PA) is lost in intuitionist arithmetic, it is only transformed into a distinct theorem that obeys the principles of Heyting's arithmetic (HA). It is not a question of distinguishing between classical and intuitionistic logic in terms of what each of them can prove, but of distinguishing them as *notations* with different expressive capacities:

> [T]he mappings from classical logic into intuitionistic logic show that *relative to an appropriate translation* all theorems of classical logic are intuitionistically provable. Hence, it is often argued that classical logic and intuitionistic logic do not differ in *deductive strength* but in *expressive richness*. Intuitionistic logic allows, it is suggested, for more distinctions among formulas than does classical logic. (Bell et al. 2001, 210)

In the 1933 paper in which the translation of classical arithmetic into intuitionistic arithmetic is introduced, Gödel already interpreted this result as showing that the latter is only apparently more restrict than the former (Gödel 1986, 295). A clear expression of the idea that intuitionistic arithmetic is richer than the classical version (precisely for distinguishing formulas which are classically considered equivalent) can be found in a 1949 paper by the Dutch mathematician Johan de Iongh entitled *Restricted forms of intuitionistic mathematics*. In this article, de Iongh claims that

> [T]he most important advantage of intuitionistic mathematics is, that it distinguishes in every instance between directly and indirectly proved propositions and analyses the mathematical concepts into sequences of concepts with different degree of indirectness. (De Iongh 1949, 746)

9 I maintain that there is a close connection between Wittgenstein' and Griss's views on negation in mathematics, but I shall not defend this here.

To illustrate this idea, De Iongh takes as example the proposition: "the real number *a* is the limit of the sequence of real numbers *a*1, *a*2, ...". In the language of classical mathematics, this proposition can be written as:

$$\forall_m \exists_N \forall_n \; [n \geq N \rightarrow |a_n - a| < 1/m]$$

The same formula can be classically written in 41 different ways, if we limit ourselves to using at most two successive negation signs (e.g. $\neg\neg\forall_m\neg\forall_N\neg\neg\exists_n \; \neg[n \geq N \rightarrow |a_n - a| < 1/m]$). In intuitionistic logic, however, these 41 formulas are grouped into 9 groups of formulas, according to the following scheme:

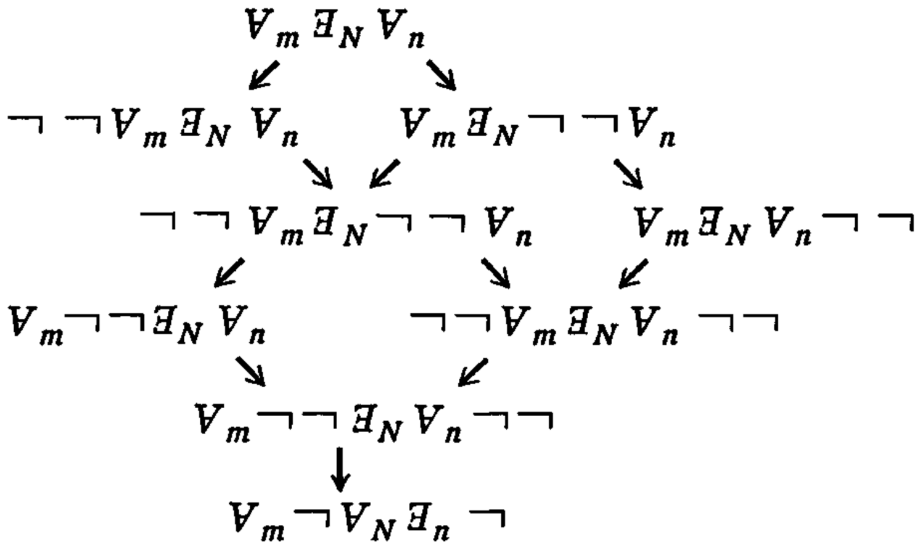


Figure 2: Grouping of different formulas in intuitionistic logic, all of them being equivalent in classical logic. Source: De Iongh 1949, 746.

In this scheme, the arrows indicate that one proposition follows from the other, but not vice-versa (as it occurs in classical logic). As a consequence, many proofs that are distinct proofs of the same proposition in classical arithmetic, are proofs of different propositions in intuitionistic arithmetic. For instance, many proofs that, in classical arithmetic, are distinct proofs of the formula $\forall_m\exists_N\forall_n \; [n \geq N \rightarrow |a_n - a| < 1/m]$, are in fact proofs of different and non-equivalent formulas when seen from the point of view of intuitionistic arithmetic.

Therefore, the outcome of the absence of excluded middle in intuitionistic logic is a drastic reduction of what the classical mathematician calls “independent proofs of the same theorem”. Intuitionist logic, thus, “teaches us differences”. This point is also made clear when the intuitionist distinguishes between the formulas $\exists n\varphi n$ and $\neg\forall n\neg\varphi n$. The classical mathematician normally identifies them, calling a proof in which one gives an example of a number satisfying the property φ a “constructive” proof of existence and a proof in which one reduces the assumption of non-existence to absurdity a “non-constructive” proof of existence. This is because he or she supposes that both proofs prove *the same* end-result. If these two proofs are conceived as proving two different things, this way of speaking becomes inappropriate. Moreover, if we are presented with the result of a classical proof of $\exists n\varphi n$, we cannot know precisely *what* was proved only by looking at the result. Instead, in order to know what was proved we shall, as Wittgenstein emphasizes, look at the proof.

Intuitionistic logic, however, still considers indirect reasoning as the canonical way of proving a negative statement, which does not fit well with Wittgenstein’s idea, which I presented in §6, about *reductio ad absurdum* proofs. In the next section, I will show how this idea of translation between logics can be developed to eliminate indirect proofs.

7 More on translation: negationless interpretations. The best-known results of the early development of intuitionistic logic are: the formal system published by Heyting in 1930, and the interpretation of the logical constants known today as the "proof-interpretation" or "BHK-interpretation". The fragment of this interpretation for propositional logic runs as follows (see Troelstra and Van Dalen 1988, vol. 1, 9):

(H1) A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

(H2) A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

(H3) A proof of $A \rightarrow B$ is a construction which permits us to transform any proof of A into a proof of B.

(H4) Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

However, not all early contributors to the development of intuitionistic logic agreed with this interpretation of logical constants. The *crux* of the disagreement lies in the interpretation of the conditional $A \rightarrow B$, that is, of hypothetical reasoning in intuitionistic logic. In a short article dated from 1937, Freudenthal criticizes (H3), saying that "a *supposition* that, say, A is proved is no material for construction."[10] As any construction that proves B may start only from previously constructed material, never from supposed constructed material, a proof of $A \rightarrow B$ could only consist of a proof of B by means of an actual proof of A. From this point of view, unrealized suppositions are banned from intuitionistic mathematics. As van Atten argues, Freudenthal arrived at this interpretation, because he rejected the separation between propositions and assertions, and criticized Heyting's interpretation of propositions as "intentions" and Kolmogorov's interpretations of them as "tasks" (see van Atten 2009, 125).

But van Atten is wrong in thinking that, according to Freudenthal, the only interpretation that can be given to $A \rightarrow B$ is "I have actually obtained a proof of A and building on that proof I have then actually obtained a proof of B" (ibid, 125). In the same article, Freudenthal suggests that we interpret $A \rightarrow B$ as "a proposition that deals with two predicates A and B of the same subject, two predicates whose content is the restriction of the free becoming of the subject."[11] That is, according to Freudenthal, the conditional appears in intuitionistic mathematics only in the context of general propositions and is interpreted as a relation of inclusion between species.

A similar result is achieved by Griss in his negationless intuitionistic mathematics, as I shall now show. Griss published a series of papers from 1946 to 1951 with the purpose of developing a negationless intuitionistic mathematics. Griss's main idea was to banish negative (indirect or apagogical) reasoning from mathematics. According to him, "the making of a supposition that a proof is given while this proof appears to be impossible is incompatible

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10 In the original: "Ein *Annahme*, etwa die, daß a bewiesen sei, ist kein Konstruktionsmaterial." (Freudenthal 1937, 114)

11 In the original: "ein Satz, der von zwei Prädikaten a and b desselben Subjekts handelt, zwei Prädikaten, deren Inhalt die Einschränkung des freien Werdens des Subjekts ist." (Freudenthal 1937, 115) The "free becoming of the subject" (freien Werdens des Subjekts) refers in this context to an intuitionistic understanding of mathematical objects such as functions, sets, choice sequences, and so on. These objects are conceived not as "finished objects", but as objects *in statu nascendi*.

with the constructive and evident starting point." (Griss 1948, 71) For "one cannot have a clear conception of a supposition that eventually proves to be a mistake." (Griss 1946, 1127)

The program of Griss's negationless intuitionistic mathematics consists basically of two things. The first is the replacement of negative notions by positive ones. For instance, the definition of parallel lines as lines having no point in common is replaced by the following: "two lines are *parallel* if each point of the first is different from each point of the other line," (Griss 1951, 455) the notion of difference between points being ultimately reducible to the difference between natural numbers. This last notion is considered to be as fundamental as the notion of equality between natural numbers, and not built secondarily by means of negation of equality.[12] The second idea is that false assertions are excluded in mathematics, even in cases where they are part of a true disjunction. For instance, the disjunction "7 is prime or 7 is not prime" is not allowed as a construction in negationless intuitionistic mathematics, since "7 is not prime" is a false assertion. Disjunctions only have a role when tied to generality, e.g. when we say that "A natural number is either prime or divisible". This assertion is not to be interpreted as saying that, for each particular number n, n is either prime or divisible (since this interpretation makes use of false assertions), but as saying that the union of the species of prime numbers and divisible numbers is identical with the species of natural numbers. That is, disjunction never occurs in assertions of the form "p or q is true" but only in assertions of the form "p or q is true for all elements of the set A", meaning that "the property p holds for a subspecies B and property q holds for a subspecies C, A being the union of B and C" (see Griss 1950, 457).

In this interpretation of logical constants tied with generality, negation is interpreted as the complementary subspecies of a given (proper) subspecies. So, if we start with a given species u (e.g., the set of natural numbers) and construct a subspecies $a \neq u$, we can further construct the subspecies $\neg a$ consisting of all elements of u that are distinguishable from all the elements of a. The conditional is in turn interpreted, as suggested in Freudenthal's 1937 article, as a relation of inclusion between species. Logical reasoning, thus, operates mainly in the context of the predicate calculus. This whole reasoning echoes Wittgenstein's idea in *Philosophical Remarks* that a certain generality is needed to make negation (and disjunction) in arithmetic interesting to us (Wittgenstein 1975, 247).[13]

Based on Griss's work, several attempts were made to formalize a negationless intuitionistic predicate logic (the best known being those of Vredenduin (1953), Gilmore (1953), Valpola (1955), Nelson (1966, 1973) and Krivtsov (2000a, 2000b). The work of Krivtsov is particularly interesting for our purposes, because he not only presents a formal system of negationless intuitionistic predicate logic and arithmetic, but also a translation of HA into negationless arithmetic (NA), and extends the strategy to the case of intuitionistic arithmetic in higher types and analysis. So, from the translatability of PA into HA and of HA into NA, we know that each theorem of classical arithmetic can be recovered in a negationless system. That is, each theorem of classical arithmetic can be interpreted in a way compatible with Griss's (and Wittgenstein's) requirements for constructive proofs. In saying this, however, I am not claiming that what I called above the proper interpretation of a given statement is given by the (mechanical) translation of the classical proof into a negationless system. I am arguing that there is a formal guarantee of the existence of at least one interpretation for a certain class of mathematical statements that satisfies a certain class of constructivist requirements.

12 According to Griss, this replacement is also appealing when it comes to the application of mathematical system in an empirical science, since experience always gives positive results: the outcome of an experiment is always about real things and not about nonexistent things.

13 I abstain from drawing a complete parallel between Wittgenstein and Griss on negation and falsity in arithmetic, since that would lead us to far afield.

8 The interpretation-strategy and the end of analysis. I suggested in §6 that, following the interpretation-strategy, we could regard intuitionistic logic as a sort of tool for analysis of classical results. A consequence of this analysis is that many proofs that, in classical arithmetic, are distinct proofs of the same proposition, in intuitionistic arithmetic are proofs of different propositions. Now, according to Wittgenstein's constructivist views in his middle period, the proof is not just *one* criterion of identity of a mathematical statement, but the *only* criterion available. In this vein, Wittgenstein says in the early thirties that the mathematical proposition is essentially the last link in a chain of proof (he compares the mathematical proposition with the immediately visible surface of a body of proof, see Wittgenstein 1975, 192).[14] A corollary of this reasoning is that, if we have two proofs, we have two propositions, two senses. Understanding how a proposition is proved is, therefore, a necessary and sufficient condition to understand what the proposition says. As a result, there cannot be two independent proofs of the same mathematical proposition (see Wittgenstein 1975, 184), where "independent proofs" means that they cannot be translated into each other.[15]

According to this view, the end of the process of analysing the current mathematical practice is attained when, for each formulation of a mathematical theorem, there exists just a single proof of it, such that the formulation serves as a *name* for the proof and could *catalog* the proof. Thus, if we want to analyse the current mathematical practice in order to understand what is proved based only on the result, we have to devise a notation in which a mathematical theorem designates its proof. That is, we need a notation in which, given a certain proposition *p*, if someone claimed that he has proved *p*, there would be no doubt about *how* he or she proved *p*. Intuitionistic logic is a step towards such a notation, though, it is true, it fails to exclude every mistake. Apart from considering indirect reasoning as a canonical way of proving a negative sentence (which, as we have seen in §6–7, is objectionable to Wittgenstein), intuitionistic arithmetic treats induction in mathematics as a way of *proving* a generality. Once this is accepted, it becomes impossible not to have two independent proofs of the same mathematical proposition. Consider, *e.g.*, the proposition "there is a digit '3' in the fifth place of the division of 1 by 3". Consider, then, the following proofs for this proposition:[16]

10   : 3
 10   0.3333**3**
  10
   10
    10
	<u>10</u> : 3
	 <u>10</u> 0.3333**3**

In the proof at left, the calculation is performed following the rules of the division operation until the fifth place. In the proof at right, it is "perceived" that the number on the left repeats in one step (or, formally, the induction is proved) and the result is obtained by

14 Freudenthal also arrived at this conclusion. See Freudenthal (1937, 112): "each proposition, once formulated in an intuitionistically irreproachable way, contains automatically the totality of its proof. Intuitionistically speaking, the proposition is thus only a provisional guide, a kind of heading, while only the proof is the real proposition." In the original: "jeder Satz, wenn man ihn erst einmal intuitionistisch einwandfrei formuliert, automatisch seinen ganzen Beweis enthält. Intuitionistisch gesprochen ist also der Satz nur eine kurze vorläufige Orientierung, eine Art Überschrift, während erst der Beweis der eigentliche Satz ist."

15 See Wittgenstein (1975, 179); also (1984, 109; 1994b, 321).

16 In the first line, a '0' appears after the '1' which is divided by 3 as a result of the application of the algorithm for division.

instantiation of the generality demonstrated. According to intuitionistic arithmetic, we would have here two independent proofs for the same proposition. In LFM, Wittgenstein called the kind of proof at the right a "proof by means of a shortcut", and said that this is "the queerest thing in the world":

> For instance, we can prove f(1000) by proving f(1), and that if f(1) then f(2), and that if f(2) then f(3) and so on. Or else, having proved f(1) and f(n) ⊃ f(n+1), one can make a short cut. And this is the queerest thing in the world: that one should have a short cut through logic. For if the proof of the proposition is the step-by-step proof, how can anything else be a proof of it? How can it be certain that the one will reach the same result as the other? Aren't we really rash?
>
> How can there be a short cut through logic? A proof ought to be a proof, and everything cutting it short should be rash. (Wittgenstein 1976, 266)

In the writings from the middle-period, Wittgenstein thought that there could be no shortcuts through logic, that every proof proves something different and that, therefore, there could not be two proofs proving the same end-result. Instead of seeing the "inductive proof" presented earlier as an alternative and shorter proof of the proposition "there is a digit '3' in the fifth place of the division of 1 by 3", Wittgenstein thought that the seeing of an induction motivates the adoption (stipulation) of an algebraic formula that could be applied to arithmetic, in the sense that the formula could be used to correct wrong arithmetical calculations (for instance, if someone had arrived at a digit '4' in any place of the division, then the rule will tell us to regard it as a mistake). But when this rule is recognized as applicable to arithmetical division, the very meaning of the concept "arithmetic division" is changed, since he regards the identity of a mathematical concept as determined by *all* the criteria for the correct and incorrect application of the concept (see Wittgenstein 1975, 182). The "inductive proof", therefore, would not provide a shorter proof of an old mathematical proposition, but would change the meaning of the concepts involved.

This would be the proper interpretation of what is normally called in mathematical jargon a "proof by induction". The very fact that an inductive procedure is commonly regarded as a *proof* of a proposition would then reveal that something outside the pure calculations is entering into the standard conception of what had been proved. Wittgenstein used the German term "*Prosa*", in this connection, to refer to the result of conceiving mathematical induction as a rule of inference.[17] Incidentally, it is relevant to point out that Wittgenstein used the term "*Prosa*" for the first time in his manuscripts in a context where he criticizes the idea that a mathematical proposition can have two independent proofs.[18] He takes the distinction between proof (or calculus) and prose to be a way to solve the dilemma between constructivism and anti-revisionism, as a way to criticize loose jargon then current in mathematics (which treats, for instance, induction as a rule of inference), while preserving the results of the mathematicians' calculations.

In sum, this way of conciliating constructivism and anti-revisionism in Wittgenstein's writings about mathematics in his middle period takes the form of a strategy of interpretation of mathematical results based on constructivist principles. This strategy in turn can be seen as the basis for an analysis of mathematical practice, having as horizon a notation in which the constructive character of the classical mathematical reasonings is exhibited on the surface of

17 See Wittgenstein (1984, 137). See also (1984, 33), where the expression "*Privatangelegenheit*" is used instead of "*Prosa*".

18 See Wittgenstein (1994b, 138). Most of the remarks made in this context are used in (1975, 183–184) as well.

the mathematical language and in which every formulation of a mathematical theorem catalogs its proof. I am not suggesting that Wittgenstein was *committed* to this kind of analysis. He was apparently satisfied with his understanding of the *actual* practice, and with his recommendation that statements of mathematical theorems are to be read together with their proofs to avoid confusion and misunderstanding about what was being proved. I am suggesting that the project of devising a notation in which the perils of prose are removed is possible and coherent.

Note that in this notation, in which a theorem cannot have more than one proof and the path is constitutive of the result, the consistency of mathematics follows: if we arrive at p by a certain path, and at $\neg p$ by another path, this situation does not count as a "contradiction", but only as an ambiguity of the sign "p". Hilbert's question whether transfinite methods in mathematics could introduce an inconsistency at the finite level presupposes, on the other hand, that these methods are not constitutive of their results. If instead the result is seen as a non-detachable part of the technique, the question of the consistency of arithmetic does not arise. It is now understandable why Wittgenstein in the following passage attributes the emergence of these questions to the "prose" that accompanies the calculus:

> It is a strange error of mathematicians that some of them think that through a critic of the foundations something could get lost in mathematics. Some mathematicians have the completely correct instinct: what we have calculated once cannot get lost or disappear. Indeed, what the criticism makes disappear are the names and allusions that occur in the calculus, hence what I wish to call *prose*. It is very important to distinguish as strictly as possible between the calculus and this kind of prose. Once people have become clear about this distinction, all these questions, such as those about consistency, independence, etc., will be removed. (Wittgenstein 1984, 149)

9 Conclusions. In this paper, I have argued for the view that, in his middle-period writings, Wittgenstein had a very clear and precise solution to the problem of how to make compatible his anti-revisionary views about mathematical practice with his constructivist and anti-realist reading of mathematical results. This solution calls for the reinterpretation of that part of mathematical discourse that is not entirely faithful to the calculation "actually going on in the proof". For instance, mathematicians say "Let us suppose…" where, in fact, no supposition is actually involved. They say that they proved by induction some general proposition where, in fact, what was really shown (according to Wittgenstein) was just the applicability of algebraic formulas to arithmetic.

I have also tried to substantiate this solution by showing how the translation of mathematical results, given by the possibility of translation between logics, can be seen as a tool for partially implementing the solution Wittgenstein had in mind. To repeat, I am not suggesting that Wittgenstein had a project of analysis of mathematical practice that involved the search for a notation in which the calculation "actually going on in the proof" was immediately visible on its surface. However, this project is not at variance with what he says and would not count as a *revision* of mathematical practice in the relevant sense, namely, in the sense that some mathematical calculations are considered to be dubious or not "true mathematics".

Finally, as I said in §1, the idea that the proof is the sole criterion for the sense of a mathematical proposition is characterized as a "half-truth" in Wittgenstein's later writings. He later recognized that the proposition had a certain autonomy with regard to its possibly multiple proofs. In this sense, there can well be multiple independent proofs of the very same mathematical proposition. However, the main idea of the solution remains (namely, to separate the mathematical result from the "standard interpretation" of it). This can be seen from the various critical remarks Wittgenstein makes concerning the interpretation of well-established mathematical results (like Cantor's diagonal proof and Gödel's incompleteness theorem). I tend to think, therefore, that he elaborates his earlier solution with an eye to doing more justice to the actual practice of mathematics.

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