GÖDEL ON TRUTH AND PROOF:
Epistemological Proof of Gödel's Conception of the Realistic Nature of Mathematical Theories and the Impossibility of Proving Their Incompleteness Formally

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> No calculus can decide a philosophical problem. A calculus cannot give us information about the foundations of mathematics. (Wittgenstein, 1933-34: 296)

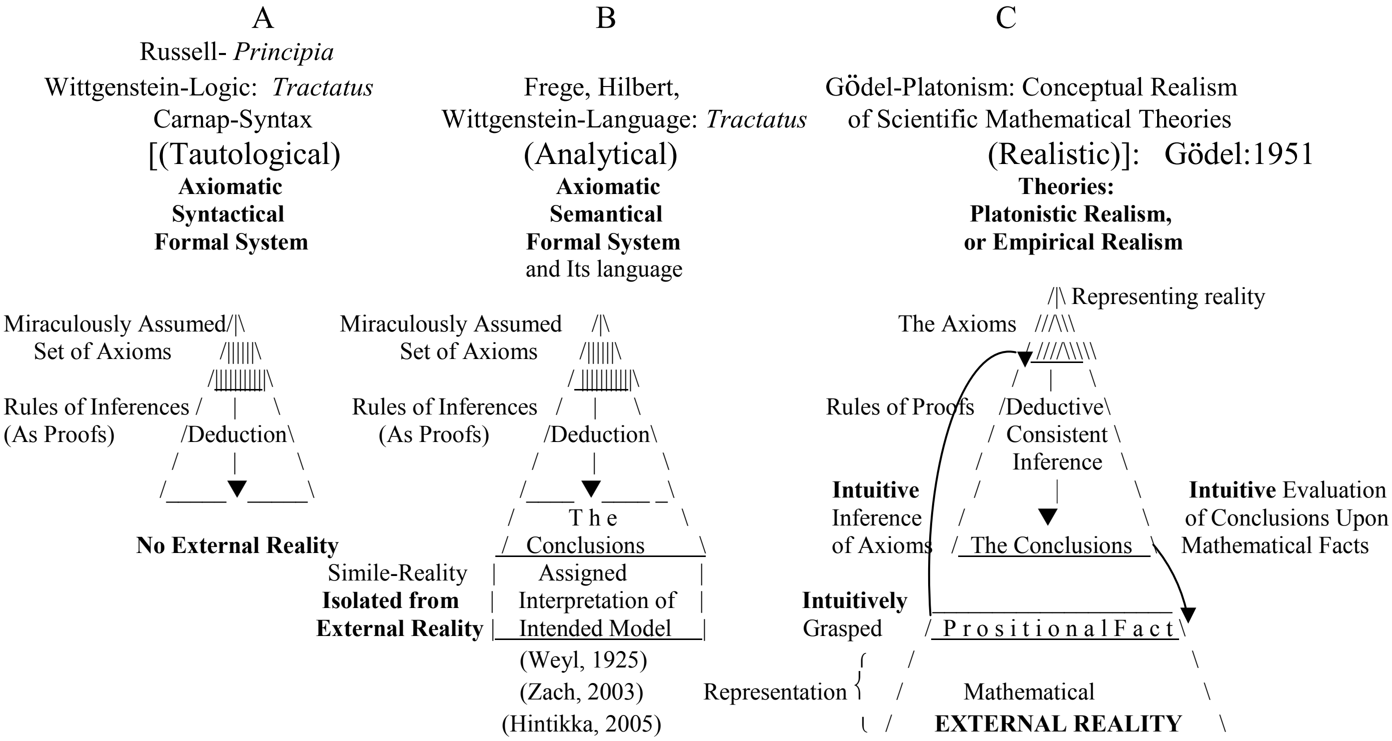
1. Introduction: Pragmaticist Epistemological Proof of Gödel's Insight of the Realistic Nature of Mathematical Theories and the Impossibility of Proving Their Incompleteness Formally

In this article, I attempt a pragmaticist epistemological proof of Gödel's conception of the realistic nature of mathematical theories representing facts of their external reality. Gödel generated a realistic revolution in the foundations of mathematics by attempting to prove formally the distinction between complete formal systems and incomplete mathematical theories. According to Gödel's Platonism, mathematical reality consists of eternal true ideal facts that we can grasp with our mathematical intuition, an analogue of our sensual perception of physical facts. Moreover, mathematical facts force us to accept intuitively mathematical true axioms, which are analogues of physical laws of nature, and through such intuition we evaluate the inferred theorems upon newly grasped mathematical facts. However, grasping ideal abstractions by means of such mysterious pure intuitions is beyond human cognitive capacity. Employing pragmaticist epistemology, I will show that formal systems are only *radical abstractions* of human cognitive operations and therefore cannot explain how we represent external reality. Moreover, in formal systems we cannot prove the truth of their axioms but only assume it dogmatically, and their inferred theorems are logically isolated from external reality. Therefore, if Gödel's incompleteness of mathematical theories holds, then we cannot know the truth of the basic mathematical facts of reality by means of any formal proofs. Hence Gödel's formal proof of the incompleteness of mathematics cannot hold since the truth of basic facts of mathematical reality cannot be proved formally and thus his unprovable theorem cannot be true. However, Gödel separates the *truth* of mathematical facts from mathematical *proof* by assuming that mathematical facts are eternally true and thus, the unprovable theorem seems to be true. Pragmatistically, realistic theories represent external reality, not by formal logic and not the abstract reality, but by the *epistemic logic* of the complete proof of our perceptual propositions of facts and realistic theories. Accordingly, it can be explained how all our knowledge starts from our perceptual confrontation with reality without assuming any *a priori* or "given" knowledge. Hence, mathematics is also an empirical science; however, its represented reality is neither that of *ideal objects* nor that of *physical objects* but our operations of counting and measuring physical objects which we perceptually quasi-prove true as mathematical basic facts (Nesher, 2002: V, X).

2. Gödel's Platonism and the Conception of Mathematical Reality with Its True Conceptual Facts

Gödel's basic insight of the realistic nature of mathematics that it is a science represents mathematical reality and not just a conventional formal system. Yet, Gödel's Platonist mathematics is an abstract science representing ideal true mathematical reality though analogical to the empirical sciences (Gödel, 1944). As a *metaphysical realist*, Gödel separates the mathematical reality of abstract true facts from formal proofs, and it is only by pure intuition that we can grasp these facts. Figure 1 presents a schema of Gödel's different conceptions of logic and mathematics:

[1] The Gödelian Epistemology of Three Conceptions of Logic and Mathematics:



Gödel's tri-partitions are between (A) *Complete* Analytic Formal Systems with their formal syntactic *tautologies*, (B) *Complete* Formal Semantic *analyses*, and (C) the *Incomplete* Realistic Theories of *conceptual* mathematics (Gödel, 1951: 319-323; Poincaré, 1902: Chap. I).

> The two significations of the term *analytic* might perhaps be distinguished as tautological and analytic (Gödel, 1944:139, n. 46).

Epistemologically the *tautological* and *analytic* of *complete formal systems* are, respectively, ***syntactically*** **closed** upon their fixed axioms and formal rules of inference and ***semantically closed*** upon axioms, formal rules, and the assigned model. The *realist incomplete theory* is only ***relatively closed*** upon its relative proof-conditions, the formal proofs, the operations of pure intuition, and conceptual facts of external reality (Nesher, 2002: X). Since Gödel's mathematical theories are regarded as axiomatic formal systems with formal inferences, yet their external reality can be grasped only by pure intuition (Gödel, 1931a: 203, 1964: 268).

For Gödel, pure mathematical intuition has three functions: (1) to grasp the true ideal mathematical facts of mathematical reality, (2) to enforce by these ideal facts to accept the true axioms of mathematical theories in order to infer the theorems formally, and (3) to evaluate how the theorems represent truly facts of mathematical reality (Gödel, 1953-54?: fn. 34; Nesher, 2001a, 2010). Gödel's conception of mathematical intuition is based on his mathematical experience, which he calls the "psychological fact of the existence of an intuition," but as a "given" without any explanation.

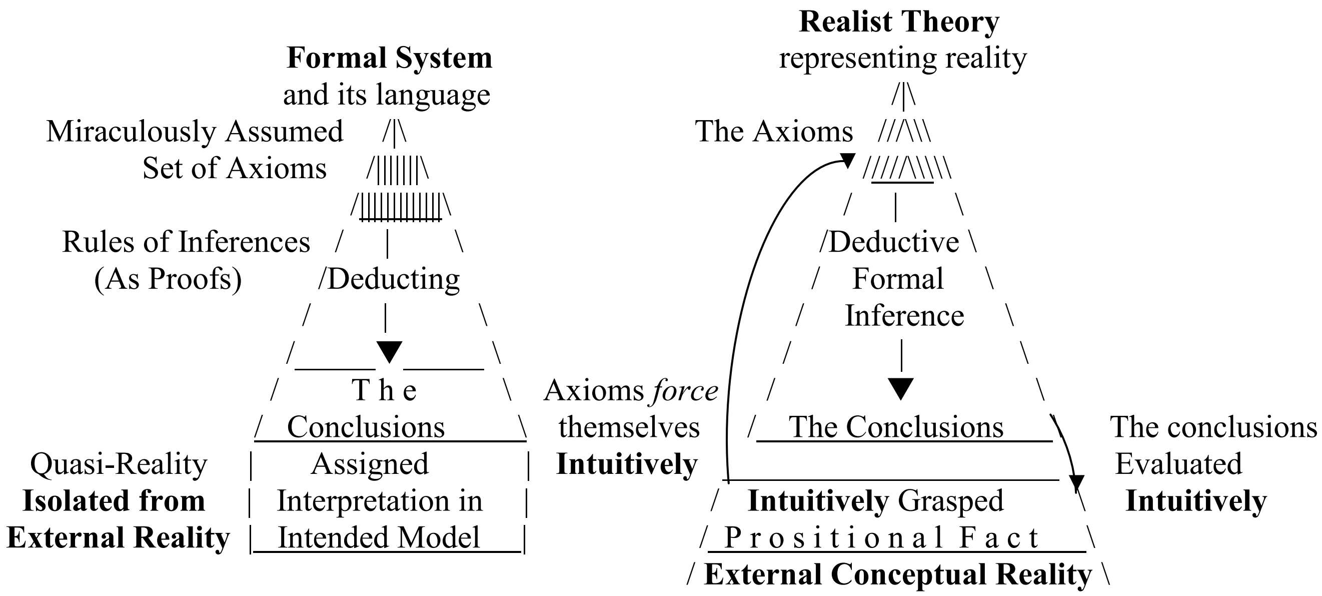
> However, the question of the objects of mathematical intuition (which incidentally is a replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis (Gödel, 1964: 268).

How with mathematical intuition we grasp pure meanings of mathematical propositions is the essential problem to the possibility of Gödel's conceptual realism (Gödel, 164:268).

3. Gödel's Incomplete Distinction between Formal Systems and Realistic Theories

Gödel revolutionized the conception of the nature of mathematics through his distinction between complete logical formal systems and incomplete mathematical theories (Gödel, 1931:195, 1964). However, he did not conclude this revolution, because of his acceptance the formalist methods of mathematical proofs and the subjective conception of pure intuition owing to his Platonist realism that motivated this revolution (Gödel, 1931:#1).

[2] **Epistemological Gap between Logical Formal Systems and Mathematical Theories**



The difference between formal system and realist theory lies in their proof-conditions when the formal system is by definition **hermetically closed** upon its fixed formal proof-conditions without relation to external reality; the mathematical realistic theory is **relatively closed** upon its proof-conditions: the mathematical facts of external reality, the formal inferences, and the pure intuitions that complete the representation of reality, while the axioms change by our continually grasping new mathematical facts. Yet, the formal systems are *artificially* abstracted from human mathematical operations and cannot explain them, and thus they can never be "ideal machines" by lacking any human cognitive self-consciousness and self-controlled operations upon reality (Gödel, 1931: 195 & n. 70; 1951: 310; Feferman, 2006; Putnam, 2011; Penrose, 2011). Apparently Gödel did not completely conceive his epistemological revolution of the realistic nature of mathematics and considered the three classes of logico-mathematics**,** A, B, and C, as formal systems, while neglecting the essential distinction between formal systems and mathematical theories.

> The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules. The most comprehensive formal systems that have been set up hitherto are the system of *Principia mathematica* (*PM*) on the one hand and the Zermelo-Frankel axiom system of set theory . . . on the other. These two systems are so comprehensive that in them all methods of proof today used in mathematics are formalized, that is, reduced to a few axioms and rules of inference. (Gödel, 1931: #1; cf. 1931a; Kleene, 1967: 253)

Gödel's incompleteness theorem essentially shows that *PM* and ZF are mathematical theories, not formal systems; however, since they use formal inferences, then without the help of mathematicians' conceptual intuition, those systems are isolated from mathematical reality. According to pragmaticist epistemology, the formal inference is only one component of the epistemic logic which includes also the Abductive and Inductive material inferences of the complete proof enable also to prove the basic mathematical facts of external reality. Yet, even after proving the incompleteness of mathematics, Gödel still oscillated between mathematics as axiomatic formal systems and as scientific theories, and thus he could not complete his realistic revolution of mathematics (Gödel, 1953-54? II; Feferman, 1984: 9-11).

4. Gödel's Paradoxical Formal Proof of Incompleteness, Based on Separating Truth from Its Proof

If Gödel's incompleteness holds, then mathematics is theory and a not formal system so, can Gödel prove formally his incompleteness in mathematical theory that cannot prove formally true theorems (Hintikka, 2000: V)? Gödel's formal proof of incompleteness is actually an "arithmetization of syntax," which attempts to prove his epistemological conception of the nature of mathematics. But Gödel's incompleteness is a general claim that can be proved only epistemologically, and not through any specific theory about itself. It could be that in respect of a special mathematical theorem it can be prove that a specific theory (e.g., PM or ZF) is incomplete in respect to specific propositions and the given true mathematical facts; but it cannot provide a general proof of the nature of mathematics (Gödel, 1944:121).

Gödel arithmeticized the proof of the undecided proposition G1: "I am unprovable," by means of a *metamathematical description* in order to prove this unprovable mathematical proposition, "We therefore have before us a proposition that says about itself that it is not provable [in *PM*]" (Gödel, 1931: 151). The question is whether this formal proof can be considered proof of G1: "I am unprovable"? There are two problems here: (1) Can at all there be metalanguages, since meta-descriptions of mathematical languages can , at most, describe physical-syntactical signs, following Tarski, and not their meaning-contents, which we can only interpret, yet not in abstract models but in respect to experience (Wittgenstein, 1921, 1933-34: II.12; Gödel, 1953-54?: fn.34, p.203: Nesher, 1987, 2002: V)? (2) Can G1 be meaningful and "contentually true" that eventually represents a mathematical true fact (Gödel, 1931a: 203)?

If G1: "I am unprovable" is proved formally *true* in *PM*, then its claim of being unprovable is *false* because it was proved true [in *PM*] and cannot be unprovable, but when G1 is *false* then being unprovable in *PM* is *true* as it claims, and thus presenting a paradox like the liar paradox, and Gödel's trick of using a kind of paradoxical argument fails.

> The analogy of this argument with the Richard antinomy leaps to the eye. It is closely related to the "Liar" too.[14] (Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions). (Gödel, 1931: 149)

Since any epistemological antinomy is void of truth, this means that its proof is also void of truth. It seems that Gödel felt this difficulty, and his way out of this paradoxical situation is to locate the *proof* at the metamathematical arithmetical language and thus separate this formal proof from the language of G1 with the assuming truth of its bizarre meaning.

> From the remark that [R(q);q] says about itself that it is not provable, it follow at once that [R(q);q] is true, for [R(q);q] is indeed unprovable (being undecidable). Thus, the proposition that is undecidable *in the system PM* still was decided by metamathematical considerations.
> (Gödel, 1931: 151)

Why did Gödel take recourse in this "epistemological antinomy" as a trick and not proving the incompleteness of *PM* by showing that propositions "of the type of Goldbach or Fermat" are unprovable in it (Gödel1931a: 203)? It seems that Gödel intended a general proof of the nature of all mathematical theories in respect of their infinite mathematical reality (Agazzi, 1974: 24). Gödel's Platonist realism leads him to formulate his proof with the suffix *able* as his "provable" and "unprovable" terms. This means that since there are eternal and infinite true mathematical facts that eventually can be grasped by pure intuition, they are either provable or unprovable in any mathematical theory (Hintikka, 2000:29). In such Platonic epistemology, *truth* in reality and *proof* in theories are *separated*, which enables Gödel to separate the *proof* of G1 from the *truth* of the mathematical fact it is to represent, in order to avoid the paradox in proving his incomplete theorem of being "closely related to the 'Liar.'"

> Finally it should be noted that the heuristic principle of my construction of undecidable number theoretical propositions in the formal systems of mathematics is the highly transfinite concept of 'objective mathematical truth' as *opposed* to that of 'demonstrability' . . . , with which it was generally confused before my own and Tarski's work (Gödel in a letter to Wang, Dec. 7, 1967, in Wang, 1974: 9; Feferman, 1984: 106-107; Franzèn, 2005: 2.4).

Hence, Gödel leans on the distinction between the liar proposition P^L: "I am lying" and the unprovable proposition P^U: "I am unprovable" since in the former we reach the liar paradox that if it is true then it is false and vice versa, whereas there is no such paradox of truth and falsity in the latter, since proof and truth are separated (Gödel, 1934 #7, 1951: 322-323; Hintikka, 2000:35-36; Devlin, 2002).

> So we can see that the class α of numbers of true formulas cannot be expressed by the propositional function of our system, whereas the class β of provable formulas can. Hence $\alpha \neq \beta$ and if we assume $\beta \subseteq \alpha$ (i.e., every provable formula is true) we have $\beta \subset \alpha$, i.e., there is a proposition A which is true but not provable. ~A then is not true and therefore not provable either, i.e., A is undecidable (Gödel, 1934: 363).

Generally, Gödel separates the truth of mathematical facts, which can be grasped intuitively, from the formal *proof* of propositions in mathematical theories and thus also, he can separate the attempted formal *proof* of G1 from its seemingly representing the *truth* of a fact in the mathematical reality of *PM*. Leaning on his Platonistic realism he could do it in order to avoid the possibility that G1 would be both true and false like the Tarskian liar proposition.

> Thus if truth for number theory *were* definable within itself, one could find a precise version of the liar statement, giving a contradiction. It follows that truth is not so definable. But provability in the system *is* definable, so the notions of provability and truth must be distinct. In particular if all provable sentences are true, there must be true non-provable sentences. The self-referential construction applied to provability (which is definable) instead of truth, then leads to a specific example of an undecidable sentence (Feferman, 1984: 106).

However, if the notions of truth and proof are not separated there are no "true non-provable sentences" and "the self-referential construction" of G1 leads to an "epistemological antinomy," a kind of the liar paradox. Metaphysical realists, such as Platonists and formal semanticists (e.g., Tarski), assume that truth is independent of proof and, by the bivalence of truth values, the principle of excluded middle, identify truth with reality, yet, not for complete formal systems (Gödel, 1929: 63; Penrose, 2011: 342-343).

Pragmaticists, however, show that for humans the truth and falsity of propositions consist only of that which we have already proved as such, since we cannot know truth from a Godly perspective (Nesher, 2002: V). Since there is no separation between truth and being proved, then we have to drop the expressions "provable" and "unprovable" from our epistemology. This terminology belongs to Metaphysical Realism, such as Gödel's Conceptual Realism, Popper's absolute truth, among others, in distinction from Pragmaticist Representational Realism (Nesher, 2002: III, V, VIII).

Therefore, without being proved true or false, propositions remain doubtful, and since no one has proved the truth or the falsity of the *liar* proposition, it is doubtful and there cannot be any paradox (Nesher, 2002: V). Hence the separation of truth from proof is epistemologically untenable and so also the separation between the liar paradox and the unprovable-provable antinomy, and thus, with the doubtful *unprovable* proposition we cannot prove anything (Hintikka, 2000:31-35). Although Wittgenstein sensed the paradoxical difficulty in Gödel's alleged proof of incompleteness, he could not explain it without having an epistemology of truth (Wittgenstein, 1937; Nesher, 1992; Floyd and Putnam, 2000; Floyd, 2001; Berto, 2009: # 9).

> How can Gödel prove that his crucial proposition is not logically provable by using the very same logic? And how we can know that the proposition in question is true if we cannot prove it? (Hintikka, 2000:29)

What, then, is the meaning of G1 if it were proved to represent a conceptual true fact in mathematical reality? And can we specify this true fact that the alleged meaning-content of G1 represents? Indeed, there is no mathematical fact that G1 represents, since it is not a proposition with real subject matter and clear content and if anything at all, it has only a shadowing meaning (Gödel, 1931a: 203; Weyl, 1949: 51; Feferman, 1984: 106). However, if G1: "G1 is unprovable" is void of real meaning and thus cannot be "contentually true" then it cannot represent any intended "mathematical objects or facts exist," according to Gödel's criticism of the syntactic conception of mathematics (Gödel, 1931a: 203, Gödel, 1953-54?: #30; Agazzi, 1974: 24; Feferman, 1984: 103). Hence the arithmeticized proof of G1 is only mechanically connected to the object language and has nothing to do with its meaning (Tarski, 1944; Nesher, 1987, 2002: V; Floyd, 2001: III). Then if G1 can be proved formally, any sentence can be proved emptily and the system or theory in which it is proved is inconsistent (Gödel 1931a: 203).

> This formulation of the non-feasibility of the syntactic program (which also applies to finitary mathematics) is particularly well suited for elucidating the question as to whether mathematics is void of content [in the sense that no mathematical objects or facts exist]. For, if *prima facie* content of mathematics were only a wrong appearance, it would have to be possible to build up mathematics

satisfactorily without making use of this "pseudo" content. (Gödel, 1953-54?: #30; Hintikka, 2000: 29)

However, the meaning-contents of scientific theories are based on our experiential confrontation with external reality and mathematical reality, as well. Thus, the basic facts of mathematical reality cannot be proved formally in theory from its axioms and the question is how we prove their truths and whether we can grasp their truths by pure mathematical intuition (Gödel, 1944: 21).

> It is turned out that (under the assumption that modern mathematics is consistent) the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic, i.e., the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception. (Gödel, 1944: 121; cf. Gödel, 1953: #34)

This Gödel insight fits the pragmaticist understanding of the role of epistemic logic proofs in all empirical sciences, mathematics included (Gödel, 1947: 182-183, 1964: 268-269; Nesher, 2002, 2007; Chihara, 1982). The central problem in the epistemology of mathematical theories concerns an explanation of mathematical reality: What is it and how do we prove the propositional facts of mathematics (Kitcher, 1984; Nesher, 2002: X)? Since this reality cannot be known by any axiomatic mathematical theory, there may be other methods to know it, such as Gödel's mathematical intuition grasping mathematical true facts, or rather the epistemic logic we operate to quasi-prove the truth of our perceptual judgments representing mathematical reality (Agazzi, 1974: 24).

> (Assuming the consistency of classical mathematics) one can even give examples of propositions (and in fact of those type of Goldbach or Fermat) that, while contentually true, are unprovable in the formal system of classical mathematics. Therefore, if one adjoins the negation of such a proposition to the axioms of classical mathematics, one obtains a consistent system in which a contentually false proposition is provable. . . . (Gödel 1931a: 203).

The discrepancy between Gödel's intuition about the realistic nature of mathematics and his attempt to prove propositional facts formally can be resolved by the Peircean epistemic logic of complete proofs. Through it, we can prove the truth of the basic propositional facts of mathematics, discover hypothetical axioms, and evaluate their truth upon the true facts of mathematical reality.

The question is, why nevertheless did Gödel's formal proof of the incompleteness of mathematical theories were accepted almost without questioning the problematic "epistemological antinomy?" It may be that the generation of Frege and Hilbert, and the next one, were captivated by the deductivist-formalist agenda and the analytic formal semantic epistemology with the metalanguages hierarchies, which could not seriously reevaluate this proof (Dawson, 1984). Since the realistic conception of mathematics expresses mathematicians' intuition about their work, then what Gödel offered about the incompleteness of mathematical theories is accepted naturally: i.e., that there are "contentually true" propositions in the language of theory that cannot be proved except by extended axiomatic theories (Hintikka, 2000: V).

5. The Pragmaticist Epistemology of Cognitive Empirical Representations of External Reality

The deviation of formal systems from human working with mathematical theories can be explained by suggesting that *formal systems are only realistic theories in disguise* or *utopian; i.e., impossibly "ideal machines*" of different degrees (Dawson, 1984:79; Nesher, 2001b).

> By the turn of this century mathematics, 'the paradigm of certainty and truth', seemed to be the real stronghold of orthodox Euclideans. But there are certainly some flaws in the Euclidean organization even of mathematics, and these flaws caused considerable unrest. Thus the central problem of all foundational schools was: 'to establish once and for all the certitude of mathematical methods'. [1] ([1] Hilbert, 1925). However, foundational studies unexpectedly led to the conclusion that a Euclidean reorganization of mathematics as a whole may be impossible; that at least the richest mathematical theories were, like scientific theories, quasi-empirical. Euclideanism suffered a defeat in its very stronghold (Lakatos, 1978: 30).

The formal systems with their formal proofs, though aiming to increase the power of formal computations, yet as far as they estranged from human cognitive operations representing reality their efficiency is decreased. The advantage of human cognitive operations lies in its having self-consciousness and self-control in confronting the mathematical, physical, and other realities, which enable correcting errors and evolving human knowledge (Gödel, 1972a: 305-6; Nesher, 1990, 1999; Hintikka, 1997: 5.7, 2000: X; Putnam, 2011: 15.4). In this perspective, we can understand the epistemology of the "Exact Sciences," the issue of the Königsberg Conference in September 1930, in which Gödel announced his discovery of incompleteness; namely, that even mathematics is not pure science and is only relatively exact (Nesher, 2002: X).

> . . . as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. (Einstein, 1921)

Gödel's incompleteness theorem is about the relativity of any mathematical theory in respect to its proof-conditions in representing mathematical reality.

> There is in fact in the light of hindsight a major puzzle about Gödel's insights and about the way he put them to use. One of his greatest achievements, arguably the greatest one, was to show the deductive incompleteness of elementary arithmetic. (Hintikka, 2005: 536)

Hintikka obscures the issue that the incompleteness of any scientific theory, including elementary arithmetic, is due not only to the incompleteness of formal deductive inferences; scientific theories with their complete epistemic logical proofs are also incomplete and are true only upon their specific proof-conditions and therefore, they are incomplete in respect to reality we endeavor to represent. Since all our knowledge of reality is based on perceptual experience in confrontation with reality, so also is our mathematical experience in confrontation with its reality, which cannot comprise Platonist abstract objects. The distinction between completeness of axiomatic formal systems and the incompleteness of mathematical and other scientific theories is not logical but, rather, epistemological and can be proved with pragmaticist epistemic logic (Nesher, 2002, 2007; Wittgenstein, 1933-34: 296).

> The nontriviality of the proof of completeness for limpid logic must be forcefully presented the possibility to Platonist Gödel that there were propositions that are *arithmetically* true but not provable within a formal system of arithmetic. (Goldstein, 2005: 154)

Thus, Gödel's "evident without proof" of true propositions that were not proved in specific formal systems, illustrates that cognitive confrontation with external reality cannot be formalized. According to Gödel the basic true mathematical facts can be grasped intuitively and from them the axioms are intuitively accepted as true without proofs.

> Of course, the task of axiomatizing mathematics proper differs from the usual conception of axiomatics insofar as the axioms are not arbitrary, but must be correct mathematical propositions, and moreover, evident without proof. There is no escaping the necessity of assuming some axioms or rules of inference as evident without proof, because the proofs must have some start point. (Gödel, 1951: 305).

However, since there is no human truths without proofs this can be undertaken only by quasi-proofs of basic perceptual judgments representing reality in complete epistemic logic, the trio sequence of the material logical inference of Abductive discovery, the Deductive necessary inference and the material inference of Inductive evaluation (Nesher, 2002: V, X). Hence, the impossibility of proving formally in metamathematics the theorem of unprovability is also due to the impossibility of proving formally the truth of propositional facts of external mathematical reality, "because the proofs must have some start point" and their proved truth is the "start point." This is hinted by Russell about the empirical assumptions of mathematics, and so Gödel, too, cannot prove G1 formally in an incomplete mathematical theory (Russell, 1914; Nesher, 2002: V). With the cognitive epistemic logic, we start from the quasi-proof of the basic perceptual facts of our knowledge of reality without any miraculous "given." Thus, we can discard the transcendental *a priorism* while all our knowledge is empirical (Nesher, 2007).

[3] The Entire Perceptual Operation: Complete *Trio* of Abduction, Deduction, and Induction:

Abduction$((C^{Ab}(A^{Ab}\rightarrow C^{Ab})\Rightarrow A^{Ab})$+**Deduction**$((A\rightarrow C)^{Ab}\ A^{Ab})\rightarrow C^{Dd})$+**Induction**$((A^{Ab}, C^{In})\sim> (A^{Ab}\rightarrow C^{In}))$

Where: => is the Abductive *plausibility connective* suggesting the concept A^{Ab}, → is the Deductive *necessity connective* from which the abstract object C^{Dd} is inferred, and ~> is the Inductive *probability* *connective* evaluating the relationship between the concept A^{Ab} and the new experiential object C^{In} , when C^{In} is similar to C^{Dd}. From this epistemological position, it is amazing that Gödel, by using pure intuition and thus admitting the limitation of formal proofs, nevertheless attempted to prove the incompleteness of mathematical theories by incomplete formal inference (Gödel, 1931: #1, 1951: 304-306; Dawson, 1984: #2; Hintikka, 2005: 536).

Indeed, Lakatos and Putnam's conception of the *quasi-empirical* proofs in mathematics seem analogical to Gödel's mathematical proofs with intuitive grasps of true facts and his other intuitive inferences. Howevr, the Peircean epistemic logic of the *trio* inferences is the solution to the limitation of formal logic, yet not as the *quasi-empirical method* based on convention but *empirically quasi-proving* the *truth* of the basic propositions upon mathematical *external reality*. Thish is the only way to reach *convention* and for realism in human knowledge including mathematical knowledge (Lakatos, 1967[1978]: 36; Putnam, 1975: 63-77). The Pragmaticist overcoming of Gödel's Platonism is that all our knowledge develops from our sense-perception confrontation with external reality, and therefore conceptual realism with its pure intuition is only *disguised* empirical knowledge of reality. Since for Gödel mathematical reality consists of abstract entities, the analogy with empirical sciences is incomplete. The following is a schema of perceptual quasi-proof of perceptual judgment representing external reality (cf. [3]):

[4] **Perceptual Experience of Interpreting Cognitive Signs in Representing Physical Objects: Quasi-proof of the Truth of Perceptual Judgment**

I n t e r p r e t a t i o n relations evolve hierarchically
From Pre-verbal Sensorimotor Signs to Propositional Judgment

Interpretational relations

Percept-Sign ➔Iconic Presenting ➔Indexical Operating ➔Symbol: Perceptual Judgment

Object Shapes — ***Immediate Object*** — ***Concept of Object***

Feeling — Reaction — Thought

Replicas — **Iconic** *Feeling* — **Indexical** Reaction **Iconic** *Feeling* — The **Meaning-Content** of **Symbol-Concept**

Represented **Real Physical Object**

The signs representing a **Real Object** constitute the **Iconic** Feeling of **Object Shapes**, the **Indexical** reaction to it being the ***Immediate Object*** pre-symbolic representation, and their synthesis in the **Symbolic *Concept*** *represents* the ***Real Object*** by the true **Perceptual Judgment**. Recognizing that our knowledge starts from perceptual confrontation with reality, we can understand Gödel's problem with *grasping* ideal entities through pure intuition, like the Kantian *Intellectual Intuition* in grasping *supersensible objects*, which only a *supernatural being* can do (Gödel, 1951; Dummett, 1981: 251-252). It is upon such basic knowledge that all our theories develop through the discovery of hypotheses (Nesher, 2008).

> But despite their remoteness from sense-experience, we do have something like a perception of the objects of set theory, as it seems from the fact that the axioms *force* themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense-perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. (Gödel, 1964: 268; emphasis added; Weyl, 1949: 235)

We can compare this feeling of *force* to Frege's feeling the *force* of truth in indicative sentences:

> We declare the recognition of truth in the form of an indicative sentence. We do not have to use the word "true" for this. And even when we do use it, *the real assertive force* lies not in it but in the form of the indicative sentence, and where this loses its assertive force the word "truth" cannot put it back again. (Frege, 1918: 89-90, emphasis added; cf. Nesher, 2002: VI.5.)

Such a feeling of the force of truth is the feeling of the self-controlled perceptual quasi-proofs of our perceptual judgments, and "the fact that the axioms *force* themselves upon us" is the feeling of the Abductive discovery and Inductive evaluation of the axioms as hypotheses, through the instinctive, practical and rational operation of epistemic logic. Thus, mathematical theories are also based on perceptual experience confronting its external reality. The question is how mathematical reality differs from physical reality (Putnam. 1975: #4, 1994: # 12).

6. What, Therefore, Is the *Mathematical Reality* That Mathematical Theories Represent?

Since all our knowledge of reality is based on perception and introspection, then basic mathematical knowledge is also based on such experiences (Wang, 1974: VII.3; Nesher, 2002: III). The basic *Mathematical reality* that we initially represent consists of *our operations of counting, grouping, and measuring physical objects* when confronting our environment (Nesher, 1990, 2002: V, 2007).

> . . . the primitive man could count only by pointing to the objects counted, one by one. Here the object is all-important, as was the case with early measures of all peoples. The habit is seen in the use of such units as the foot, ell (elbow), thumb (the basis for our inch), hand, span, barleycorn, and furlong (furrow long). In due time such terms lost their primitive meaning and we think of them as abstract measures. In the same way the primitive words used in counting were at first tied to concrete groups, but after thousands of years they entered the abstract stage in which the group almost ceases to be a factor. (Smith, 1923: 7)

Hence, arithmetic and geometry were historically basic human modes of quantitative operations on physical objects. With our sensual perception, we represent these operations, yet not the engaged physical objects and not the involved conceptual number signs, but their combination in these operations themselves. Hence, the perceptual representation of these operations, being our basic representation of mathematical reality, is "a kind of visual justification which the Egyptian employed" (Gittleman, 1975: 8, 27-31; Parsons, 1995: 61). The arithmetical numbers are neither *physical objects* nor *abstract concepts*, but the *conceptual components of our quantitative operations with physical objects*. We assign numbers to these intentional cognitive operations *cum* physical maneuvers as signs of these operations. The *discovery* of the first concepts of these operations of enumeration consist of natural numbers; and the further *discovering* of their expansion through abstractions and generalizations constitutes our new mathematical hypotheses, which will be evaluated upon the extended mathematical reality (Gödel, 1944:128, 1964:268; Martin, 2005: 207; Spinoza, 1663).

> But consider a physical law, e.g., Newton's Law of Universal Gravitation. To say that this law is true . . . one has to quantify over such non-nominalistic entities as forces, masses, distances. Moreover, as I tried to show in my book, to account for what is usually called 'measurement' – that is, for the numericalization of forces, masses, and distances – one has to quantify not just over forces, masses, and distances construed as physical properties . . . , but also over *functions from* masses, distances etc. *to real* numbers, or at any rate to rational numbers. In short – and this is the insight that, in essence, Frege and Russell already had – a reasonable interpretation of the *application* of mathematics to the physical world *requires* a realistic interpretation of mathematics. (Putnam, 1975: 74)

The realistic understanding of mathematics that I suggest here is that mathematical reality is not an interpretation in the physical reality the physical sciences represent but it is the human operations of counting, groping, and measuring physical objects and their relations, being the basic mathematical reality upon its true representation the mathematical abstract and generalized theories are developed (Putnam, 1975: 77-78; Weyl, 1949: 235).

These basic operations are known by their perceptual representations; however, when we abstract, generalize, and further recombine the arithmetical components of these operations with our intellectual intuition, we continue to self-control them perceptually. Although the new mathematical structures are based on our perceptual confrontation with the reality of operations, when we elaborate them into more complicated kinds of mathematical structures they seemed detached from their reality as abstract conceptual entities grasped by pure intuition. Actually they are evolving in hierarchical relations between *sense-perception* and *intellectual intuitions* in our knowledge of mathematical reality without this reality being divided into "two separate worlds (the world of things and the world of concepts") (Gödel, 1951: 321).

> On the other hand, we have a debate between Realism—mathematical things exist objectively, independently of our mathematical activity—and Constructivism—mathematical things are created

by our mathematical activity. We want to know how much of this can be regarded as continuous with the practice itself. (Maddy, 1997: 191)

The question is about the relationship of our mathematical activity with mathematical structures such that if they are external mathematical reality how we know them, and if they are our constructions, how can we apply them in our empirical theories (Heyting, 1931: 52-53; Dedekind, 1901:15-16)? The solution to this predicament between Metaphysical Realism and Phenomenological Constructivism is that mathematical reality *exists objectively,* yet *not independently* of our mathematical activity. Mathematical reality is our intentional self-controlled mathematical operations on physical objects, such as 1 apple and 1 apple are 2 apples, which are connected with our perceptual representation of this operation as a certain behavioral reality. Hence, we perceptually quasi-prove the truth of our perceptual judgment that "1 + 1 = 2," representing a mathematical operation, and thereby discover the structures of arithmetical numerical signs. Then, by discovering and proving the true representation of new mathematical operations, we hypothesize general theories, such as Peano's Arithmetic; finally, by evaluating them, we extend our knowledge of mathematical reality (Smith, P., 2007: #28.3). In this way we discover the construct of mathematical theories although the Constructivists consider the theories themselves as mathematical reality and not as representations of mathematical operations reality (Resnik, 1997). Hence, only by quasi-proving the truth of perceptual facts representing mathematical operations do we represent mathematical reality.

[5] The Double Layer of Mathematical Operations: (1) Counting Physical Objects; (2) Perceptual Quasi-proving the Truth of Discovering the Numerical Signs of the Operation (Peirce, 7.547)

I n t e r p r e t a t i o n Relations evolve From Pre-verbal Signs to Propositional Judgment

The Cognitive Representation of Mathematical Reality*: Discovering and Operating Numerical *Signs

Reflective Interpretational Relations

(2) Percept-Sign➔Iconic Presenting➔Indexical Operating➔Symbolic Notion: Perceptual Judgment
Object Shapes ***Immediate Object*** Representing Reality **Numerical Counting**

(1) *Human Self-Controlling of Numerical Operations* of *Counting and Measuring Physical Objects*

Mathematical Reality

Gödel considers abstract mathematical theories analogous to physical theories such that mathematical axiomatic theories representation of mathematical abstract reality precedes their application to the empirical world but it is not the reality of human mathematical operations themselves on physical objects:

> ". . . the applications of mathematics to the empirical world, which formerly were based on the intuitive truth of the mathematical axioms, . . ." (Gödel, 1953:#12)

In contrast to Gödel's role of intuition to grasp the truth of mathematical abstract facts, we can perceptually prove the truth of propositional facts representing the reality of mathematical operations (Wittgenstein, 1956: III, 44). By understanding that mathematical reality consists of perceptually self-controlled operations, we can see how Gödel confuses the meaning-contents of mathematical symbols, which are the immediate modes representing numerical operations, with his Platonist mathematical abstract objects. These immediate modes of representation are the Peircean indexical representations of real objects which in mathematics are the factual operations of mathematical reality. Here we can discern Gödel's close insight of Peirce's conception of the perceptual "immediate object" component of symbols representing mathematical reality (Peirce, *CP*: 8.183, 8.343 [1908]; Nesher, 2002: II).

> It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which is immediately given. Only this something else here is *not* or not primarily, the sensations. That something beside the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only | reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (Gödel, 1964: 268)

Here Gödel's distinction between sensual perceptions and mathematical intuitions of the reality of abstract mathematical objects is the Pragmaticist distinction between the immediate iconic-sensual sign and the indexical-reaction being the "immediate object," the "abstract element" which is only the sign *representing* the *real object*. This Gödel's distinction is based on a confused epistemology that replaces the *meaning-contents* of such mathematical propositions with the *external reality they represent* (Gödel, 1953/54?: #35). It is Peirce's conception of the cognitive "immediate object," representing the real object that Descartes calls "objective reality" in distinction from "formal reality," the real object, without being able to explain it as perceptual cognitive representation of external reality (e.g., Peirce, CP: 8.183, 8.343; Nesher, 2002: II, III, V; Feferman, 1998; Parsons, 2008: Chap. 6). The following is a schema of a mathematical reality operation represented by the perceptual *immediate object* as the meaning-content of the symbolic sign of mathematics:

[6] Perceptual Representation of the Cognitive Operation of Counting Physical Objects by Quasi-proving the Truth of Its Perceptual Judgment of Mathematical Operation

I n t e r p r e t a t i o n relations evolve From Pre-verbal Signs to Propositional Judgment

The Cognitive Representation of Mathematical Reality*: Discovering and Operating Numerical *Signs

Reflective Interpretational Relations

Percept-Sign➔Iconic Presenting➔Indexical Operating➔Symbolic Sign: Perceptual Judgment of Counting: "2 & 2 are 4"

Feeling — Reaction — Thought

Objects Shapes — ***Immediate Object*** — **Represent Objects**

Replicas { **Iconic** *Feeling* } — { **Indexical** Reaction **Iconic** *Feeling* } — The **Meaning-Content** of **Symbol-Concept**

▼ { Relation of Representation

Human Self-Controlling of Numerical Operations* of *Counting and Measuring Physical Objects

Mathematical Reality

An echo of this explanation is noticed in Gödel's insight into the realist nature of mathematics:

> . . . [mathematics] in its simplest form, when the axiomatic method is applied, not to some hypothetico-deductive system as geometry (where the mathematician can assert only the conditional truth of the theorems), but mathematical proper, that is, to the body of those mathematical propositions, which hold in an absolute sense, without any further hypothesis. There must exist propositions of this kind, because otherwise there could not exist any hypothetical theorems | either. For example, *some* implications of the form:
>
> > If such and such axioms are assumed, then such and such theorems hold, must necessarily be true in the absolute sense. Similarly, any theorem of finitistic number theory, such as 2 + 2 = 4, is no doubt, of this kind. (Gödel, 1951: 305; cf. 322)

The perceptual representation of basic mathematical operations is the quasi-proved true empirical facts of mathematical reality, but not an ideal one. Yet this seems to be an unbridgeable gap for Penrose.

> . . . real numbers are called 'real' because they seem to provide the magnitudes needed for the measurement of distance, angle, time, energy, temperature, or of numerous other geometrical and physical quantities. However, the relationship between the abstractly defined 'real' numbers and the physical quantities is not as clear-cut as one might imagine. Real numbers refer to *mathematical idealization* rather than to any actual physically objective quantity. (Penrose, 1989: 112-113; cf. Penrose, 2011: 16:1)

Hence, Popper's amazement as to why mathematics can be applicable to reality is resolved by explaining that mathematics indeed originated in human perceptual true representations of mathematical reality, the "empirical basis" of mathematical theory being more abstract component of this empirical science (Popper, 1963: #9; Dedekind, 1901: 17; Poincare, 1902: Author's Preface, Chap. II).

7. Mathematics Is an Empirical Science Based on True Propositional Facts of Mathematical Reality

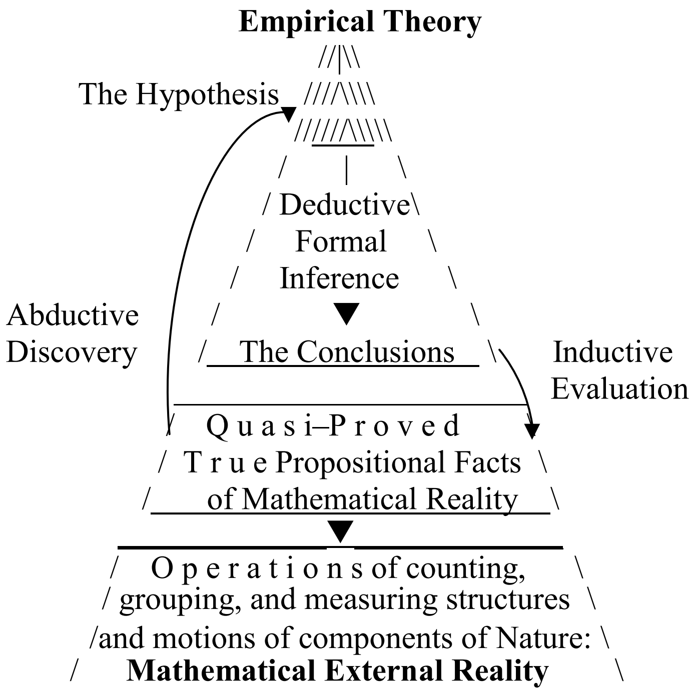
Hence the problem is to explain the nature of mathematical science and what are the "data," the basic facts upon them the mathematical theories develop and evaluated?

> . . . mathematics has always presented itself, throughout the history, as an abstract discipline, but has nevertheless always dealt with specific subject matter of its own. Considering mathematics in this light one might ask: what kind of knowledge can be attained through it? How can it be said to deal with contents and objects which are offered as 'data,' and yet are not data at all from the point of view of sensible experience? We are here confronted with the problem of mathematical intuition, considered as a real source of knowledge, to be clearly distinguished from that further form of mathematical activity which consists in the systematic construction of various theories. Indeed, the most delicate point of this problem is precisely the comparison between the intuitive moment and the moment of theoretical construction, since it is impossible to deny that, in many cases at least, mathematical theories are in fact an exact and systematic codification of what is known intuitively, and that, on the other hand, intuition is not sufficiently reliable unless it is supported by logical proof (Agazzi, 1974: 9-10).

The formal logical proof cannot support or replace the intuitive grasp of the mathematical basic true fact in Gödelian Platonism, and only the epistemic logic of Peircean trio can quasi-prove the truth of the perceptual judgments as the basic mathematical propositional facts (Nesher, 2002: X). Only this logic can replace the mysterious unexplainable intuition of mathematical facts and can prove mathematical truths by the epistemic complete proof. Thus it also replaces the assuming roles of such intuition for discovery and evaluation of the axioms of mathematical theories (Agazzi, 1974: 12).

From the quasi-proof of the truth of the basic mathematical propositional facts of mathematical reality, the mathematical hypotheses are Abductively *discovered* to infer Deductively their *predicted* theorems and *evaluated* Inductively upon empirically newly discovered and proved mathematical facts. The following is a pragmaticist epistemological explanation of the general structure and operation of the theories of mathematical empirical science:

[7] Pragmaticist Epistemological Presentation of Mathematical Empirical Theory:



The *proof-conditions* of **mathematical empirical theory** are the *epistemic logic*, the *trio* comprising *inferential rules of the complete proof* of the truth of basic *propositional facts* representing **external reality**. With this epistemic logic we also prove the truth of scientific hypotheses (Gödel's axioms), through their Abductive discovery, Deductive formally inferred theorems and their Inductive evaluation upon the basic *propositional facts*. Yet, Gödel's conception of mathematical intuition covers those different components of the Pragmaticist epistemic logic which though he felt their operations but could not explain the truth of these basic propositional facts of mathematical reality and the truth of the axioms which the epistemic logical complete proof can do (Feferman, 1998: #1; Parsons, 2008: #5). Hence, empirical theories are only relatively true by being "closed" upon their proof-conditions, which can change with newly discovered facts of reality (Heisenberg, 1971:43-44; Nesher, 2002: V.5, X.10).

> Yet if mathematical facts are facts, they must be facts about something; if mathematical truths are true, something must make them true. Thus arises the first important question: what is mathematics about? If 2 plus 2 is so definitely 4, what is it that makes it so? (Maddy, 1990:1)

Although mathematical theory is about mathematical operations of counting, grouping, measuring, and so on, the question is, how do we prove the mathematical facts representing such operations; i.e., "what is it that makes it so" that 2 plus 2 are definitely 4? We operate in such a manner that we count with our indexical ostensions while representing this operation in our perceptual judgment as a true fact of such arithmetical counting. Since all our basic knowledge comprises such quasi-proofs of our perceptual judgments, so too do the truths of our basic mathematical facts represent such operations of mathematical reality (comp. Hempel, 1945).

Indeed, we do not create on our will the patterns of mathematical reality, but we discover the mathematical concepts of our counting, grouping, and measuring operations with physical objects in the operations of mathematical reality, and this is "[mathematics] in its simplest form, . . . mathematical proper, that is, to the body of those mathematical propositions, which hold in an absolute sense, without any further hypothesis" (Gödel, 1951: 305; Dedekind, 1901:15-16). Epistemologically we can understand that when we intuit the force of the truth of our basic mathematical propositions we feel that they "hold in an absolute sense" but without conceiving the epistemic logic we cannot explain them as our own empirically quasi-proved true mathematical propositions (Steiner, 2000: 337-339).

> Namely, it is correct that a mathematical proposition says nothing about the physical or psychical reality existing in space and time, because it is true already owing to the meaning of the terms occurring in it, irrespectively of the world of real things (Gödel, 1951: 320).

Yet Gödel is right that mathematical reality consists of neither physical nor psychical realities but it is the specific connection between them; namely, the mathematical "world of real things" is our cognitive operations of quantifying components of physical reality, and the meaning-contents of mathematical signs evolve in this perceptual experience (Wittgenstein, 1956: III, 44; Benacerraf, 1973; Tait, 1986; Resnik, 1992: #1; Martine, 2005: 210).

> To mention another example, the Pitta-Pitta, a tribe [of aborigines] in Queensland, are able to count the fingers and toes without a system of numerals, but only by the aid of marks in the sand. . . (Smith, D., 1923: 7; Gullberg, 1997: Ch. 4).

This is evidence of arithmetical facts that are iconic cum indexical sensori-motoric operations of counting and grouping with pre-conceptual signs of properties and relations that eventually develop into conceptual components, the numerical symbols involving in mathematical facts (Gödel, 1951: 320).

> From its earliest beginnings science has used mathematics. Counting, measuring, ordering, and estimating are basic mental operations necessary for science as well as for many other human activities, and their nature is mathematical (Bos, 1993: 165).

Hence, mathematics, from "the ubiquitous use of elementary mathematics" to "the great variety of high level applications of mathematics" (Bos, 1993: 165-166), is an empirical science of the operational quantification of physical components of nature. Its development is from the use of elementary to the variety of high level mathematics evolved from the elaboration of abstract mathematical theories related to their advance applications by scientists working toward the advancement of scientific theories.

8. Conclusion: Mathematics Is an Empirical Science Representing Its Own Reality, Being Neither Queen Nor Servant of Other Empirical Sciences but Their Quantitative Backbone

The problem is to explain the difference between mathematical science and other sciences and their collaboration, when all are empirical sciences representing different realities and with different roles in developing our knowledge of nature (Wang, 1974: VII). Thus, in mathematics we cannot have true theories without proving them upon mathematical reality. Mathematicians develop their theories by discovering general hypotheses as mathematical formulations of theoretical models, typically of physics, like of fields of forces and topology of fluid flows, but of all other sciences, and evaluate them upon mathematical reality of quantitative operations on predicted physical observations.

> The rich interplay between mathematics and physics predates even their recognition as separate subjects. The mathematical work that in some sense straddles the boundaries between the two is commonly referred to as *mathematical physics*, though a precise definition is probably impossible. (Jaffe & Quinn, 1993: 4)

Mathematical theories formularize models for theoretical physical hypotheses, but there is a distinction between proving the truth of mathematical theories and proving the truth of the relevant physical theories themselves (Feferman, 1998).

> Fore as far as verifiable consequences of theories are concerned the mathematical axioms are exactly as necessary for obtaining them as the laws of nature (cnf. footn. 41). If, e.g., the impredicative axioms of analysis are necessary for the solution of some problem of mathematical physics, these axioms will imply predictions about observable facts not obtainable without them. Moreover it is perfectly conceivable that an inconsistency with observation may be due to not to some wrong physical assumptions but to an inconsistency of these axioms. (Gödel, 1953-54 II: #44, p. 188)
>
> That it is arbitrary to call mathematics void of content because, without laws of nature, it has no verifiable consequences also appears from the fact that the same is true for the laws of nature without mathematics or logic. cf. also #44. (Gödel, 1953-54 II: fn. 41, p. 207)

Thus, physicists and mathematicians have different realities to represent with their theories, and the mathematical theory which proved true in the measurement of observed physical facts is only the condition for the evaluation of physical theories. Thus, in distinction from Gödel's conceptual epistemology of mathematics, according to the above explanation, the mathematical reality is also empirical. The truth of mathematical theory enables proving experimentally the truth but also the falsity of physical theories. In this way, we can understand the Gödelian epistemic intuition about the nature of mathematical theories, yet not the Quinean "mathematical naturalism," which confuses mathematics with other sciences and identifies mathematical reality with physical reality.

When there are difficulties with a physical picture of reality and the mathematical model for it, such that it becomes impossible to make measurable predictions, then the problem is to inquire what is wrong that we are unable to evaluate experimentally the physical hypothesis (Woit, 2007: x-xiii, Ch. 14; Feferman, 1998: #2, #4).

> I can't say whether string theory will ever get past its most serious hurdle–coming up with a testable prediction and then showing that the theory actually gives us the right answer. (The math part of things, as I have said, is already on a much firmer ground.) Nevertheless, I do believe the best chance for arriving at a successful theory lies in pooling the resources of mathematicians and physicists, combining the strengths of the two disciplines and their different ways of approaching the world. (Yau & Nadis, 2010: 304)

Hence, mathematics without operational measuring the predicted and eventually observed true facts of reality cannot be true and cannot be "on a much firmer ground" than physics without "a testable prediction." Both have to prove their own truths upon "their different ways of approaching the world."

> However mathematical intuition in addition creates the conviction that, if these formulas express observable facts and were obtained by applying mathematics to verified physical laws (or if they express ascertainable mathematical facts), then these facts will be brought out by observation (or computation) (Gödel, 1953/9-III: #16; cf. ##13-15 & n. 34).

How may one understand this hinted explication of the relationship between intuitive mathematical truth representing its own reality and its application to physical theories to enable observable predictions of them (Gödel, 1953II: #15)? In the end, mathematics is neither the *queen* of science nor its *servant* but its *quantitative backbone*—that is, the quantified formulations of scientific theoretical models and their operations on scientific observations—without which physical and other theories cannot be evaluated experimentally (Bos, 1993: #10). The explanation to the puzzlement why mathematics is considered *exact* or *pure* science while being empirical like other experimental sciences, is the relative simplicity of its represented reality in respect to the physical and the psychological realities.

> Mathematics may be the queen of the science and therefore entitled to royal prerogatives, but the queen who loses touch with her subjects may lose support and even be deprived of her realm. Mathematicians may like to rise into the clouds of abstract thought, but they should, and indeed they must, return to earth for nourishing food or else die of mental starvation. They are on safer and saner ground when they stay close to nature. (Kline, 1959: 475)

This is a poetic metaphor that illustrates the above explanation of the empirical nature of mathematical reality, upon which mathematical theories can be evaluated and be proved true. This empirical explanation can be seen in Gödel's late philosophical writings on the foundations of mathematics:

> If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics. . . . This whole consideration incidentally shows that the philosophical implications of the mathematical facts explained do not lie entirely on the side of rationalistic or idealistic philosophy, but that in one respect they favor the empiricist viewpoint. It is true that only the second alternative points in this direction. (Gödel, 1951: 313)

Hence, we can know experientially the mathematical facts of the mathematical empirical reality.

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