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Decision methods for linearly ordered Heyting algebras

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Abstract. The decision problem for positively quantified formulae in the theory of linearly ordered Heyting algebras is known, as a special case of work of Kreisel, to be solvable; a simple solution is here presented, inspired by related ideas in Gödel-Dummett logic.

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1. Introduction

This paper presents a simple decision method for positively quantified formulae of the classical first-order theory **LOHA** of linearly ordered Heyting algebras, where by *positively quantified* we mean that universal quantifiers appear only in positive positions and existential quantifiers appear only in negative positions. Π_1-formulae are examples. In particular, word problems, either in **LOHA** or in the more restrictive theory **LOL** of linearly ordered lattices, with or without bounds 0 and 1, are solvable by this method, as are word problems in Gödel algebras (i.e. those Heyting algebras that satisfy the condition $\forall xy.(x^y \vee y^x = 1)$, but are not necessarily linearly ordered). Since zero-order Gödel-Dummett logic **LC** has free Gödel algebras as its Lindenbaum algebras [17], the method can be used to decide formulae in that logic, e.g. by interpreting each formula A as a term $h(A)$ of **LOHA** and showing that $1 \leqslant h(A)$ is derivable.

Our presentation is, as befits the subject, in algebraic terminology. Key technical contributions are already made in the logical setting by various authors, as discussed below; the algebraic presentation however is novel, compact and easily implemented. It is also slightly more general, in the absence of a method to interpret such quantified formulae into Gödel-Dummett logic. The decidability of the full first-order theory was first shown by Kreisel [21]; this was later rediscovered ([12], [24]), as pointed out in [29]. We do not claim to cover the full first-order theory; but we give a method that offers relative simplicity in important but restricted cases.

The first-order definability of lattice operations means that (e.g.) any atom $r \leqslant s \vee t$ can be rewritten using a quantifier as $\forall x.((s \leqslant x \wedge t \leqslant x) \supset r \leqslant x)$. The same applies to the exponentiation operator in Heyting algebras, where (e.g.) $r^s \leqslant t$ can be rewritten as $\forall x.(s \wedge x \leqslant r \supset x \leqslant t)$. However, applied to an atomic formula in negative position, this creates a universal formula in negative position; in effect, this increases the logical complexity of the problem to be solved. In the linear case, use of quantifiers can be avoided, as shown below, and so the logical complexity is not increased. In fact, for reasons associated with space complexity, discussed below, there are good reasons for using some rewritings by means of quantifiers, observing the positivity constraint overall, by a different method.

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2. Background

Posets (partially ordered sets) $(X, \leqslant)$ are *linearly ordered* just when the order relation $\leqslant$ is *linear*, i.e. satisfies all instances of the disjunctive formula $r \leqslant s \ \vee \ s \leqslant r$. A poset is *bounded* when it has elements $0, 1$ satisfying $\forall x.(0 \leqslant x)$ and $\forall x.(x \leqslant 1)$. The notion of *bounded linear order* is similarly defined; the corresponding first-order theory we call **BLO**.

Lattices are posets with (associative and commutative) operations "meet" ($\wedge$) and "join" ($\vee$) satisfying all instances of the equivalences

$$(t \leqslant r \ \wedge \ t \leqslant s) \iff (t \leqslant r{\wedge}s)$$
$$(r \leqslant t \ \wedge \ s \leqslant t) \iff (r{\vee}s \leqslant t).$$

The symbols for meet and join are small, to be distinguished from the larger symbols used for conjunction and disjunction. A lattice is *linearly ordered* (resp. *bounded*) just when the underlying order relation is linear (resp. bounded). Linearly ordered lattices are easily shown to be distributive. **LOL** will denote the theory of linearly ordered lattices.

Heyting algebras are bounded lattices with an "exponentiation" operation, t^s, satisfying all instances of the equivalence

$$r \leqslant t^s \iff r{\wedge}s \leqslant t.$$

Such an algebra is *non-trivial* provided that $1 \leqslant 0$ is false. We deal only with non-trivial algebras henceforth.

A *linearly ordered Heyting algebra* is a Heyting algebra of which the order relation is linear. A *Gödel algebra* is a Heyting algebra satisfying $\forall xy.(1 \leqslant (x^y{\vee}y^x))$. It is known [17] that Gödel algebras — also called "L-algebras" [17], also called "linear Heyting algebras" [20], and also called "relative Stone algebras with 0" [7] — form a variety, consisting of the subdirect products of linearly ordered Heyting algebras. **LOHA** will denote the theory of linearly ordered Heyting algebras.

We now present more formally the syntactical notions implicit in the above description of various theories. Let X be an arbitrary set, the elements of which we call *variables*. The *language* $L(X)$ *of Heyting algebras over* X is constructed as the set of *terms* freely generated by the syntax definition: All variables $x, y, ...$ are terms; the *constants* $0, 1$ are terms; if s, t are terms, then $s{\wedge}t$, $s{\vee}t$ and s^t are terms.

Atoms (i.e. atomic formulae) are just inequalities $s \leqslant t$ between terms. Since equality $s = t$ can be defined as the conjunction of $s \leqslant t$ and $t \leqslant s$, we ignore it. We define the constants *true* and *false* using (e.g.) the inequality $1 \leqslant 1$ and its negation; variations appropriate to the fragment of our language without $0, 1$ are left to the reader.

Atoms of the form $x \leqslant y$, where x, y are constants or variables, are called *simple atoms.*

(Zero-order) formulae A are built up from atoms using the zero-order logical operations $\wedge, \vee, \supset, \neg$. A *$\Pi_1$-formula* is of the form $\forall x_1 \dots x_n.A$ where A is a zero-order formula. More generally, a *positive* formula is a first-order formula built up from atoms and with all its universal (resp. existential) quantifiers occurring only in positive (resp. negative) position.

The method presented below decides validity of such positive formulae, in any of the theories mentioned that include linearity.

A *sequent* is a formal expression $\Gamma \Rightarrow \Delta$ where Γ, Δ are multisets of formulae. It is *simple* iff every formula therein is a simple atom.

3. Bounded linear order

The theory **BLO** of "bounded linear order" is fundamental. For this section, we forget the lattice and Heyting algebra operations, but keep the constants $0, 1$ and the associated conditions. Sequents in this language are, of course, simple. The following definition is taken from the literature, e.g. [2] and [29], adapted to deal with the constants $0, 1$.

Definition 1. *A sequent $\Gamma \Rightarrow \Delta$ is* cyclic *iff there exists a finite sequence $t_1, \dots, t_n$ of terms, such that, for each i, either $t_i \leqslant t_{i+1}$ is in Δ or its* converse *$t_{i+1} \leqslant t_i$ is in Γ, with at least one such inequality in Δ (so $n > 1$), and we have at least one of the following:*

$$t_n \equiv t_1 \qquad t_1 \equiv 0 \qquad t_n \equiv 1.$$

Note that every sequent having one of the following forms is cyclic:

$$\Gamma \Rightarrow \Delta, t \leqslant t \qquad \Gamma \Rightarrow \Delta, 0 \leqslant t \qquad \Gamma \Rightarrow \Delta, t \leqslant 1 \qquad s \leqslant t, \Gamma \Rightarrow \Delta, s \leqslant t.$$

It is known from (e.g.) [29] (except for non-consideration of 0 and 1) that a simple sequent is valid in **BLO** iff it is cyclic. Efficient tests for cyclicity are mentioned in Section 8 below. The terminology "cyclic" is appropriate when 0 and 1 are not used; by abuse of language, we allow it in the more general case.

Lemma 1. *Cyclic sequents are closed under* Cut, *i.e. if $\Gamma \Rightarrow \Delta, s \leqslant t$ and $s \leqslant t, \Gamma' \Rightarrow \Delta'$ are cyclic then so is $\Gamma', \Gamma \Rightarrow \Delta, \Delta'$.*

Proof. Routine: see Theorems 5.1 and 7.2 of [29], where neither 0 nor 1 is considered. (The argument is just that if the cut atom plays an essential role in both cycles, the two cycles are glued together (with omission of the links given by the atom); otherwise, the conclusion $\Gamma', \Gamma \Rightarrow \Delta, \Delta'$ already contains a cycle from at least one premiss.)

It follows that the standard rules [28] of the Gentzen-Ketonen-Kleene sequent calculus **G3cp** for classical propositional logic are valid, invertible and complete for the zero-order theory of bounded linear order, provided that cyclic sequents (rather than those of the form $A, \Gamma \Rightarrow \Delta, A$) are taken as the initial sequents.

4. Non-trivial bounded linear order

As noted above, a bounded linear order is non-trivial when $1 \leqslant 0$ is false. The results of the previous section extend to non-trivial bounded linear orders (the theory **NBLO** provided we change the definition of *cyclic* to read as follows:

Definition 2. *A sequent $\Gamma \Rightarrow \Delta$ is* cyclic *iff there exists a finite sequence $t_1, \ldots, t_n$ of terms, such that (i) for each i, either $t_i \leqslant t_{i+1}$ is in Δ or its* converse *$t_{i+1} \leqslant t_i$ is in Γ and (ii) either at least one such inequality in Δ (so $n > 1$) and we have at least one of the following:*

$$t_n \equiv t_1 \qquad t_1 \equiv 0 \qquad t_n \equiv 1$$

or all inequalities $t_{i+1} \leqslant t_i$ are in Γ, with $t_n \equiv 1$ and $t_1 \equiv 0$.

Since our focus in what follows is on linearly ordered Heyting algebras, this is what we mean by *cyclic* in such a context; the earlier definition should be used for tackling problems in the theory **LOL**.

5. Properties of linearly ordered Heyting algebras

The following fact (especially the equivalence between (1), (5), (6), and (8)) is crucial:

Proposition 1. *In any linearly ordered Heyting algebra, the following are equivalent for all r, s, t:*

1. $r^s \leqslant t$
2. $\forall x.(s \wedge x \leqslant r \supset x \leqslant t)$
3. $\forall x.((s \leqslant r \vee x \leqslant r) \supset x \leqslant t)$
4. $\forall x.((s \leqslant r \supset x \leqslant t) \wedge \forall x.(x \leqslant r \supset x \leqslant t)$
5. $r \leqslant t \wedge (s \leqslant r \supset 1 \leqslant t)$
6. $(r \leqslant t \supset s \leqslant r) \supset 1 \leqslant t$
7. $(\forall x.((r \leqslant x \wedge x \leqslant t) \supset s \leqslant x)) \supset 1 \leqslant t$
8. $\exists x.(((r \leqslant x \wedge x \leqslant t) \supset s \leqslant x) \supset 1 \leqslant t)$

Proof. (1) and (2) are equivalent in any Heyting algebra, since $s \wedge x \leqslant r \iff x \leqslant r^s$. (2) and (3) are equivalent, since, by linearity, $s \wedge x \leqslant r \iff s \leqslant r \;\vee\; x \leqslant r$. (3) and (4) are equivalent, in any Heyting algebra. (4) and (5) are equivalent, since (a), by transitivity of $\leqslant$, $\forall x.(x \leqslant r \supset x \leqslant t) \iff r \leqslant t$ and (b), by $\forall x.x \leqslant 1$, $\forall x.(s \leqslant r \supset x \leqslant t) \iff s \leqslant r \supset 1 \leqslant t$. (5) and (6) are equivalent, using only the properties of bounded linear order, including the fact that $1 \leqslant t \supset r \leqslant t$. (7) implies (6), since the antecedent of (6) easily implies the antecedent of (7). (6) implies (8): just instantiate the bound variable x by r. (8) implies (7), by classical first-order logic. □

6. A rewriting system for linearly ordered Heyting algebras

We first describe the method, for simplicity, in the case of Π_1-formulae. Our method of deciding the validity of such a formula consists (in essence) of stripping off the universal quantifiers and eliminating (by rewriting) the lattice and Heyting operations. We then use the logical rules of the calculus **G3cp** [28] to reduce the problem to simple sequents, of which the cyclicity (and thus validity in **NBLO**) can then be decided. The rewriting steps and the logical steps can, of course, be interleaved.

If universal quantifiers are in the formula to be decided, then, provided they appear only positively, the usual technique of choosing a fresh variable to replace the bound variable suffices. Similarly for negative occurrences of existential quantifiers.

The essence of our method is the following rewriting system, where the *validity* of a rule means the equivalence between its two sides:

1. $t \leqslant 1 \longrightarrow 0 \leqslant 1$
2. $0 \leqslant t \longrightarrow 0 \leqslant 1$
3. $r \leqslant s{\wedge}t \longrightarrow r \leqslant s \ \wedge\ r \leqslant t$
4. $r{\wedge}s \leqslant t \longrightarrow r \leqslant t \ \vee\ s \leqslant t$
5. $r \leqslant s{\vee}t \longrightarrow r \leqslant s \ \vee\ r \leqslant t$
6. $r{\vee}s \leqslant t \longrightarrow r \leqslant t \ \wedge\ s \leqslant t$
7. $r \leqslant t^s \longrightarrow r \leqslant t \ \vee\ s \leqslant t$
8. $r^s \leqslant t \longrightarrow r \leqslant t \ \wedge\ (s \leqslant r \supset 1 \leqslant t)$.

For efficiency reasons, we also use

$$8'.\quad r^s \leqslant t \longrightarrow (r \leqslant t \supset s \leqslant r) \supset 1 \leqslant t.$$

Proposition 2. *1. Rules* 1 *and* 2 *are valid in the theory* **BLO** *of bounded linear order (and in its non-trivial version* **NBLO***).*
2. Rules 3*,* 4*,* 5*, and* 6 *are valid in the theory* **LOL** *of linearly ordered lattices.*
3. Rules 7*,* 8*, and* 8′ *are valid in the theory* **LOHA** *of linearly ordered Heyting algebras.*

Proof. (7) is justified by the equivalence between $r \leqslant t^s$ and $r{\wedge}s \leqslant t$, together with (4). (8) and (8′) are justified by those between (1), (5) and (6) of Proposition 1. Others are routine. □

Zero-order formulae (and even Π_1-formulae) in the language of **LOHA** are thus reduced to zero-order formulae in the language of **NBLO** (non-trivial bounded linear order), then reduced by root-first proof search to a conjunction of simple sequents; these are decided by considering cyclicity.

It is straightforward to combine rewriting with proof search, using the rules of **G3cp**; for example, rewriting by (3) of an atom to a conjunction can be combined with an immediate analysis of the conjunction, the combination being treated as one step. In the discussion below of the complexity of our procedure we assume this is done. For example, an occurrence of an atom $r^s \leqslant t$ leads, by (8′) and (8) respectively, to proof rules that we may represent as

$$\frac{r \leqslant t, \Gamma \Rightarrow \Delta, s \leqslant r \quad 1 \leqslant t, \Gamma \Rightarrow \Delta}{r^s \leqslant t, \Gamma \Rightarrow \Delta}\ LL^* \qquad \frac{\Gamma \Rightarrow \Delta, r \leqslant t \quad s \leqslant r, \Gamma \Rightarrow \Delta, 1 \leqslant t}{\Gamma \Rightarrow \Delta, r^s \leqslant t}\ RL^*.$$

7. Complexity improvements

It is not always necessary to reduce atoms to simple atoms; heuristics to close a branch of the derivation, as soon as, for example, a non-simple atom $s \leqslant s$ appears in the succedent, may easily be added.

More seriously, consider e.g. the rewriting of an atom $r \leqslant s{\wedge}t$ occurring in the antecedent of a sequent; this leads to the sequent calculus rule

$$\frac{r \leqslant s, r \leqslant t, \Gamma \Rightarrow \Delta}{r \leqslant s \wedge t, \Gamma \Rightarrow \Delta}\ LR\wedge$$

where the two occurrences of the term r may lead to derivations of exponential depth. Similar issues arise with all rules except (1) and (2).

The following simple method, approximated by Larchey-Wendling in [22] and Fiorino in [13] (but originating in the work of Hudelmaier [18] or earlier), avoids the problem in this case; in the premiss, replace r (if compound) by a fresh variable x, and add the new atom $r \leqslant x$ to the antecedent:

$$\frac{r \leqslant x, x \leqslant s, x \leqslant t, \Gamma \Rightarrow \Delta}{r \leqslant s \wedge t, \Gamma \Rightarrow \Delta}\ LR\wedge'$$

Similar methods deal with the other rules, the new atom in each case being added to the antecedent. Of course, if the two occurrences are on different branches, nothing needs to be done. In the case just mentioned, where $r \leqslant s \wedge t$ occurs negatively, we are in essence using the easy equivalence between $r \leqslant s \wedge t$ and $\exists x.(r \leqslant x \wedge x \leqslant s \wedge x \leqslant t)$.

Consider in particular the rule LL^* for analysing an antecedent atom $r^s \leqslant t$, i.e. an atom of this form occurring negatively. Using (8) of Proposition 1, when r is compound, we can use instead of LL^* the rule (with x fresh)

$$\frac{r \leqslant x, x \leqslant t, \Gamma \Rightarrow \Delta, s \leqslant x \quad 1 \leqslant t, \Gamma \Rightarrow \Delta}{r^s \leqslant t, \Gamma \Rightarrow \Delta}\ LL^{*\prime}$$

Summarising, then, we give the rules of our calculus in full:

Definition 3. *The calculus* **G3-LOHA** *(including in (7) the six rules just obtained) is as follows:*

1. *Initial sequents are both the cyclic simple sequents and those sequents of the form* $\Gamma \Rightarrow \Delta, t \leqslant 1$ *or* $\Gamma \Rightarrow \Delta, 0 \leqslant t$*;*
2. *The rules for analysing* 0 *and* 1*, as follows:*

$$\frac{\Gamma \Rightarrow \Delta}{0 \leqslant t, \Gamma \Rightarrow \Delta}\ LL0 \qquad \frac{\Gamma \Rightarrow \Delta}{t \leqslant 1, \Gamma \Rightarrow \Delta}\ LR1$$

3. *The rules of* **G3cp** *[35] for zero-order logical operators;*
4. *The rules (in which x is not free in Γ, Δ and A is any formula)*

$$\frac{A, \Gamma \Rightarrow \Delta}{\exists x.A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x.A}\ R\forall$$

for analysis of existential quantifiers in negative position and universal quantifiers in positive position;

5. *Rules for the lattice operators (in which x is any simple term):*

$$\frac{\Gamma \Rightarrow \Delta, r \leqslant s \quad \Gamma \Rightarrow \Delta, r \leqslant t}{\Gamma \Rightarrow \Delta, r \leqslant s \wedge t}\ RR\wedge \qquad \frac{x \leqslant s, x \leqslant t, \Gamma \Rightarrow \Delta}{x \leqslant s \wedge t, \Gamma \Rightarrow \Delta}\ LR\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, r \leqslant x, s \leqslant x}{\Gamma \Rightarrow \Delta, r \wedge s \leqslant x}\ RL\wedge \qquad \frac{r \leqslant t, \Gamma \Rightarrow \Delta \quad s \leqslant t, \Gamma \Rightarrow \Delta}{r \wedge s \leqslant t, \Gamma \Rightarrow \Delta}\ LL\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, x \leqslant s, x \leqslant t}{\Gamma \Rightarrow \Delta, x \leqslant s \vee t}\ RR\vee \qquad \frac{r \leqslant s, \Gamma \Rightarrow \Delta \quad r \leqslant t, \Gamma \Rightarrow \Delta}{r \leqslant s \vee t, \Gamma \Rightarrow \Delta}\ LR\vee$$

$$\frac{\Gamma \Rightarrow \Delta, r \leqslant t \quad \Gamma \Rightarrow \Delta, s \leqslant t}{\Gamma \Rightarrow \Delta, r \vee s \leqslant t}\ RL\vee \qquad \frac{r \leqslant x, s \leqslant x, \Gamma \Rightarrow \Delta}{r \vee s \leqslant x, \Gamma \Rightarrow \Delta}\ LL\vee$$

6. *Rules for the exponentiation operator (in which x is any simple term):*

$$\frac{\Gamma \Rightarrow \Delta, r \leqslant x, s \leqslant x}{\Gamma \Rightarrow \Delta, r \leqslant x^s}\ RR^* \qquad \frac{\Gamma \Rightarrow \Delta, r \leqslant t \quad s \leqslant r, \Gamma \Rightarrow \Delta, 1 \leqslant t}{\Gamma \Rightarrow \Delta, r^s \leqslant t}\ RL^*$$

$$\frac{r \leqslant t, \Gamma \Rightarrow \Delta \quad s \leqslant t, \Gamma \Rightarrow \Delta}{r \leqslant t^s, \Gamma \Rightarrow \Delta}\ LR^* \qquad \frac{x \leqslant t, \Gamma \Rightarrow \Delta, s \leqslant x \quad 1 \leqslant t, \Gamma \Rightarrow \Delta}{x^s \leqslant t, \Gamma \Rightarrow \Delta}\ LL^*$$

7. *Rules that replace the ordinary rules when the duplicated term (r, r, t, t, t, r respectively) is compound. In each case x is a variable and must be* fresh*, i.e. not free in r, s, t, Γ, Δ.*

$$\frac{r \leqslant x, x \leqslant s, x \leqslant t, \Gamma \Rightarrow \Delta}{r \leqslant s{\wedge}t, \Gamma \Rightarrow \Delta}\ LR{\wedge}' \qquad \frac{x \leqslant r, \Gamma \Rightarrow \Delta, x \leqslant s, x \leqslant t}{\Gamma \Rightarrow \Delta, r \leqslant s{\vee}t}\ RR{\vee}'$$

$$\frac{t \leqslant x, \Gamma \Rightarrow \Delta, r \leqslant x, s \leqslant x}{\Gamma \Rightarrow \Delta, r{\wedge}s \leqslant t}\ RL{\wedge}' \qquad \frac{x \leqslant t, r \leqslant x, s \leqslant x, \Gamma \Rightarrow \Delta}{r{\vee}s \leqslant t, \Gamma \Rightarrow \Delta}\ LL{\vee}'$$

$$\frac{t \leqslant x, \Gamma \Rightarrow \Delta, r \leqslant x, s \leqslant x}{\Gamma \Rightarrow \Delta, r \leqslant t^s}\ RR^{*\prime} \qquad \frac{r \leqslant x, x \leqslant t, \Gamma \Rightarrow \Delta, s \leqslant x \quad 1 \leqslant t, \Gamma \Rightarrow \Delta}{r^s \leqslant t, \Gamma \Rightarrow \Delta}\ LL^{*\prime}$$

Definition 4 (Size of a term, of an atom, of a formula, and of a sequent).

1. *The* size $\| r \|$ *of a term r is defined as the number of operations in r; thus $\| 0 \| = \| 1 \| = \| x \| = 0$, $\| r{\wedge}s \| = \| r{\vee}s \| = \| r^s \| = 1 + \| r \| + \| s \|$.*
2. *The* size $\| r \leqslant s \|$ *of an atom $r \leqslant s$ is defined as $\| r \| + \| s \|$.*
3. *The* size $\| A \|$ *of a formula A is defined as the sum of the sizes of its atoms and the number of logical connectives, each counted with its appropriate multiplicity.*
4. *The* size $\| \Gamma \Rightarrow \Delta \|$ *of a sequent $\Gamma \Rightarrow \Delta$ is defined as the sum of the sizes of the formulae in Γ, Δ, each counted with its appropriate multiplicity.*

It follows that simple sequents are those with size 0. Note that, in contrast to [10], [13], and [18], no special weighting of the operators is involved. The logical operators in those papers correspond to our algebraic operators; the latter just have weight 1.

Proposition 3. *Derivations in the calculus* **G3-LOHA** *have depth bounded by the size of the end-sequent.*

Proof. For each rule, the size of each premiss is less than the size of the conclusion. □

Theorem 1. *The calculus* **G3-LOHA** *is complete for positively quantified formulae of the theory* **LOHA***: if such a formula A is valid in that theory, then the sequent $\Rightarrow A$ is derivable.*

Proof. By Proposition 3, root-first search for a derivation of the sequent terminates, and each step replaces a sequent by zero or more sequents whose conjunction (augmented with appropriate quantifiers) is equivalent to the replaced sequent. The result then follows by the characterisation of cyclic sequents as those valid in the theory **BLO**. □

Each simple sequent constructed by the above method involves at most n variables, where n is the the number of variables in the original sequent, plus its size, since each rewriting step or logical step (maybe combined as one step) reduces the sequent size by 1 and adds at most one new variable. (Of course, each such step may replace one sequent by two; but they are on different branches.)

Similarly, the number of atoms in any simple sequent so constructed from a sequent of size m and with k formulae is at most $2m + k$, since each rewriting step (combined if appropriate with a logical step or two) replaces one atom by at most three and each purely logical step replaces one formula by at most two.

This may be compared with rules of [22] that use logical equivalences similar to those expressed by our rules; these begin operating on a sequent (constructed by some pre-processing steps) at most about 5 times the size of the original sequent, and are subject to similar remarks about the number of steps in a branch being bounded by the size (measured after the pre-processing) of the sequent; cf. also [13].

8. Cyclicity testing

This paper proposes no new method for testing cyclicity; it is straightforward to adapt the ideas of Larchey-Wendling [22] and [23] to implement the test for cyclicity of a simple sequent in time linear in the length (i.e. the number of atoms) of the sequent. The method outlined in [22] for reducing a sequent to a simple sequent adds many new variables, and corresponding atoms, one new variable for each compound subformula (in our notation, subterm) of the original sequent. However, this is to simplify the construction of a single structure in which the cyclicity of many sequents can be checked simultaneously; it is not yet clear whether this single structure can be generalised to the algebraic setting.

9. Relation to Gödel-Dummett logic

For background on Gödel-Dummett logic, see [22]; it is intuitionistic logic extended by linearity axioms, i.e. formulae of the form $(A \supset B) \vee (B \supset A)$. As noted in [33], it has its origins in the neglected work of Skolem, who by 1919 had defined and studied those several kinds of algebra that later came to be known as Heyting algebras, as Gödel algebras, and (respectively) as linearly ordered Heyting algebras. The Lindenbaum algebra of any consistent zero-order theory in this logic is, as shown by Horn [17], a Gödel algebra, i.e. a subdirect product of linearly ordered Heyting algebras.

Formulae A of this logic are routinely interpreted as terms $h(A)$ of Heyting algebra; e.g. $h(A \supset B) = h(B)^{h(A)}$. The problem of deciding a formula A is then replaced by the problem of deciding the atom $1 \leqslant h(A)$ in the theory of Gödel algebras. As that is (in effect) an equation, it suffices to decide the atom in the theory **LOHA**, since, by universal algebra, the same equations are valid in both theories. (This is not true for arbitrary formulae, the linearity axioms being obvious examples.)

Early work on decision procedures for this logic (using a tableau calculus) is by Corsi [9]. Key contributions were made by Avron & Konikowska [2], who first identified the equivalence expressed by our rewriting rule (8). When used for an antecedent occurrence, the logical version LL^* of this rule goes back to (at least) [10]; it is the (non-invertible) rule $\supset\Rightarrow_4^{**}$ for a terminating multi-succedent calculus for intuitionistic logic. Its invertibility in Gödel-Dummett logic was noticed and exploited in [1] and [11].

Avron & Konikowska's methodology [2] of "Rasiowa-Sikorski decomposition systems" is here expressed in algebraic terms; but we have adopted nothing analogous to their variations employing hyper-sequents.

Larchey-Wendling, in [22] and [23], presents details of the state-of-the-art implementation, including sophisticated methods for cyclicity testing, using graph theory and binary decision diagrams. His methods for eliminating compound formulae are similar to some of ours (for terms), but we are unable to match them up in detail.

Baaz, Fermueller & Ciabattoni [4] present a "Sequent of relations calculus"; this has the same idea of reduction of complex terms by invertible operations to logical combinations of simpler terms. Our atoms correspond to their sequents; our disjunctions of atoms correspond to their hypersequents, with our antecedent simple atoms $x \leqslant y$ moved to the succedent in the form $y < x$. We find it simpler to deal with a single relation $\leqslant$.

Fiorino [13] presents, for Gödel-Dummett logic, tableaux calculi that suffer, like the calculi of [1], [9] and [11], from unnecessary duplication of effort in deciding the logical equivalent of our simple sequents; they do however have evident linear bounds on search depth.

10. Other related work

Gentzen-style proof systems for the theories of partial order, of linear order, and of lattice theory have been developed in [29] and [30] exploiting a variety of techniques, including a systematic method, expounded in [28], for conversion of universal axioms to inference rules. Ideas [26] based on this method led to the genesis of the present paper; various refinements designed to ensure efficient root-first proof search and simplifications based on the symmetry between left and right of the sequent arrow have disguised the application of this technique.

Jipsen [20] gives a Gentzen calculus for residuated lattices, with a completeness theorem based on non-commutative phase spaces; the underlying algorithm is attributed to Okada & Terui [31] and the calculus is attributed to Ono & Komori [32]. The idea there however is to represent an ordering $s \leqslant t$ between two terms s and t as a sequent $s \Rightarrow t$, and thus is (quite apart from the extra emphasis on semantic methods) rather different from our approach.

Other Gentzen-style methods for ortholattices, showing decidability of the word problem in the theory of free ortholattices, have been developed by Tamura [34], based on earlier work on lattice theory by Matsumoto. These, and Jipsen's work, differ from our work in that only atoms $s \leqslant t$, rather than formulae, are decided.

There is (we believe) no connection with Haiman's work [16] on proof theory of "linear lattices"; these are called "linear" despite not being linearly ordered.

It is of interest to speculate on how second-order propositional quantifiers of Gödel-Dummett logic (which are known to be eliminable [5]) can be interpreted, presumably as first-order quantifiers, and to what extent and by what means such first-order quantifiers are also eliminable.

Without the linearity condition, one has a corresponding problem for the theory **HA** of Heyting algebras. The reduction rules in Proposition 2 are now valid only in part; specifically, (1), (2), (3), and (6) are valid in **HA**, (4) can be replaced for positive occurrences by

$$r{\wedge}s \leqslant t \longrightarrow \forall x.((x \leqslant r \,\wedge\, x \leqslant s) \supset x \leqslant t),$$

(5) can be replaced for positive occurrences by

$$r \leqslant s{\vee}t \longrightarrow \forall x.((s \leqslant x \,\wedge\, t \leqslant x) \supset r \leqslant x).$$

(7) can be replaced by $r \leqslant t^s \longrightarrow r{\wedge}s \leqslant t$, and (8) can be replaced, for positive occurrences, by

$$r^s \leqslant t \longrightarrow \forall x.(x{\wedge}s \leqslant r \supset x \leqslant t).$$

But, it is an open problem to give appropriate rules for simplifying negative occurrences of $r{\wedge}s \leqslant t$, of $r \leqslant s{\vee}t$ or of $r^s \leqslant t$. Since it would suggest a simple algorithm, without backtracking, for a *PSPACE*-hard decision problem, a solution to this problem would be of general interest.

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