Linearly Stratified Models for the Foundations of Nonstandard Mathematics

Mauro Di Nasso

Abstract

Assuming the existence of an inaccessible cardinal, transitive full models of the whole set theory, equipped with a linearly-valued rank function, are constructed. Such models generalize superstructures and provide a global framework for nonstandard mathematics.

MSClassification: 03H05, 03C20.
Keywords: Nonstandard models of set theory, non-well-founded set theory.

Introduction.
Let us briefly recall the traditional superstructure approach to nonstandard analysis (full details can be found in [Z] or [CK] § 4.4.). A 1-1 mapping

$$* : V_\omega(X) \longrightarrow V_\omega(Y)$$

between ω-superstructures is a *nonstandard embedding* if for every bounded formula $\vartheta(x_1, \ldots, x_n)$ and parameters $a_1, \ldots, a_n \in V_\omega(X)$, the following *transfer principle* holds

$$V_\omega(X) \models \vartheta(a_1, \ldots, a_n) \Leftrightarrow V_\omega(Y) \models \vartheta({}^*a_1, \ldots, {}^*a_n)$$

Recall that a formula of set theory is *bounded* if all its quantifiers are of the form $\forall x(x \in y \to \ldots)$ or $\exists x(x \in y \wedge \ldots)$. The superstructure $V_\omega(X)$ is the *standard model* based on X and $V_\omega(Y)$ is the *nonstandard model* based on $Y = {}^*X$. The elements of X and Y are atoms with respect to the corresponding superstructures [1]. The mapping $*$ is *enlarging*, i.e. ${}^*A$ properly contains

[1]Precisely, $x \cap V_\omega(X) = \emptyset$ and $y \cap V_\omega(Y) = \emptyset$ for every $x \in X$ and $y \in Y$.

$\{{}^*a : a \in A\}$ for each infinite set $A \in V(X)$. *Standard sets* of $V_\omega(Y)$ are those of the form ${}^*B$ for some $B \in V_\omega(X)$. Elements of standard sets are called *internal sets*. Thus, the collection $\mathcal{I}_*$ of internal elements is the transitive closure of range$(*)$. The *internal model* relative to $*$ is the submodel of $\langle V_\omega(Y), \in \rangle$ whose universe is $\mathcal{I}_*$.

In the superstructure approach to *real nonstandard analysis*, one takes as X a copy of the real numbers $\mathbb{R}$. The set $Y = {}^*\mathbb{R}$ of hyperreal numbers is an ordered field properly containing (an isomorphic copy of) $\mathbb{R}$. In particular, infinitesimals do exist in ${}^*\mathbb{R}$. The existence of nonstandard embeddings

$$* : V_\omega(\mathbb{R}) \longrightarrow V_\omega({}^*\mathbb{R})$$

is a consequence of Mostowki's collapse theorem. Precisely, one takes the union of ultrapowers $\bigcup_{n<\omega} V_n(\mathbb{R})^I{}_D$ which is extensional (up to atoms) and well-founded. If $\varphi : \bigcup_{n<\omega} V_n(\mathbb{R})^I{}_D \to T$ is the transitive collapse, then $T \subseteq V_\omega({}^*\mathbb{R})$ where ${}^*\mathbb{R}$ is an isomorphic copy of $\mathbb{R}^I{}_D$. Denote by $d : V_\omega(X) \to \bigcup_{n<\omega} V_n(\mathbb{R})^I{}_D$ the diagonal immersion and by $\imath : T \hookrightarrow V({}^*\mathbb{R})$ the inclusion. The mapping $* = \imath \circ \varphi \circ d$ is a nonstandard embedding and the collection of internal sets is $\mathcal{I}_* = T$. If the ultrafilter D is countably incomplete, then $*$ is enlarging.

The superstructure approach to nonstandard analysis works within the usual ZFC set theory, at the price of restricting both the standard and the nonstandard models to the finite levels of the cumulative hierarchy [2]. One can overcome this limitation if the axiom of regularity is replaced by a suitable anti-foundation principle in the style of [BH].

In what follows we construct nonstandard embeddings where both the standard and the nonstandard structures are models of the whole set theory. The standard models are superstructures of inaccessible height, while nonstandard models are "generalized" superstructures, where the usual well-ordered $\in$-stratification is weakened to a linearly ordered $\in$-stratification.

Linearly $\in$-stratified models of ZFC$^-$.
The reader is assumed to be familiar with the basics of model theory and

[2] The following fact is easily proved. Suppose the mapping $* : \mathcal{S} \to \mathcal{N}$ satisfies the *transfer principle*. If $*$ is enlarging and $\mathcal{S}$ is a model of the infinity axiom, then $\mathcal{N}$ is non-well-founded.

set theory. For all unexplained notions and notation, see [CK] and [J]. Let ZFC$^-$ denote the Zermelo-Fraenkel set theory whose axioms are:

> *EXT (Extensionality); PAIR (Pairing); SEP (Separation Schema); UN (Union); PS (Power-set); REP (Replacement Schema); INF (Infinity) and AC (Choice).*

The regularity axiom is replaced by the following anti-foundation principle:

> U Boffa's axiom of Universality.
> *"Any extensional binary structure is isomorphic to a transitive model"*

Recall that a *transitive model* of the language of set theory is an $\in$-structure $\langle M\,,\, \in \rangle$ whose universe M is a transitive set. Axiom U extends the validity of Mostowski's collapse to non-well-founded structures. The relative consistency of U with respect to ZFC$^-$ was first proved in [B]. See also [FH], where other anti-foundation principles are studied.

Let I be the axiom

> I *"There exists an inaccessible cardinal"*

In this paper, we shall work within the theory $ZFC^- + U + I$.

Now, let us recall some notions and notation. $\mathcal{P}(A)$ is the power-set of A. The *cumulative hierarchy* over the set A is defined by transfinite induction. $V_o(A) = A$; $V_{\alpha+1}(A) = V_\alpha(A) \cup \mathcal{P}(V_\alpha(A))$; $V_\beta(A) = \bigcup_{\alpha<\beta} V_\alpha(A)$ if β is limit. An $\in$-model $\mathcal{M} = \langle M, \in \rangle$ is *full* if $\mathcal{P}(A) \subseteq M$ for each $A \in M$. Abusing notation, sometimes we write $A \in \mathcal{M}$ instead of $A \in M$. $\langle M^I_D, E \rangle$, or simply M^I_D, will denote the ultrapower of $\langle M, \in \rangle$ modulo the ultrafilter D over the set I. For each $f : I \to M$, let f_D be its equivalence class modulo D.

Theorem
Let κ be an inaccessible cardinal, and let $\mathcal{S} = \langle V_\kappa(\emptyset), \in \rangle$. Then there exist a model $\mathcal{N} = \langle N, \in \rangle$ and a mapping

$$* : \mathcal{S} \longrightarrow \mathcal{N}$$

such that:
(i) $\mathcal{N}$ *is a full transitive model of ZFC*$^-$*.*
(ii) $\mathcal{N}$ *is provided with a linearly-valued rank function* $R : N \to \Lambda$ [3]

$$R(A) \;=\; \begin{cases} 0 & \textit{if } A = \emptyset \;; \\ sup\{R(a)+1 : a \in A\} & \textit{otherwise.} \end{cases}$$

which is an extension of the usual ordinal-valued rank function ρ [4]*. Precisely,* $R({}^*s) = {}^*\rho(s)$ *for every* $s \in \mathcal{S}$*.*
(iii) $\mathcal{N}$ *is a generalized superstructure in the following sense. Let* $N_\lambda = \{a \in \mathcal{N} : R(a) < \lambda\}$ *and* $\mathcal{P}_{N}(N_\lambda) = \{a \in \mathcal{N} : a \subseteq N_\lambda\}$ *for each* $\lambda \in \Lambda$ [5]*. Then*

$$N = \bigcup_\lambda N_\lambda \textit{ and } N_{\lambda+1} = N_\lambda \cup \mathcal{P}_{N}(N_\lambda).$$

(iv) The corestriction of the mapping $*$ *to the transitive closure* $\mathcal{I}_*$ *of range*$(*)$ *is an elementary embedding. Hence,* $* : \mathcal{S} \to \mathcal{N}$ *satisfies tranfer.*
(v) For any given cardinal $\mu < \kappa$*, one can choose* $*$ *so that* $\langle \mathcal{I}_*, \in\rangle$ *is* μ^+*-saturated.*
(vi) $\mathcal{N}$ *is well-founded over* $\mathcal{I}_*$*, i.e. there is no infinite descending chain*

$$a \ni a_1 \ni \ldots \ni a_n \ni \ldots$$

such that $a_n \notin \mathcal{I}_*$ *for all* n*.*
(vii) $*$ *is enlarging, i.e.* $\{{}^*t : t \in s\}$ *is a proper subset of* ${}^*s$ *for each infinite set* $s \in \mathcal{S}$*.*

The above theorem has a direct application to the foundations of nonstandard mathematics. In fact,

1. The superstructure $\mathcal{S} = \langle V_\kappa(\emptyset), \in\rangle$ is a full transitive model of ZFC, given that κ is an inaccessible cardinal. Thus one can view $\mathcal{S}$ as a standard model of mathematics.

2. The linearly-stratified structure $\mathcal{N} = \langle \bigcup_\lambda N_\lambda, \in\rangle$ is a full transitive model of ZFC$^-$. Thus one can view $\mathcal{N}$ as a nonstandard model of mathematics.

[3]It is implicitly assumed that Λ has a least element 0 and that each $\lambda \in \Lambda$ has immediate successor $\lambda + 1$.

[4]Recall the definition of the von Neumann rank for well-founded sets: $\rho(A) = \min\{\alpha : A \subseteq V_\alpha(\emptyset)\}$. An equivalent definition (by $\in$-induction) is the following: $\rho(\emptyset) = 0$; $\rho(A) = sup\{\rho(a)+1 : a \in A\}$ if $A \neq \emptyset$.

[5]We remark that in general $N_\lambda \notin N$.

3. The mapping $*: \mathcal{S} \to \mathcal{N}$ is a nonstandard embedding, given that it is enlarging and satisfies transfer.

4. For any cardinal μ below some inaccessible cardinal, there are nonstandard embeddings $*$ that satisfy the following *μ-saturation property*: "Let $\mathcal{F}$ be a family of internal sets with the finite intersection property, i.e. $\bigcap \mathcal{F}_o \neq \emptyset$ for every finite $\mathcal{F}_o \subseteq \mathcal{F}$. If $|\mathcal{F}| < \mu$, then $\bigcap \mathcal{F} \neq \emptyset$".

Proof of the Theorem.
Let D be any countable incomplete ultrafilter over a set I of cardinality $< \kappa$, and consider the diagonal immersion

$$d : \langle V_\kappa(\emptyset), \in \rangle \to \langle V_\kappa(\emptyset)^I{}_D, E \rangle$$

Since d is an elementary embedding, the binary structure $\langle V_\kappa(\emptyset)^I{}_D, E \rangle$ is extensional. Thus the Universality axiom yields a transitive model $\langle T, \in \rangle$ and an isomorphism

$$\varphi : \langle V_\kappa(\emptyset)^I{}_D, E \rangle \xrightarrow{\cong} \langle T, \in \rangle$$

Notice that $T = \bigcup_{\alpha \in \kappa} T_\alpha$ where $T_\alpha = \{\varphi(f_D) : f_D E d(V_\alpha(\emptyset))\}$. Take $\mathcal{N} = \langle N, \in \rangle$ with $N = \bigcup_{\alpha \in \kappa} V_\kappa(T_\alpha)$, and let $* : \mathcal{S} \to \mathcal{N}$ be the composition $\imath \circ \varphi \circ d$, where $\imath$ is the inclusion $T \hookrightarrow N$. We claim that $*$ satisfies all the required properties.
(vii) $*$ is enlarging if and only if d is enlarging if and only if $\{f_D \in V_\kappa(\emptyset)^I{}_D : f_D E d(A)\}$ properly contains $\{d(a) : a \in A\}$ for every infinite $A \in V_\kappa(\emptyset)$. The latter is true, since the ultrafilter D is countably incomplete [6]. Properties *(iv)* and *(vi)* immediately follow by noticing that $\mathcal{I}_* = T$. Regarding the transfer principle, recall that bounded formulas are preserved under transitive extensions. The structure $\langle T, \in \rangle \cong \langle V_\kappa(\emptyset)^I{}_D, E \rangle$ is μ^+-saturated provided D is μ^+-good. Since μ^+-good countably incomplete ultrafilters exist over any set of cardinality μ [7], *(v)* is proved.
Now, let us consider property *(ii)*. Let ρ the usual ordinal-valued rank function $\rho : V_\kappa(\emptyset) \to \kappa$ and consider its canonical ultrapower-extension $\widetilde{\rho} : V_\kappa(\emptyset)^I{}_D \to \kappa^I{}_D$ defined by $\widetilde{\rho}(f_D) = \langle R(f(i)) : i \in I \rangle_D$. By the elementary embedding property of ultrapowers, it is immediately proved that $\langle \kappa^I{}_D, E \rangle$ is a linear order and that $\widetilde{\rho}$ satisfies the following

[6] See [CK] § 4.3.
[7] See [CK] § 6.1.

$$\widetilde{\rho}(f_D) \;=\; \begin{cases} d(0) & \textit{if } f_D = d(0)\ ; \\ sup\{\widetilde{\rho}(g_D)+1 : g_D E f_D\} & \textit{otherwise.} \end{cases}$$

Thus the composition $R_T = \varphi \circ \widetilde{\rho} \circ \varphi^{-1} : T \to \Theta$ is a linearly-valued rank function such that

$$R_T(A) \;=\; \begin{cases} 0 & \textit{if } A = \emptyset\ ; \\ sup\{R_T(a)+1 : a \in A\} & \textit{otherwise.} \end{cases}$$

Notice that $\Theta = \text{range}(\varphi\lceil {\kappa^I}_D) \subset T$ is linearly ordered by the membership relation $\in$, given that $\langle \Theta, \in \rangle$ is isomorphic to ${\kappa^I}_D$. We extend R_T to a rank function R for N. To do so, in the order completion $\langle \Theta', < \rangle$ of $\langle \Theta, \in \rangle$ replace each new element $x \in \Theta' \setminus \Theta$ with a copy $\{\alpha_x : \alpha \in \kappa\}$ of κ, so to get a linear order $\langle \Lambda, < \rangle$. By identifying x with 0_x, we can assume $\Lambda \supset \Theta' \supset \Theta$. For every $\alpha \in \kappa$, define $R_\alpha : V_\kappa(T_\alpha) \to \Lambda$ as follows

$$R_\alpha(A) \;=\; \begin{cases} R_T(A) & \textit{if } A \in T_\alpha\ ; \\ sup\{R_\alpha(a)+1 : a \in A\} & \textit{otherwise.} \end{cases}$$

Let $\Lambda^{(\beta)} \doteq \Theta' \cup \{\{0_x, 1_x, \ldots, \beta_x\} : x \in \Theta' \setminus \Theta\}$ be the subset of Λ containing only the initial segment of lenght $\beta < \kappa$ for each $x \in \Theta' \setminus \Theta$. By transfinite induction on β, one can prove that

$$A \in V_{\beta+1}(T_\alpha) \Rightarrow R_\alpha(A) \in \Lambda^{(\beta)} \text{ and } A \in V_\beta(T_\alpha) \cap T \Rightarrow R_\alpha(A) = R_T(A)$$

Thus R_α is well-defined and $R_\alpha(A) = R_T(A)$ for all $A \in V_\kappa(T_\alpha) \cap T$. Given $\alpha, \alpha' < \kappa$, one can also prove by transfinite induction on β that $R_\alpha(A) = R_{\alpha'}(A)$ for every $A \in V_\beta(T_\alpha) \cap V_\kappa(T_{\alpha'})$. Thus the following is well-defined

$$R \doteq \bigcup_{\alpha \in \kappa} R_\alpha : N \to \Lambda$$

The mapping R is a rank function for N such that $R({}^*s) = {}^*\rho(s)$ for all $s \in \mathcal{S}$. Property *(iii)* directly follows from the definition of R.
We still have to prove *(i)*. Since $\mathcal{N}$ is a full transitive model, we have $\mathcal{N} \models$ EXT,SEP,UN,AC. Moreover, $\mathcal{N} \models$ PAIR,PS because N is closed under pairing and power-set. We also have $\mathcal{N} \models$ INF because $\omega \in \kappa \subset N$. Regarding the Replacement schema, recall that κ is inaccessible and $|I| < \kappa$, so $|T_\alpha| = |{V_\alpha(\emptyset)^I}_D| < \kappa$ and also $|V_\beta(T_\alpha)| < \kappa$ for every $\alpha, \beta < \kappa$. Hence $|A| < \kappa$ for each $A \in N$. Now, take any $A \in N$ and let F be a functional relation such

that $\{F(a) : a \in A\} \subseteq N$. For each $a \in A$, let $\alpha_a = \min\{\alpha \in \kappa : a \in V_\kappa(T_\alpha)\}$ and $\beta_a = \min\{\beta \in \kappa : a \in V_\beta(T_{\alpha_a})\}$. Now take $\alpha = \sup\{\alpha_a : a \in A\}$, $\beta = \sup\{\beta_a : a \in A\}$. By regularity of κ and $|A| < \kappa$, we have $\alpha, \beta < \kappa$. Therefore $\{F(a) : a \in A\} \in V_\beta(T_\alpha) \subset N$. $\dashv$

References

[B] M. Boffa, *Sur la Théorie des Ensembles sans Axiome de Fondement*, **Bull. Soc. Math. Belg.**, vol. **21** (1969), pp. 16-56.

[BH] D. Ballard and K. Hrbáček, *Standard Foundations for Nonstandard Analysis*, **J. Symb. Logic**, vol. **57** (1992), pp. 741-748.

[CK] C.C. Chang and H.J. Keisler, **Model Theory**, 3rd edition, North-Holland, Amsterdam, 1990 (1st edition 1973).

[FH] M. Forti and F. Honsell, *Set Theory with Free Construction Principles*, **Ann. Scuola Norm. Sup. Pisa**, **Cl. Sci.** (IV), vol. **10** (1983), pp. 493-522.

[J] T. Jech, **Set Theory**, Academic Press, New York, 1978.

[Z] E. Zakon, *A New Variant of Nonstandard Analysis*, in **Victoria Symposium on Nonstandard Analysis** (A.E. Hurd and P.A. Loeb, eds.), Lecture Notes in Mathematics, Vol. **369**, Springer-Verlag, Berlin, 1974, pp. 313-339.

Dipartimento di Matematica, Università di Siena.
Via del Capitano, 15. 53100 Siena, Italy.
e-mail: dinasso@unisi.it