LATTICES OF THEORIES IN LANGUAGES WITHOUT EQUALITY

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Abstract. If $\mathbf{S}$ is a semilattice with operators, then there is an implicational theory $\mathcal{Q}$ such that the congruence lattice $\mathrm{Con}(\mathbf{S})$ is isomorphic to the lattice of all implicational theories containing $\mathcal{Q}$.

Imprudent will appear our voyage since none of us has been in the Greenland ocean. - Bjarni Herjulfson

The author and Kira Adaricheva have shown that lattices of quasi-equational theories are isomorphic to congruence lattices of semilattices with operators [1]. That is, given a quasi-equational theory $\mathcal{Q}$, there is a semilattice with operators $\mathbf{S}$ such that the lattice $\mathrm{QuTh}(\mathcal{Q})$ of quasi-equational theories containing $\mathcal{Q}$ is isomorphic to $\mathrm{Con}(\mathbf{S})$. There is a partial converse: if the semilattice has a largest element 1, and under strong restrictions on the monoid of operators, then $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ can be represented as a lattice of quasi-equational theories. Any formulation of a converse will necessarily involve some restrictions, as there are semilattices with operators whose congruence lattice cannot be represented as a lattice of quasi-equational theories. In particular, one must deal with the element corresponding to the relative variety $x \approx y$, which has no apparent analogue in congruence lattices of semilattices with operators.

In this note, it is shown that if $\mathbf{S}$ is a semilattice with operators, then $\mathrm{Con}(\mathbf{S}, +, 0, \mathcal{F})$ is isomorphic to a lattice of implicational theories in a language that may not contain equality. The proof is a modification of the previous argument [1], but not an entirely straightforward one. *En route,* we also investigate atomic theories, the analogue of equational theories for a language without equality.

For classical logic without equality, see Church [4] or Monk [12]. More recent work includes Blok and Pigozzi [2], Czelakowski [5], and Elgueta [6]. The standard reference for quasivarieties is Viktor Gorbunov's book [7].

The rules for deduction in implicational theories are given explicitly in section 4. Our main result, Corollary 16, of course depends on these. It

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does *not* depend on the model theory used to interpret how it applies to structures, and indeed there are options in this regard. So there are two versions of this paper. The longer one includes a suitably weak model theory to interpret the results, while the shorter one proceeds more directly to the main theorem. This is the longer version; both are available on the author's website: www.math.hawaii.edu/∼jb.

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1. Atomic theories

1.1. **Language.** Let us work in a language $\mathcal{L}$ that has a set of variables X, constants, function symbols, relation symbols, and punctuation, but no primitive equality relation. Constants are regarded as nullary functions, but assume that $\mathcal{L}$ has no nullary relations.

Despite the setting of a language without equality, the logic used is conservative, with boolean truth values and functions.

1.2. **Structure.** An $\mathcal{L}$-*structure* is $\mathbf{A} = \langle A, \mathcal{F}^{\mathbf{A}}, \mathcal{R}^{\mathbf{A}} \rangle$ with the following interpretation. The carrier set A is nonempty. For f a k-ary function symbol, $f^{\mathbf{A}} \subseteq A^k \times A$ satisfies $\forall \mathbf{a} \exists b \; f(\mathbf{a}, b)$. As we can no longer distinguish elements, b is not required to be unique, but there must be at least one such element. We write $f(\mathbf{a})$ as a shorthand for any element b such that $f(\mathbf{a}, b)$. For each nullary function c, there exists at least one $b \in A$ with $c^{\mathbf{A}}(b)$ holding. A k-ary relation of $\mathbf{A}$ is a subset $R^{\mathbf{A}} \subseteq A^k$. Relations of $\mathbf{A}$ are allowed to be empty.

1.3. **Substructure.** Given an $\mathcal{L}$-structure $\mathbf{A}$, the subset $\mathbf{S} = \langle S, \mathcal{F}^{\mathbf{S}}, \mathcal{R}^{\mathbf{S}} \rangle$ is a *substructure* if $S \subseteq A$, for each function $f^{\mathbf{S}} = f^{\mathbf{A}}|_{S^{k+1}}$ is defined on S^k taking values in S, and for each relation $R^{\mathbf{S}} = R^{\mathbf{A}}|_{S^k}$. We do not require that $\mathbf{s} \in S^k$ and $f^{\mathbf{A}}(\mathbf{s}, b)$ implies $b \in S$, but only that $f^{\mathbf{A}}(\mathbf{s}, b)$ for some $b \in S$. If there are no constants in the language, then S may be empty.

Explicitly,

- $S \subseteq A$,
- $\mathbf{s} \in S^k$ implies $f(\mathbf{s}, b)$ for some $b \in S$,
- for each function symbol, $f^{\mathbf{S}} = f^{\mathbf{A}}|_{S^{k+1}}$, and
- for each relation symbol, $R^{\mathbf{S}} = R^{\mathbf{A}}|_{S^k}$.

Because substructures are only weakly closed, and we might have, for example, $h(s, t)$ and $h(t, u)$ with s, $u \in S$ but $t \notin S$. This may be counterintuitive, but in fact the weak closure is all that need be required.

1.4. **Direct products.** These are done component-wise, and require no reinterpretation.

1.5. **Honumorphism.** We need to modify the notion of homomorphism to something more appropriate for this setting. A subset $h \subseteq \mathbf{A} \times \mathbf{B}$ is a *honumorphism* if

(1) for each a there exists b with $h(a, b)$,
(2) for each function symbol,

$$f^{\mathbf{A}}(a_1, \ldots, a_k, c) \,\&\, h(a_1, b_1) \,\&\, \ldots \,\&\, h(a_k, b_k) \,\&\, h(c, d)$$

implies $f^{\mathbf{B}}(b_1, \ldots, b_k, d)$,
(3) for each relation symbol, $R^{\mathbf{A}}(a_1, \ldots, a_k) \,\&\, h(a_1, b_1) \,\&\, \ldots \,\&\, h(a_k, b_k)$ implies $R^{\mathbf{B}}(b_1, \ldots, b_k)$,
(4) for each function symbol,

$$f^{\mathbf{B}}(b_1, \ldots, b_k, d) \,\&\, h(a_1, b_1) \,\&\, \ldots \,\&\, h(a_k, b_k) \,\&\, h(c, d)$$

implies there exists c' such that $h(c', d)$ and $f^{\mathbf{A}}(a_1, \ldots, a_k, c')$.

Given $h \in \mathrm{Honu}(\mathbf{A}, \mathbf{B})$, let $h(\mathbf{A}) = \{b \in B : \exists a \in A \; h(a, b)\}$.

Lemma 1. *If* $\mathbf{S} \leq \mathbf{A}$ *and* $h \in \mathrm{Honu}(\mathbf{A}, \mathbf{B})$*, then* $h(\mathbf{S}) \leq \mathbf{B}$.

1.6. **Kernel.** The *kernel* of a honumorphism h is the set of relations

$$\ker h =$$
$$\{R(a_1, \ldots, a_k) : R \in \mathcal{R} \text{ and } h(a_1, b_1) \,\&\, \ldots \,\&\, h(a_k, b_k) \implies R^{\mathbf{B}}(b_1, \ldots, b_k)\}.$$

1.7. **Examples.** Part of the philosophy here is that if elements of a structure have different properties, then they are distinct, but in a language without equality the converse is false. This is the idea behind the Leibniz congruence, which however plays no direct role here; see, e.g., [2].

Given a structure $\mathbf{A}$, we can form an *expansion* $\mathbf{E}$ of $\mathbf{A}$ thusly. For each $a \in A$, let X_a be a nonempty set. Let $E = \bigcup_{a \in A} X_a$. Define the operations and relations of $\mathbf{E}$ by

$$f^{\mathbf{E}}(x_1, \ldots, x_k, y) \text{ iff } x_1 \in X_{a_1} \,\&\, \ldots x_k \in X_{a_k} \,\&\, y \in X_b \,\&\, f^{\mathbf{A}}(a_1, \ldots, a_k, b),$$
$$R^{\mathbf{E}}(x_1, \ldots, x_k) \text{ iff } x_1 \in X_{a_1} \,\&\, \ldots x_k \in X_{a_k} \,\&\, R^{\mathbf{A}}(a_1, \ldots, a_k).$$

Then the relation h given by $h(a, x)$ iff $x \in X_a$ is in $\mathrm{Honu}(\mathbf{A}, \mathbf{E})$.

At the other extreme, we can perform the *contraction* $\breve{A}$ of $\mathbf{A}$ that identifies all indistinguishable elements. Define $d \equiv d'$ if

$$F^{\mathbf{A}}(a_1, \ldots, d, \ldots, a_k, b) \text{ iff } F^{\mathbf{A}}(a_1, \ldots, d', \ldots, a_k, b)$$
$$F^{\mathbf{A}}(a_1, \ldots, a_k, d) \text{ iff } F^{\mathbf{A}}(a_1, \ldots, a_k, d')$$
$$R^{\mathbf{A}}(a_1, \ldots, d, \ldots, a_k) \text{ iff } R^{\mathbf{A}}(a_1, \ldots, d', \ldots, a_k).$$

Choose a system S of representatives of the $\equiv$-classes. With the inherited operations and relations, this becomes a substructure $\mathbf{S} \leq \mathbf{A}$. Moreover, the relation h defined by $h(a, s)$ iff $a \equiv s \in S$ is a honumorphism: $h \in \mathrm{Honu}(\mathbf{A}, \mathbf{S})$.

1.8. **Icemorphism and isomorphism.** There are two notions of equivalence for structures.

Say that $\mathbf{A}$ is *icemorphic* to $\mathbf{B}$, denoted $\mathbf{A} \simeq \mathbf{B}$, if there is a honumorphism $h \in \mathrm{Honu}(\mathbf{A}, \mathbf{B})$ such that

(1) for each $a \in A$ there exists $b \in B$ with $h(a, b)$,
(2) for each $b \in B$ there exists $a \in A$ with $h(a, b)$,
(3) for each function symbol, if $h(a_1, b_1) \,\&\, \dots \,\&\, h(a_k, b_k) \,\&\, h(c, d)$, then $f^{\mathbf{A}}(a_1, \dots, a_k, c)$ iff $f^{\mathbf{B}}(b_1, \dots, b_k, d)$,
(4) for each relation symbol, if $h(a_1, b_1) \,\&\, \dots \,\&\, h(a_k, b_k)$, then $R^{\mathbf{A}}(a_1, \dots, a_k)$ iff $R^{\mathbf{B}}(b_1, \dots, b_k)$.

An icemorphism is a honumorphism, for the fourth condition for a honumorphism is included in the strengthened condition for function symbols.

Regular *isomorphism*, denoted $\mathbf{A} \cong \mathbf{B}$, means that there exists a bijection $h : \mathbf{A} \to \mathbf{B}$ such that

(1) for each function symbol, $f^{\mathbf{A}}(a_1, \dots, a_k, c)$ iff $f^{\mathbf{B}}(h(a_1), \dots, h(a_k), h(c))$,
(2) for each relation symbol, $R^{\mathbf{A}}(a_1, \dots, a_k)$ iff $R^{\mathbf{B}}(h(a_1), \dots, h(a_k))$.

The philosophy is that icemorphism is the natural equivalence in a language without equality, where you cannot necessarily distinguish elements. But in looking at models from outside the system, isomorphism remains the appropriate equivalence.

These two notions are connected thusly.

Lemma 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{L}$*-structures, with contractions* $\check{\mathbf{A}}$ *and* $\check{\mathbf{B}}$*, respectively. Then* $\mathbf{A} \simeq \mathbf{B}$ *if and only if* $\check{\mathbf{A}} \cong \check{\mathbf{B}}$.

Looking ahead, since an expansion of $\mathbf{A}$ satisfies the same implications as $\mathbf{A}$, a standard result quoted below yields this.

Lemma 3. *Let* $\mathbf{A}$ *be an* $\mathcal{L}$*-structure and* $\hat{\mathbf{A}}$ *an expansion of* $\mathbf{A}$*. Then* $\hat{\mathbf{A}}$ *is isomorphic to a substructure of a reduced power of* $\mathbf{A}$.

Likewise, we have two notions of embedding:

- $\mathbf{A} \sqsubseteq \mathbf{B}$ if $\mathbf{A}$ is icemorphic to a substructure of $\mathbf{B}$,
- $\mathbf{A} \leq \mathbf{B}$ if $\mathbf{A}$ is isomorphic to a substructure of $\mathbf{B}$.

The two lemmate, however, explain why these distinction play no major role in our analysis.

1.9. **Atomic theories.** As usual, form the absolutely free structure $\mathbf{F} = \mathbf{F}_{\mathcal{L}}(X)$. No relations hold on $\mathbf{F}$, but we can form $\mathcal{R}(\mathbf{F})$, the set of all potential relation instances on $\mathbf{F}$. The elements of $\mathbf{F}$ are called *terms*, and members of $\mathcal{R}(\mathbf{F})$ are *atomic formulae.*

Note that $\mathbf{F}$ is an algebra in the usual sense, and any map $\sigma : X \to \mathbf{F}$ can be extended to a homomorphism in the usual way. We refer to these endomorphisms as *substitutions*, and use $\mathrm{Sbn}(\mathbf{F})$ to denote the monoid of all substitutions.

A subset $\Sigma \subseteq \mathcal{R}(\mathbf{F})$ is an *atomic theory* if whenever $R(\mathbf{t}) \in \Sigma$ and $\sigma \in \mathrm{Sbn}(\mathbf{F})$, then $R(\sigma\mathbf{t}) \in \Sigma$. That is, atomic theories are just sets of relations on $\mathbf{F}$ that are closed under substitution.

By general principles, the lattice of all atomic theories of $\mathcal{L}$ forms an algebraic lattice $\mathrm{ATh}(\mathcal{L})$.

2. Models and satisfaction for atomic theories

2.1. **Satisfaction.** An $\mathcal{L}$-structure $\mathbf{A}$ is said to *satisfy* the atomic formula $R(t_1(\mathbf{x}), \ldots, t_k(\mathbf{x}))$ if for every $\mathbf{a} \in A^n$, $t_1(\mathbf{a}, b_1) \,\&\, \ldots \,\&\, t_k(\mathbf{a}, b_k)$ implies $R^{\mathbf{A}}(b_1, \ldots, b_k)$. Here the composition of terms is defined as the composition of relations in $\mathbf{A}$.

As always, satisfaction determines a Galois connection between $\mathcal{L}$-structures and atomic formulae. The respective closed sets are called *atomic classes* and *atomic theories.*

Now let Σ be an atomic theory. The structure $\mathbf{A}$ is a *model* of Σ, or $\mathbf{A} \in \mathrm{Mod}\ \Sigma$, if $\mathbf{A}$ satisfies Φ for every $\Phi \in \Sigma$. In this case, we also say that $\mathbf{A}$ *satisfies* Σ.

The *free* Σ*-structure on* X is $\mathbf{F}_{\mathcal{L}}(X)$ with the relations Σ, denoted $\mathbf{F}_{\Sigma}(X)$. The definition of a theory insures that $\mathbf{F}_{\Sigma}(X)$ is in $\mathrm{Mod}\ \Sigma$.

2.2. **Closure.** The usual closures go through.

Theorem 4. *Let* Σ *be an atomic theory.*

1. $\mathbf{F}_{\Sigma}(X) \in \mathrm{Mod}\ \Sigma$.
2. *If* $\mathbf{A}_i \in \mathrm{Mod}\ \Sigma$ *for all* $i \in I$*, then* $\prod_{i \in I} \mathbf{A}_i \in \mathrm{Mod}\ \Sigma$.
3. *If* $\mathbf{A} \in \mathrm{Mod}\ \Sigma$ *and* $\mathbf{S} \leq A$*, then* $\mathbf{S} \in \mathrm{Mod}\ \Sigma$.
4. *If* $\mathbf{A} \in \mathrm{Mod}\ \Sigma$ *and* $h \in \mathrm{Honu}(\mathbf{A}, \mathbf{B})$*, then* $h(\mathbf{A}) \in \mathrm{Mod}\ \Sigma$.

2.3. **Satisfaction and Honumorphisms.** The standard results extend to atomic classes and honumorphisms.

Theorem 5. *Let* Σ *be an atomic theory. TFAE for a structure* $\mathbf{A}$*.*

1. $\mathbf{A} \in \mathrm{Mod}\ \Sigma$.
2. *For every* $h \in \mathrm{Honu}(\mathbf{F}_{\mathcal{L}}(X), \mathbf{A})$*,* $\Sigma \subseteq \ker h$.
3. *There exists* $h \in \mathrm{Honu}(\mathbf{F}_{\Sigma}(Y), \mathbf{A})$ *for some* Y *with* $h(\mathbf{F}) = \mathbf{A}$.

A key step in the proof is this claim. *Let* $\mathbf{A} \in \mathrm{Mod}\ \Sigma$*. If* $h_0 \subseteq X \times A$*, then* h_0 *can be extended to a honumorphism* $h \in \mathrm{Honu}(\mathbf{F}_{\mathcal{L}}(X), \mathbf{A})$ *with* $\Sigma \subseteq \ker h$*.* This is done recursively: if

- the term $t = f(s_1, \ldots, s_k)$ identically,
- $h(s_1, a_1), \ldots, h(s_k, a_k)$ are given, and
- $f^{\mathbf{A}}(a_1, \ldots, a_k, b)$ holds, then (and only then) let $h(t, b)$ hold.

Corollary 6. *A class of* $\mathcal{L}$*-structures is an atomic class if and only if it is closed under honumorphisms, substructures and direct products.*

This combines the theorem above with the observation that the free structure generated by a class $\mathcal{K}$ can be constructed as a substructure of a direct product.

2.4. **Caveat.** Consider the statement

(4) Every map $h_0 : \mathbf{F}_\Sigma(X) \to \mathbf{A}$ can be extended to a honumorphism.

When the operations are not necessarily functions, a honumorphism is not determined by its action on a set of generators. Statement (4) is *not* equivalent to the statements of the theorem.

In a language with one unary operation and one unary relation, consider the atomic theory Σ generated by the formula $R(f(x))$, and the structure $\mathbf{A}$ on $\{a, b, b'\}$ with $f(a, b)$, $f(a, b')$, $f(b, b)$, $f(b', b)$ and $R(b)$. Then $\mathbf{A}$ does not satisfy $R(f(x))$, but the mapping property holds for maps from $\mathbf{F}_\Sigma(X)$ to $\mathbf{A}$.

This example can be fixed by changing *map* in (4) to *relation*, but then a slightly more complicated example shows that is not equivalent.

We have been intentionally vague about the set X. For while we generally want X to be countably infinite, there are times when a finite set is sufficient, e.g., for languages with one unary predicate and one unary function. If we insist that X be infinite, then (4) seems to be again equivalent, as we can put all the elements involved in the range. Alternatively, we could talk about extending partial honumorphisms. Both these options are far removed from the original meaning of (4) for algebras. So perhaps this line should be abandoned.

2.5. **Kongruences and factors.** With linguistic apologies, a *kongruence* is a honumorphism kernel. Indeed, any extension of the relations of $\mathbf{A}$ gives a kongruence (associated with the identity map). So the set of all kongruences on $\mathbf{A}$ forms a boolean lattice, denoted as Kon $\mathbf{A}$. This is not very exciting: the more important notion will be a K-congruence, where K is an implicational theory.

If φ is a kongruence, then $\mathbf{A}/\varphi$ denotes $\mathbf{A}$ with the relations φ. This is consistent with the usual notation, though not how one normally thinks of it. However, in the setting without equality, the kernel does not carry enough information and there is no clear analogue of the first isomorphism theorem. Indeed, distinct honumorphisms may have the same kernel! These ideas remain useful, though.

Meanwhile, the second isomorphism theorem holds trivially.

3. Lattices of atomic theories

3.1. **Fully invariant kongruences and lattices of atomic theories.** A *fully invariant kongruence* is a set of relations closed under substitution endomorphisms. These again form an algebraic lattice Fikon $\mathbf{F}$.

The collection of all atomic theories extending a given theory Σ is also an algebraic lattice, denoted by $\mathrm{ATh}(\Sigma)$. Without a primitive equality, the only means of deduction for atomic formulae is substitution. Evidently:

Theorem 7. *For an atomic theory Σ, the lattice* $\mathrm{ATh}(\Sigma)$ *is isomorphic to* Fikon $\mathbf{F}_\Sigma(X)$ *with X countably infinite.*

The structure of the lattices $\mathrm{At}(\Sigma)$ is the topic of the fourth part of this series citeHKNT, with T. Holmes, D. Kitsuwa and S. Tamagawa. In particular, these lattices are completely distributive and coatomic.

4. Implicational theories

Formally, an *implication* is an ordered pair $\langle F, Q\rangle$ with F a finite set of atomic formulae and Q an atomic formula. Thus each $P \in F$ and Q are of the form $A(\mathbf{t})$ with A a relational symbol and $\mathbf{t} \in \mathbf{F}^n$. To reflect the intended interpretation, we write an implication $\langle F, Q\rangle$ with $F = \{P_0, \dots, P_m\}$ as either $F \implies Q$ or $\& P_i \implies Q$. The antecedent is allowed to be empty: $\emptyset \implies P$ is equivalent to P. The formal definition insures that conjunction is idempotent, commutative and associative.

A collection T of implications is an *implicational theory* if

(i) $F \implies P$ is in T whenever $P \in F$,
(ii) if $F \implies Q$ is in T, then $F \cup \{R\} \implies Q$ is in T for any R,
(iii) whenever $F \implies Q$ is in T for all $Q \in G$, and $G \implies R$ is in T, then $F \implies R$ is in T.
(iv) T is closed under substitutions: if $\Phi \in T$ and $\sigma : X \to \mathbf{F}$, then $\sigma\Phi \in T$.

Note that condition (iii), transitivity, implies *modus ponens*:

(v) if $P_i \in T$ for all i and $\& P_i \implies Q$ is in T, then $Q \in T$.

The *free T-structure on X* is $\mathbf{F}_{\mathcal{L}}(X)$ with the purely atomic relations of T, denoted $\mathbf{F}_T(X)$. Thus $A(\mathbf{t})$ holds in $\mathbf{F}_T(X)$ if and only if $A(\mathbf{t})$ is in T.

5. Models and satisfaction for implicational theories

The structure $\mathbf{A}$ is a *model* for an implicational theory T if, for every honumorphism $h \in \mathrm{Honu}(\mathbf{F}, \mathbf{A})$ and every $\Phi \in T$, if $P_i \in \ker h$ for all i then $Q \in \ker h$. Again we say that $\mathbf{A}$ *satisfies* T.

The definition of a theory insures that $\mathbf{F}_T(X) \in \mathrm{Mod}\ T$. Moreover, it has the mapping property: for any $\mathbf{A} \in \mathrm{Mod}\ T$, any map $h_0 : X \to \mathbf{A}$ can be extended to a honumorphism.

A class $\mathcal{Q}$ of $\mathcal{L}$-structures is an *implicational class* if $\mathcal{Q} = \mathrm{Mod}(T)$ for some implicational theory T. As usual, satisfaction induces a Galois correspondence between structures and implications, and hence a dual isomorphism between implicational classes and implicational theories. Thus we can talk about the implicational class generated by a class of structures, and so forth.

In this case, the standard results for quasivarieties and quasi-equational classes carry over without change to languages without equality.

Lemma 8. *If $\mathcal{K}$ is a class of structures, and the structure $\mathbf{A}$ satisfies every implication $F \implies Q$ satisfied by all members of $\mathcal{K}$, then $\mathbf{A}$ is isomorphic to a substructure of a reduced product of structures in $\mathcal{K}$.*

Theorem 9. *A class $\mathcal{Q}$ of $\mathcal{L}$-structures is an implicational class if and only if it is closed under isomorphism, substructures and reduced products.*

Recall briefly the standard proof of the lemma. The *extended diagram* of $\mathbf{A}$ is the collection of all atomic formulae $\varphi(\mathbf{a})$ and negations $\neg\varphi(\mathbf{a})$, with $\mathbf{a} \in A^k$, that hold in $\mathbf{A}$. Assume that $\mathbf{A}$ satisfies all the implications satisfied by $\mathcal{K}$, and let F be a finite subset of the extended diagram of $\mathbf{A}$. Then there are a finite direct product $\mathbf{P} = \prod_{i=1}^{n} \mathbf{K}_i$ of members of $\mathcal{K}$, and a mapping h of the elements appearing in F, that embeds F into $\mathbf{P}$. Combining these, $\mathbf{A}$ is isomorphic to a substructure of a reduced product indexed by the finite subsets of the extended diagram of $\mathbf{A}$. This proof works in any first-order language.

6. Relative kongruences

Let T be an implicational theory. A kongruence θ on $\mathbf{A}$ is a T-kongruence if $\mathbf{A}/\theta \in \operatorname{Mod} T$, that is, $\mathbf{A}$ with the relations θ satisfies all the formulae of T. So a kongruence θ, regarded as a set of relations, is a T-kongruence if, whenever $\& P_i \implies Q$ is in T and $P_i(\alpha\mathbf{x}) \in \theta$ for some substitution $\alpha : X \to \mathbf{A}$ and all i, then $Q(\alpha\mathbf{x}) \in \theta$. Again, the set of all T-congruences on $\mathbf{A}$ forms an algebraic lattice $\operatorname{Kon}_T(\mathbf{A})$, as the closure operator $\operatorname{kon}_{\mathrm{T}}$ is finitary in nature.

For any set M of atomic formulae, let $\operatorname{kon}_T(M)$ denote the T-kongruence generated by M, i.e., the smallest T-kongruence containing M. Thus $\operatorname{kon}_T(M)$ contains M and all its T-consequences.

Consider the substitution endomorphisms of the free algebra $\mathbf{F}_T(X)$, that is, the homomorphisms ε generated by maps $\varepsilon_0 : X \to \mathbf{F}$. These maps form a monoid, denoted $\operatorname{Sbn}(\mathbf{F})$. (Since the relational part of an endomorphism is not determined by the substitution for the variables, $\mathbf{F}$ may have other endomorphisms.)

The substitution endomorphisms of $\mathbf{F}$ act naturally on the compact kongruences of $\operatorname{Kon}_T(\mathbf{F})$. For $\varepsilon \in \operatorname{Sbn} \mathbf{F}$, define

$$\widehat{\varepsilon}(\operatorname{kon}_T(R(\mathbf{s}))) = \operatorname{kon}_T(R(\varepsilon\mathbf{s}))$$
$$\widehat{\varepsilon}(\bigvee_j \varphi_j) = \bigvee_j \widehat{\varepsilon}\varphi_j.$$

Lemma 10 below checks the crucial technical detail that $\widehat{\varepsilon}$ is well-defined, and hence join-preserving, because $\psi \leq \bigvee_j \varphi_j$ implies $\widehat{\varepsilon}\psi \leq \bigvee_j \widehat{\varepsilon}\varphi_j$ for principal kongruences ψ and φ_j in $\operatorname{Kon}_T(\mathbf{F})$. Also note that $\widehat{\varepsilon}$ is zero-preserving: the least T-kongruence Δ_T of $\mathbf{F}$ contains exactly those relations $A(\mathbf{t})$ such that $A(\mathbf{t})$ is in T, and $\widehat{\varepsilon}(\Delta_T) = \Delta_T$ because T is closed under substitution. Let $\widehat{\mathcal{E}} = \{\widehat{\varepsilon} : \varepsilon \in \operatorname{Sbn} \mathbf{F}\}$.

The next lemma reflects the interpretation that $\operatorname{kon}_T(M)$ consists of M and all its T-consequences.

Lemma 10. *If T is an implicational theory, then $\operatorname{kon}_T(Q) \leq \bigvee_i \operatorname{kon}_T(P_i)$ holds in $\operatorname{Kon}_T(\mathbf{F})$ if and only if $\&_i P_i \implies Q$ is in T.*

7. Lattices of implicational theories

Form the lattice $\mathrm{ITh}(T)$ of all implicational theories extending T, an algebraic lattice.

Theorem 11. *For an implicational theory T,*

$$\mathrm{ITh}(T) \cong \mathrm{Con}\ \mathbf{S}$$

where $\mathbf{S} = \langle \mathbf{U}, \vee, 0, \widehat{\mathcal{E}} \rangle$ with $\mathbf{U}$ the semilattice of T-kongruences that are compact in $\mathrm{Kon}_{\mathrm{T}}(\mathbf{F})$, $\mathcal{E} = \mathrm{Sbn}(\mathbf{F})$, and $\mathbf{F} = \mathbf{F}_T(X)$ with $|X| = \aleph_0$.

At one point, we use a technical variant, with the same proof.

Theorem 12. *Let T be an implicational theory and $n \geq 1$ an integer. The lattice of all implicational theories that*

1. *contain T, and*
2. *are determined relative to T by implications in at most n variables*

is isomorphic to $\mathrm{Con}\ \mathbf{S}_n$, where $\mathbf{S}_n = \langle \mathbf{U}, \vee, 0, \widehat{\mathcal{E}} \rangle$ with $\mathbf{U}$ the semilattice of T-kongruences that are compact in $\mathrm{Kon}_{\mathrm{T}}(\mathbf{F})$, $\mathcal{E} = \mathrm{Sbn}(\mathbf{F})$, and $\mathbf{F} = \mathbf{F}_T(n)$.

For the proof of this theorem, and for its application, it is natural to use two structures closely related to the congruence lattice instead [1]. For an algebra $\mathbf{A}$ with a join semilattice reduct, let $\mathrm{Don}\ \mathbf{A}$ be the lattice of all reflexive, transitive, compatible relations R such that $\geq \subseteq R$, i.e., $x \geq y$ implies $x\,R\,y$. Let $\mathrm{Eon}\ \mathbf{A}$ be the lattice of all reflexive, transitive, compatible relations R such that

1. $R \subseteq \leq$, i.e., $x\,R\,y$ implies $x \leq y$, and
2. if $x \leq y \leq z$ and $x\,R\,z$, then $x\,R\,y$.

Lemma 13. *If $\mathbf{A} = \langle A, \vee, 0, \mathcal{F} \rangle$ is a semilattice with operators, then* $\mathrm{Con}\ \mathbf{A} \cong \mathrm{Don}\ \mathbf{A} \cong \mathrm{Eon}\ \mathbf{A}$.

The proof of the lemma is fairly straightforward, and can be found in Part I of [1].

Proof. Define the map $\kappa : \mathrm{ITh}(T) \to \mathrm{Don}\ \mathbf{S}$ by $(\theta, \psi) \in \kappa(K)$ if and only if there are $P_0, \ldots, P_m, Q_0, \ldots, Q_n$ such that

- for each j, the implication $\& P_i \implies Q_j$ is in K,
- $\theta = \bigvee_i \mathrm{kon}_{\mathrm{T}}(P_i)$ in $\mathrm{Kon}_{\mathrm{T}}(\mathbf{F})$, and
- $\psi = \bigvee_j \mathrm{kon}_{\mathrm{T}}(Q_j)$ in $\mathrm{Kon}_{\mathrm{T}}(\mathbf{F})$.

In the other direction, define $\tau : \mathrm{Don}\ \mathbf{S} \to \mathrm{ITh}(T)$ such that $\& P_i \implies Q$ is in $\tau(R)$ if and only if $(\bigvee \mathrm{kon}_{\mathrm{T}}(P_i), \mathrm{kon}_{\mathrm{T}}(Q))$ is in R.

The proof of the theorem is mostly routine checking, modulo Lemma 10.

First, we check that $\kappa(K) \in \mathrm{Don}\ \mathbf{S}$. Reflexivity follows from property (i) of K.

The transitivity of $\kappa(K)$ requires some care. Suppose $\theta\,\kappa(K)\,\psi\,\kappa(K)\varphi$, where

$$\begin{aligned}\theta &= \bigvee_i \operatorname{kon}_T(P_i)\\ \psi &= \bigvee_j \operatorname{kon}_T(Q_j) = \bigvee_k \operatorname{kon}_T(R_k)\\ \varphi &= \bigvee_\ell \operatorname{kon}_T(S_\ell)\end{aligned}$$

with the corresponding implications being in K. Now $\& Q_j \implies R_k$ is in T for all k by Lemma 10, and $T \subseteq K$. Thus $\& P_i \implies R_k$ is in K for all k by (iii). Apply (iii) once more to obtain $\& P_i \implies S_\ell$ in K for all ℓ, whence $\theta\,\kappa(K)\,\varphi$.

The compatibility of $\kappa(K)$ with join, $\theta\,\kappa(K)\,\psi$ implies $\theta \vee \varphi\,\kappa(K)\,\psi \vee \varphi$, follows from conditions (i) and (ii). Compatibility with substitutions is condition (iv).

That $\theta \geq \psi$ implies $\theta\,\kappa(K)\,\psi$ follows from Lemma 10 and $T \subseteq K$.

We conclude that $\kappa(K) \in \operatorname{Don} \mathbf{S}$. It is also clear that κ is order-preserving.

Next, given $R \in \operatorname{Don} \mathbf{S}$, check that $\tau(R)$ is an implicational theory. Properties (i) and (ii) follow from $\geq \subseteq R$ and the transitivity of R. For property (iii), note that if $(\bigvee_i \operatorname{kon}_T(P_i), \operatorname{kon}_T(Q_j)) \in R$ for all j, then $(\bigvee_i \operatorname{kon}_T(P_i), \bigvee_j \operatorname{kon}_T(Q_j)) \in R$ since R is compatible with respect to joins. If in addition $(\bigvee_j \operatorname{kon}_T(Q_j), \operatorname{kon}_T(S)) \in R$, then $(\bigvee_i \operatorname{kon}_T(P_i), \operatorname{kon}_T(S)) \in R$ by the transitivity of R. Finally, closure under substitution, (iv), is immediate from the definition of $\widehat{\varepsilon}$.

Moreover, $\tau(R) \supseteq T$ by Lemma 10, and τ is order-preserving.

Finally, using Lemma 10 again, we note that $\kappa\tau(R) = R$ and $\tau\kappa(K) = K$ for all appropriate R and K. □

Recall that there is a natural equa-interior operator on lattices of quasi-equational theories. Given a quasi-equational theory $\mathcal{Q}$ and a theory $\mathcal{T}$ in $\operatorname{QuTh}(\mathcal{Q})$, define $\eta(\mathcal{T})$ to be the implicational theory generated by $\mathcal{Q}$ and all the equations valid in $\mathcal{T}$. This interior operator has the following properties [7].

(I1) $\eta(x) \leq x$
(I2) $x \geq y$ implies $\eta(x) \geq \eta(y)$
(I3) $\eta^2(x) = \eta(x)$
(I4) $\eta(1) = 1$
(I5) $\eta(x) = u$ for all $x \in X$ implies $\eta(\bigvee X) = u$
(I6) $\eta(x) \vee (y \wedge z) = (\eta(x) \vee y) \wedge (\eta(x) \vee z)$
(I7) The image $\eta(\mathbf{L})$ is the complete join subsemilattice of $\mathbf{L}$ generated by $\eta(\mathbf{L}) \cap \mathbf{L}_c$.
(I8) There is a compact element $w \in \mathbf{L}$ such that $\eta(w) = w$ and the interval $[w, 1]$ is isomorphic to the congruence lattice of a semilattice. Thus the interval $[w, 1]$ is coatomistic.

In view of (I5), let $\tau(x) = \bigvee\{z : \eta(z) = \eta(x)\}$. A ninth property was added in [1].

(I9) For any index set I, if $\eta(x) \leq c$ and $\bigwedge \tau(z_i) \leq \tau(c)$, then $\eta(\eta(x) \vee \bigwedge_{i\in I} \tau(x \wedge z_i)) \leq c$.

There is also a natural interior operator defined on the congruence lattice of any semilattice with operators, where $\eta(\theta)$ is the congruence generated by the 0-class of θ. This operator satisfies properties (I1)–(I7) and (I9). However, it need not satisfy (I8), which for lattices of quasi-equational theories refers to the relative variety determined by $x \approx y$.

These ideas fit into our current setting thusly. Let $\mathrm{ATh}^*(T)$ denote the lattice of implicational theories generated by T and a set of purely atomic formulae. Note that $\mathrm{ATh}^*(T)$ is a complete join subsemilattice of $\mathrm{ITh}(T)$.

In the representation of Theorem 11, relatively atomic theories of T correspond to congruences $\eta(I)$ with I an $\widehat{\mathcal{E}}$-closed ideal of $\mathbf{S}$. Thus $\mathrm{ATH}^*(T)$ is isomorphic to the lattice $\eta(\mathrm{Con}\ \mathbf{S})$ for the natural interior operator, which in turn is isomorphic to the lattice of $\widehat{\mathcal{E}}$-closed ideals of $\mathbf{S}$. In particular, $\mathrm{ITh}(T)$ has a natural interior operator satisfying properties (I1)–(I7) and (I9), and all the consequences of that apply (see [1]).

Under the circumstances, the special role of property (I8) for implicational theories in languages with equality invites further analysis.

8. Restoring equality

At this point, we pause to note that T could contain implications saying that a binary relation $\approx$ is an equivalence relation and, moreover, a congruence in the usual sense. That is, T could contain the laws

(1) $x \approx x$
(2) $x \approx y \Longrightarrow y \approx x$
(3) $x \approx y \ \&\ y \approx z \Longrightarrow x \approx z$
(4) $x \approx y \Longrightarrow f(x, \mathbf{z}) \approx f(y, \mathbf{z})$ for all functions f,
(5) $x \approx y \Longrightarrow (R(x, \mathbf{z}) \Longleftrightarrow R(y, \mathbf{z}))$ for all predicates R.

This relation can then be regarded as equality. In this case, T-kongruences correspond to regular congruences, $\mathrm{EqTh}(T) \cong \mathrm{ATh}^*(T)$ and $\mathrm{QuTh}(T) \cong \mathrm{ITh}(T)$.

Björn Kjos-Hanssen points out that while there may be no such relation, there is at most one, in view of (5).

9. Representation

Now we provide a converse to Theorem 11.

Theorem 14. *Let $\mathcal{B}$ be an implicational theory in a language $\mathcal{L}$ with the following restrictions and laws.*

(1) *$\mathcal{L}$ has only unary predicate symbols.*
(2) *$\mathcal{L}$ has only unary function symbols.*
(3) *$\mathcal{L}$ has one constant symbol e.*

(4) *$\mathcal{B}$ contains the laws $P(f(e))$ for every predicate P and every formal composition f of functions of $\mathcal{L}$.*

Then every implication holding in a theory extending the theory of $\mathcal{B}$ is equivalent (modulo the laws of $\mathcal{B}$) to a set of implications in only one variable. Hence the lattice of theories of $\mathcal{B}$ is isomorphic to $\mathrm{Con}(\mathbf{S})$ *where* $\mathbf{S} = \langle \mathbf{T}, \vee, 0, \widehat{\mathcal{E}} \rangle$ *with* $\mathbf{T}$ *the semilattice of compact kongruences of* $\mathrm{Kon}_{\mathcal{B}}(\mathbf{F})$, $\mathcal{E} = \mathrm{Sbn}(\mathbf{F})$, *and* $\mathbf{F} = \mathbf{F}_{\mathcal{B}}(1)$.

Proof. The atomic formulae of $\mathcal{L}$ are of the form $A(h(u))$, where A is a predicate, h is a formal composition of functions, possibly empty, and u is a variable or e. In an implication $\& P_i \implies Q$, the conclusion involves at most one variable. A law that is equivalent, modulo the laws of $\mathcal{B}$, is obtained by replacing every other variable occurring in the antecedent by e. □

Theorem 15. *Let* $\mathbf{S}$ *be a join semilattice with* 0, *and let* $\mathbf{M}$ *be a monoid of operators acting on* $\mathbf{S}$. *Then there is an implicational theory* $\mathcal{C}$ *such that* $\mathrm{Con}(\mathbf{S}, +, 0, \mathbf{M})$ *is isomorphic to* $\mathrm{Con}(\mathbf{T}, \vee, 0, \widehat{\mathcal{E}})$ *with* $\mathbf{T}$ *the semilattice of compact kongruences of* $\mathrm{Kon}_{\mathcal{C}}(\mathbf{F})$, $\mathcal{E} = \mathrm{Sbn}(\mathbf{F})$, *and* $\mathbf{F} = \mathbf{F}_{\mathcal{C}}(1)$.

Proof. Our language will include unary predicates A for each nonzero element a of $\mathbf{S}$, unary operations f for each $f \in \mathbf{M}$, and a constant e.

Again, $\mathcal{L}$-terms are of the form $A(h(u))$, where A is a predicate, h is a formal composition of functions, and u is a variable or e. Denote the single variable by x.

The construction begins by assigning a set of predicates to each nonzero element of $\mathbf{S}$. For each $a \in \mathbf{S}$ and formal composition $h = f_1 \dots f_k$, assign the predicate $A(h(x))$ to $h^\star(a)$, where $h^\star$ denotes h evaluated in $\mathbf{M}^{\mathrm{opp}}$, that is, $h^\star = f_k \dots f_1$. In this way each element of $\mathbf{S}$ may be assigned multiple predicates, but they will all be of the form $B(g(x))$ for different predicates B and sequences g. For $s \in \mathbf{S}$, let $\mathcal{P}(s)$ denote the set of predicates assigned to s. Thus $\mathcal{P}(s) = \{A(h(x)) : h^\star(a) = s\}$.

Define $\mathcal{C}$ to be the quasivariety determined by these laws.

(1) $P(f(e))$ for every predicate P and every formal composition f of functions of $\mathcal{L}$.
(2) $A(i(x)) \iff A(x)$ for every A, where i is the identity element of $\mathbf{M}$.
(3) $A(h(x)) \iff A(h^\star(x))$ for every formal composition.
(4) $\beta \implies \alpha$ whenever $a \le b$, $\alpha \in \mathcal{P}(a)$, $\beta \in \mathcal{P}(b)$.
(5) $\& \beta_j \implies \alpha$ whenever $a \le \sum b_j$, $\alpha \in \mathcal{P}(a)$, $\beta_j \in \mathcal{P}(b_j)$ for each j.

The laws (1) ensure that Theorem 14 applies, so that we may work with $\mathbf{F}_{\mathcal{C}}(1)$. Note that the laws (4) are redundant as a special case of (5).

The universe of $\mathbf{F} = \mathbf{F}_{\mathcal{C}}(1)$ is all terms $h(u)$ with h a sequence of operations, and u either x or e. The operations correspond to elements of $\mathbf{M}$, and there is a unary predicate for each nonzero element of $\mathbf{S}$. Note that $A(t)$ holds in the free structure only for $t = e$ or $t = h(e)$. The substitution endomorphisms of $\mathbf{F}$ are determined by the image of x. For a term t, let ε_t denote the endomorphism with $x \mapsto t$.

Since $\mathcal{C}$ satisfies $\&\beta_j \implies \alpha$ whenever $a \le \sum b_j$, $\alpha \in \mathcal{P}(a)$, and each $\beta_j \in \mathcal{P}(b_j)$, we see that every $\mathcal{C}$-kongruence of $\mathbf{F}$ is the set of predicates assigned to some ideal of $\mathbf{S}$, along with $A(h(e))$ for every A and h. Conversely, every ideal determines such a $\mathcal{C}$-congruence, and principal ideals determine compact congruences. In fact, the congruence corresponding to $\downarrow s$ is $\mathrm{kon}_{\mathcal{C}}(\alpha)$ for any $\alpha \in \mathcal{P}(s)$. Thus the semilattice of compact $\mathcal{C}$-congruences of $\mathbf{F}_{\mathcal{C}}(1)$ is isomorphic to $\mathbf{S}$, as desired.

As a matter of notation, let Θ_s denote the kongruence that has all the relations $\bigcup\{\mathcal{P}(t) : t \le s\}$, plus the base relations of the form $B(g(e))$ given by (1). Denote the set of base relations by $\mathbb{B}$. Thus $\Theta_s = \{A(h(x)) : h^\star(a) \le s\} \cup \mathbb{B}$.

It remains to show that the action of $\widehat{\mathcal{E}}$ on $\mathbf{T}$, the semilattice of compact $\mathcal{C}$-kongruences of $\mathbf{F}_{\mathcal{C}}(1)$, mimics the action of $\mathbf{M}^{\mathrm{opp}}$ on $\mathbf{S}$. The relevant facts are these.

- $\widehat{\varepsilon}_{h(e)}(\Theta_s) \subseteq \mathbb{B}$ for any h, whence $\mathbb{B}$ is the zero kongruence. So this operator does not affect the congruences of $\mathbf{T}$.
- $\widehat{\varepsilon}_{h(x)}(\Theta_s) = \Theta_{h^\star(s)}$ for any sequence h and element s.
- If f and g are sequences, then $\widehat{\varepsilon}_{f(x)} = \widehat{\varepsilon}_{g(x)}$ if and only if $\Theta_{f^\star(s)} = \Theta_{g^\star(s)}$ for all s, if and only if $f^\star = g^\star$.
- $\widehat{\varepsilon}_{f(x)}\widehat{\varepsilon}_{g(x)} = \widehat{\varepsilon}_{(gf)(x)}$.

The crucial calculation is the second one. Note that

$$\begin{aligned}\widehat{\varepsilon}_{h(x)}(\Theta_s) &= \mathrm{kon}_{\mathcal{C}}\{\varepsilon_{h(x)}A(f(x)) : f^\star(a) \le s\} \cup \mathbb{B}\\ &= \mathrm{kon}_{\mathcal{C}}\{A(f(h(x))) : f^\star(a) \le s\} \cup \mathbb{B}.\end{aligned}$$

Now $f^\star(a) \le s$ implies $(fh)^\star(a) = h^\star f^\star(a) \le h^\star(s)$, so $\widehat{\varepsilon}_{h(x)}(\Theta_s) \subseteq \Theta_{h^\star(s)}$. But the LHS includes $\varepsilon_{h(x)}S(x) = S(h(x))$, and that's a generator for $\Theta_{h^\star(s)}$, whence equality holds.

This completes the proof of the theorem. □

Combining these results (which now avoid problems that occurred in the presence of equality in Part II of [1]), we obtain the desired result.

Corollary 16. *Let* $\mathbf{S}$ *be a join semilattice with* 0*, and let* $\mathbf{M}$ *be a monoid of operators acting on* $\mathbf{S}$*. Then there is an implicational theory* $\mathcal{C}$ *such that the lattice of implicational theories of* $\mathcal{C}$ *is isomorphic to* $\mathrm{Con}(\mathbf{S}, +, 0, \mathbf{M})$.

10. Overview

It is useful to step back and consider the situation from a distance. There are (at least) four plausible settings.

(IA) Algebras, language with equality.
(IB) Pure relational structures, language with equality.
(II) Structures with functions and relations, language with equality.
(III) Structures with functions and relations, language without equality.

Likewise, there are three types of theories.

(1) Atomic theories ATh(T).
(2) Implicational theories ITh(T).
(3) Relative atomic theories ATh*(T).

That makes twelve combinations, not all equally interesting.

The traditional setting for equational theories is algebras (IA). There we have the results of McKenzie [11], Newrly [13], Nurakunov [14] leading to Lampe's Zipper Condition [9, 10] and its generalizations. Are there any versions of this that apply in other settings?

The historic setting for quasi-equational theories is general structures (II), though pure relational structures (IB) played a role. The results of [1] are the apparent analogues here, and in this note we see how this generalizes to the setting (III). The ultimate goal is still to deal with case (II), and to discern what is special about the quasi-equational theory of algebras (IA).

Atomic theories of structures with equality (II) can be viewed as relative atomic theories of structures without equality (III). This seems an odd viewpoint, but perhaps it explains some of the complexity of lattices of equational theories.

References

[1] K. Adaricheva and J.B. Nation, *Lattices of quasi-equational theories as congruence lattices of semilattices with operators*, Parts I and II, preprint available at www.math.hawaii.edu/∼jb.
[2] W.J. Blok and D. Pigozzi, *Algebraic semantics for universal Horn logic without equality*, in Universal Algebra and Quasigroup Theory (Jadwisin, 1989), vol. 19 of Res. Exp. Math., Heldermann, Berlin, 1992, pp. 1–56.
[3] A. Christie, Q. Wang and S.L. Wismath, *The inflation class operator*, Alg. Univ. **56** (2007), 107–118.
[4] A. Church, Introduction to Mathematical Logic, Princeton Univ. Press, 1956.
[5] J. Czelakowski, Protoalgebraic Logics, vol. 10 of Trends in Logic - Studia Logica Library, Kluwer, Dordrecht, 2001.
[6] R. Elgueta, *Characterizing classes defined without equality*, Studia Logica **58** (1997), 357–394.
[7] V. Gorbunov, Algebraic Theory of Quasivarieties, Siberian School of Algebra and Logic, Plenum, New York, 1998.
[8] T. Holmes, D. Kitsuwa, J.B. Nation and S. Tamagawa, *Lattices of atomic theories in a language without equality*, preprint available at www.math.hawaii.edu/∼jb.
[9] W.A. Lampe, *A property of the lattice of equational theories*, Alg. Univ. **23** (1986), 61–69.
[10] W.A. Lampe, *Further properties of lattices of equational theories*, Alg. Univ. **28** (1991), 459–486.
[11] R. McKenzie, *Finite forbidden lattices*, Universal Algebra and Lattice Theory, Lecture Notes in Mathematics **1004**(1983), Springer-Verlag, Berlin, 176–205.
[12] D. Monk, Mathematical Logic, Springer-Verlag, Berlin, 1976.
[13] N. Newrly, *Lattices of equational theories are congruence lattices of monoids with one additional unary operation*, Alg. Univ. **30** (1993), 217–220.
[14] A.M. Nurakunov, *Equational theories as congruences of enriched monoids*, Alg. Univ. **58** (2008), 357–372.

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