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An analysis of dimensional theoretical properties of some affine dynamical systems

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Contents

1. Introduction

In this work we study dimensional theoretical properties of some affine dynamical systems.

By dimensional theoretical properties we mean Hausdorff dimension and box-counting dimension of invariant sets and ergodic measures on theses sets. Especially we are interested in two problems. First we ask whether the Hausdorff and box-counting dimension of invariant sets coincide. Second we ask whether there exists an ergodic measure of full Hausdorff dimension on these invariant sets. If this is not the case we ask the question, whether at least the variational principle for Hausdorff dimension holds, which means that there is a sequence of ergodic measures such that their Hausdorff dimension approximates the Hausdorff dimension of the invariant set. It seems to be well accepted by experts that these questions are of great importance in developing a dimension theory of dynamical systems (see the book of Pesin about dimension theory of dynamical systems [PE2]).

Dimensional theoretical properties of conformal dynamical systems are fairly well understood today. For example there are general theorems about conformal repellers and hyperbolic sets for conformal diffeomorphisms (see chapter 7 of [PE2]). On the other hand the existence of two different rates of expansion or contraction forces problems that are not captured by a general theory this days. At this stage of development of the dimension theory of dynamical systems it seems natural to study non conformal examples. This is the first step to understand the mechanisms that determine dimensional theoretical properties of non conformal dynamical systems.

Affine dynamical systems represent simple examples of non conformal systems. They are easy to define, but studying their dimensional theoretical properties does nevertheless provide challenging mathematical problems and exemplify interesting phenomena. We consider here a special class of self-affine repellers in dimension two, depending on four parameters (see 2.1.). Furthermore we study a class of attractors of piecewise affine maps in dimension three depending on four parameters as well. The last object of our work are projections of these maps that are known as generalized Baker's transformations (see 2.2.).

The contents of our work is the following:

In chapter two we give an overview about some main results in the area of dimension theory of affine dynamical systems and define the systems we study in this work. We will explain, what is known about the dimensional theoretical properties of these systems and describe what our new results are. In chapter three we then apply symbolic dynamics to our systems. We will introduce explicit shift codings and find representations of all ergodic measures for our systems using these codings.

From chapter four to chapter eight we study dimensional theoretical properties, which our systems generally or generically have. In chapter four we will prove a formula for the box-counting dimension of the repellers and the attractors (see theorem 4.1.). Then in chapter five we apply general dimensional theoretical results for ergodic measures found by Ledrappier and Young [LY] and Barreira, Schmeling and Pesin [BPS] to our systems. These results relate the dimension of ergodic measures to metric entropy and Lyapunov exponents. Using this approach we will be able to reduce questions about the dimension of ergodic measures in our context to questions about certain overlapping and especially overlapping self-similar measures on the line. These overlapping self-similar measures are studied in chapter six. Our main theorem extends a result of Peres and Solomyak [PS2] concerning the absolute continuity resp. singularity of symmetric self-similar measures to asymmetric ones (see theorem 6.1.3.).

In chapter seven we bring our results together. We prove that we generically (in the sense of Lebesgue measure on a part of the parameter space) have the identity of box-counting and Hausdorff dimension for the repellers and the attractors. (see theorem 7.1.1. and corollary 7.1.2.). This result suggest that one can expect that the identity of box-counting dimension and Hausdorff dimension holds at least generically in some natural classes of non conformal dynamical systems.
Furthermore we will see in chapter seven that there generically exists an ergodic measure of full Hausdorff dimension for the repellers. On the other hand the variational principle for Hausdorff dimension is not generic for the attractors. It holds only if we assume a certain symmetry (see theorem 7.1.1.). For generalized Baker's transformations we will find a part of the parameter space where there generically is an ergodic measure of full dimension and a part where the variational principle for Hausdorff dimension does not hold (see theorem 7.1.3.). Roughly speaking the reason why the variational principle does not hold here is, that if there exists both a stable and an unstable direction one can not generically maximize the dimension in the stable and in the unstable direction at the same time. In an other setting this phenomenon was observed before by Manning and McCluskey [MM].
In chapter eight we extend some results of the last section to invariant sets that correspond to special Markov chains instead of full shifts (see theorem 8.1.1.).

In the last two chapters of our work we are interested in number theoretical exceptions to our generic results. The starting point of our considerations in section nine are results of Erdös [ER1] and Alexander and Yorke [AY] that establish singularity and a decrease of dimension for infinite convolved Bernoulli measures under special conditions. Using a generalized notion of the Garsia entropy ([GA1/2]) we are able

to understand the consequences of number theoretical peculiarities in broader class of overlapping measures (see theorem 9.1.1.).
In chapter ten we then analyze number theoretical peculiarities in the context of our dynamical systems. We restrict our attention to a symmetric situation where we generically have the existence of a Bernoulli measure of full dimension and the identity of Hausdorff and box-counting dimension for all of our systems.
In the first section of chapter ten we find parameter values such that the variational principle for Hausdorff dimension does not hold for the attractors and for the Fat Baker's transformations (see theorem 10.1.1.). These are the first known examples of dynamical systems for which the variational principle for Hausdorff dimension does not hold because of number theoretical peculiarities of parameter values. For the repellers we have been able to show that under certain number theoretical conditions there is at least no Bernoulli measure of full Hausdorff dimension; the question if the variational principle for Hausdorff dimension holds remains open in this situation.
In the second section of chapter ten we will show that the identity for Hausdorff and box-counting dimension can drops because there are number theoretical peculiarities. In the context of Weierstrass-like functions this phenomenon was observed by Przytycki and Urbanski [PU]. Our theorem extends this result to a larger class of sets, invariant under dynamical systems (see theorem 10.2.1).

At the end of this work the reader will find two appendices, a list of notations and the list of references. In appendix A we introduce the notions of dimension we use in this work and collect some general facts in dimension theory. In appendix B we state the facts about Pisot-Vijayarghavan number, we need in our analysis of number theoretical peculiarities. The list of notations contains general notations and a table with a summary of notations we use to describe the dynamical systems that we study.

Acknowledgments

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2. Affine dynamical systems

2.1. Self-affine repellers

Self-affine repellers are simple examples of non conformal dynamical systems. We introduce them now. Let $T_1, \ldots, T_p : \mathbf{D} \longrightarrow \mathbf{D}$ be affine contractions of a domain $\mathbf{D}$ in $I\!R^m$. Assume that the sets $T_i(\mathbf{D})$ are disjoint . From Hutchinson [HU] we know there is an unique compact self-affine subset Λ of $\mathbf{D}$ satisfying:

$$\Lambda = \bigcup_{i=1}^{p} T_i(\Lambda).$$

Define a map T on $\bigcup_{i=1}^{p} T_i(\mathbf{D})$ by

$$T(x) = T_i^{-1}(x) \quad \text{if} \quad x \in T_i(\mathbf{D}).$$

Clearly T is a smooth expanding map. Λ is invariant and a repeller for T, which means that there is an open neighborhood V of Λ such that $\Lambda = \{x \in V | f^n(x) \in V \ \ \forall n \geq 0\}$ (see chapter 20 of [PE2]). We call Λ a **self-affine repeller**.

There is one generic result about the dimension of large classes of self-affine sets.

Theorem 2.1.1.

Let $L_1, \ldots, L_p$ be linear contractions of $I\!R^m$ with $||L_i|| < 1/2$ and let $b_1, \ldots, b_m \in I\!R^m$. If Λ is the compact self-affine set satisfying

$$\Lambda = \bigcup_{i=1}^{m} L_i(\Lambda) + b_i$$

then the identity $\dim_B \Lambda = \dim_H \Lambda$ holds for almost all $(b_1, \ldots, b_m) \in I\!R^{mp}$ in the sense Lebesgue measure and the common value is independent of $(b_1, \ldots, b_m)$.

Falconer [FA2] proved this theorem in the case $||L_i|| < 1/3$ and Solomyak [SO2] extended the proof to the case $||L_i|| < 1/2$. Moreover Solomyak showed that the statement does not longer hold if we replace $1/2$ by $1/2 + \delta$.

Of course 2.1.1. leaves many questions open. First of all the question about the existence of an ergodic measure of full Hausdorff dimension remains open. Moreover one would like to have some information about classes of self-affine repellers with larger expansion rates and there may be natural subclasses that fall in the exceptional set of 2.2.1. .

Let us discuss a very natural family of self-affine repellers that is completely understood today and proved to fall in the exceptional class of 2.1.1. .
Given integers $l \geq m \geq 2$ choose a set A of pairs of integers (i, j) with $0 \leq i < l$ and $0 \leq j < m$. Denote the cardinality of A by a. Now let T_k for $k = 1 \dots a$ be affine maps given in the following way: if k enumerates the element $(i, j) \in A$ then let

$$T_k([0,1]^2) = [i/l, (i+1)/l] \times [j/m, (j+1)/m].$$

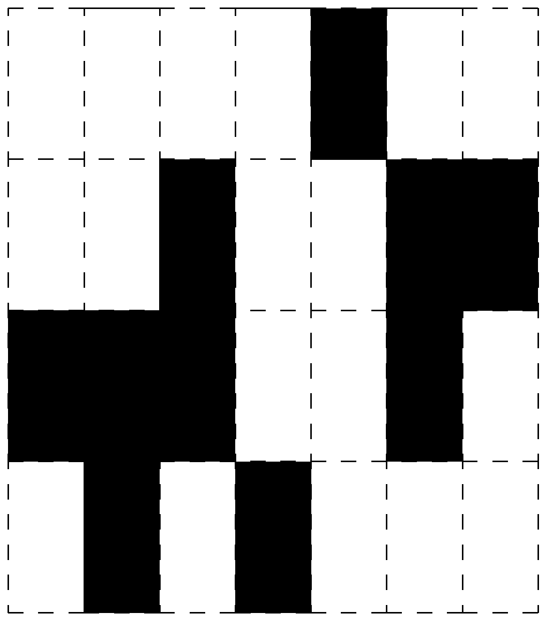

Figure 1: The images of the affine maps inducing a self-affine carpet with $l = 8$ $m = 4$ and $A = \{(4,0), (2,1), (6,1), (7,1), (0,2), (1,2), (2,2), (5,2), (1,3), (3,3)\}$

Let Λ_A be the self-affine set generated by these affine contractions. A set of this type is known as **general Sierpinski carpet**. We remark that Λ_A viewed as a subset of the Torus is invariant under the toral endomorphism given by:

$$\hat{T} : (x, y) \longrightarrow (lx, my) \mod 1.$$

Dimensional theoretical questions are answered by the following theorem of McMullen:

Theorem 2.1.2. [MC]

Let t_j be the number of those i for which $(i,j) \in A$ and let r be the number of those j for which there is some i such that $(i,j) \in A$. We have:

$$\dim_H \Lambda_A = \log_m\left(\sum_{j=0}^{m-1} t_j^{\log_l m}\right) \quad \text{and} \quad \dim_B \Lambda_A = \log_m r + \log_l(a/r).$$

Moreover there exists a Bernoulli measure of full Hausdorff dimension on Λ_A.

We remark that it is easy to see that a Bernoulli measure on the carpet is in fact an ergodic measure with respect to the map $\hat{T}$ on the torus or the expanding map T associated with the affine contractions.

Note that the theorem we implies that the Hausdorff and box-counting dimension of a general Sierpinski carpet coincide if and only if the carpet is self-similar (l=m) or the number of rectangles is constant or zero in every raw ($t_j = 0$ or $t_j = const.$ for all j).

There are some generalizations of 2.1.2. . Kenyon and Peres [KP] extended the result to analogous subsets of higher dimensional cubes, which they called self-affine Sierpinski sponges. Using this result they where also able to show the existence of an ergodic measure of full Hausdorff dimension on all compact invariant sets for endomorphisms of the d-Torus with integer eigenvalues. Gatzouras and Lalley [GL] extended the result on self-affine carpets in another direction. They considered affine contractions which map the unit square to rectangles with height greater than width such that these rectangles are lined up in rows. They calculated the Hausdorff and box-counting dimension of the limit set and found ergodic measures of full dimension.

Now we define the class of self-affine repellers we will study in our work.

Let $P^4_{all} = \{(\beta_1, \beta_2, \tau_1, \tau_2) \in (0,1)^4 | \beta_1 + \beta_2 \geq 1 \text{ and } \tau_1 + \tau_2 < 1\}$ be the set of all parameters we consider. Given $\vartheta \in P^4_{all}$ we define two affine contractions $T_{1,\vartheta}$ and $T_{-1,\vartheta}$ of the square $[-1,1]^2$ by

$$T_{1,\vartheta}(x,z) = (\beta_1 x + (1-\beta_1), \tau_1 z + (1-\tau_1))$$

$$T_{-1,\vartheta}(x,z) = (\beta_2 x - (1-\beta_2), \tau_2 z - (1-\tau_2)).$$

Let Λ_ϑ be unique compact self-affine subset of $[-1,1]^2$ satisfying

$$\Lambda_\vartheta = T_{1,\vartheta}(\Lambda_\vartheta) \cup T_{-1,\vartheta}(\Lambda_\vartheta)$$

and let T_ϑ be the smooth expanding transformation on $T_{1,\vartheta}([-1,1]^2) \cup T_{2,\vartheta}([-1,1]^2)$ defined by

$$T_\vartheta(x) = (T_{i,\vartheta})^{-1}(x) \quad \text{if} \quad x \in T_{i,\vartheta}([-1,1]^2) \quad \text{for} \quad i = 1, 2.$$

The set Λ_ϑ is an invariant repeller for the transformation T_ϑ.

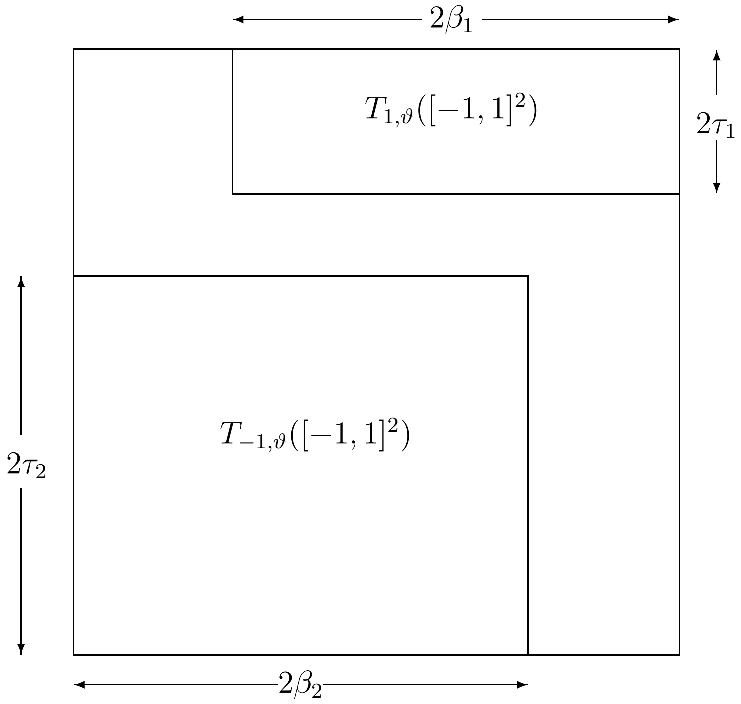

Figure 2: The transformations $T_{1,\vartheta}$ and $T_{-1,\vartheta}$ on $[-1,1]^2$

We will now describe what is known about dimensional theoretical properties of the systems $(\Lambda_\vartheta, T_\vartheta)$.

The symmetric situation $\vartheta = (\beta, \beta, \tau, \tau) \in P^4_{all}$ has been studied by Pollicott and Weiss [PW]. We need one definition to state the result. We say that $\beta \in (0,1)$ is a **Garsia-Erdös number** if

$$\exists\, C > 0\ \forall x \in I\!R : \text{card}\{(s_0, \ldots, s_{n-1}) | \sum_{k=0}^{n-1} s_k \beta^k \in [x, x + \beta^n)\} \le C(2\beta)^n\ \forall n \ge 0.$$

Examples of Garsia-Erdös numbers are the numbers $\frac{1}{\sqrt[n]{2}}$ for $n \ge 0$. Furthermore we know from appendix 3 of [PW] that for some ρ almost all $\beta \in (1-\rho, 1)$ are Garsia-Erdös numbers.

Theorem 2.1.3. [PW]

If $\vartheta = (\beta, \beta, \tau, \tau) \in P^4_{all}$ then we have

$$\dim_B \Lambda_\vartheta = 1 + \frac{\log(2\beta)}{\log(1/\tau)}.$$

If β is in addition a Garsia-Erdös number then we have

$$\dim_B \Lambda_\vartheta = \dim_H \Lambda_\vartheta$$

and the equal-weighted Bernoulli measure on Λ_ϑ has full Hausdorff dimension.

Now let us say what our new results about the dimensional theoretical properties of the systems $(\Lambda_\vartheta, T_\vartheta)$ are and where in our work the corresponding theorems can be found:

New results

First of all in theorem 4.1.1. we will find a formula for $\dim_B \Lambda_\vartheta$ for all $\vartheta \in P^4_{all}$. In fact the box-counting dimension is given by the unique positive solution of the equation

$$\beta_1 \tau_1^{x-1} + \beta_2 \tau_2^{x-1} = 1.$$

Furthermore we show or almost all $\vartheta \in P^4_{trans} := \{(\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all} | \beta_2 \leq \beta_1 \leq 0.649\}$ in the sense of four dimensional Lebesgue measure the identity $\dim_B \Lambda_\vartheta = \dim_H \Lambda_\vartheta$ and and the existence of an ergodic measure of full Hausdorff dimension for the system $(\Lambda_\vartheta, T_\vartheta)$; see corollary 7.2. . The restriction of this generic result depends on the technique we use and is due to a certain transversality condition; see chapter six. In fact our main generic result in theorem 7.1. is little bit stronger than corollary 7.2. and takes special cases into consideration. We will see that the statements in the second part of Pollicott and Weiss theorem holds for almost all $\beta \in (0.5, 1)$ in the sense of one dimensional Lebesgue measure. Our technique is different from the arguments of Pollicott and Weiss and the condition we have for the identity of Hausdorff and box-counting dimension is not the number theoretical Garsia-Erdös condition (see the remarks after 7.2.).

Let us now for a moment consider the case $\vartheta = (\beta, \beta, \tau, \tau)$ with $\tau = 0.5$. In this situation the set Λ_ϑ coincides (up to a countable number of points) with the graph of Weierstrass-like function studied by Przytycki and Urbanski [PU]. Przytycki and Urbanski were able to show that the Hausdorff dimension of these graphs is less than their box-counting dimension if β is the reciprocal of a Pisot-Vijayarghavan number (short PV number). The reader will find the definition and examples of PV numbers in appendix B.

In our work number theoretical peculiarities are also one point of main effort. We will show that if $\vartheta = (\beta, \beta, \tau, \tau) \in P^4_{all}$ and β is the reciprocal of a PV number then we have no Bernoulli measure of full Hausdorff dimension for the system $(\Lambda_\vartheta, T_\vartheta)$ (see 10.1.1.(3)) and the inequality $\dim_H \Lambda_\vartheta < \dim_B \Lambda_\vartheta$ holds (see 10.2.1.). The arguments we need to get this result in our situation with $\tau < 0.5$ are very different from the arguments of [PU].

2.2. Attractors of piecewise affine maps

Attractors of piecewise affine maps provide simple examples of generalized hyperbolic attractors. Especially the **Belykh attractors** raised great interest in the literature (see [PE1]). We want to introduce them here. Consider piecewise affine transformations on the square $[-1, 1]$ given by

$$f^{k,\rho_1,\rho_2}_{\beta_1,\beta_2}(x, y) = \{ \begin{array}{lll} (\beta_1 x + (1 - \beta_1), \rho_1 y + (1 - \rho) & \text{if} & y \geq kx \\ (\beta_2 x - (1 - \beta_2), \rho_2 y - (1 - \rho) & \text{if} & y < kx \end{array}$$

where $\beta_1, \beta_2 \in (0, 1)$ $k \in (-1, 1)$ and $\rho_1, \rho_2 \in (1, 2/(|k| + 1)$.
It is easy to see that there is a global attractor called Belykh attractor for all of these maps.

The definition we used here is due to Pesin [PE1]. Belykh [BE] himself only considered the case $\beta_1 = \beta_2$ and $\rho_1 = \rho_2$. Dimensional theoretical properties of the systems in this special case were studied by Schmeling [SCH].

In our work we are interested in another special case. We set $k = 0$ and $\rho_1 = \rho_2 = 2$ and obtain transformations

$$f_{\beta_1,\beta_2}(x, y) := f^{0,1,1}_{\beta_1,\beta_2}(x, y) = \{ \begin{array}{lll} (\beta_1 x + (1 - \beta_1), 2y - 1) & \text{if} & y \geq 0 \\ (\beta_2 x - (1 - \beta_2), 2y + 1) & \text{if} & y < 0 \end{array} .$$

of the square $[-1, 1]^2$ for all $\beta_1, \beta_2 \in (0, 1)$. We call these maps **generalized Baker's transformations**. If $\beta = \beta_1 = \beta_2$ we write f_β instead of $f_{\beta,\beta}$. Alexander and Yorke [AY] called f_β a **Skinny Baker's transformation** if $\beta < 0.5$ and a **Fat Baker's transformation** if $\beta > 0.5$. $f_{0.5}$ is known as the **Baker's transformation**.

The attractor for f_{β_1,β_2} is given by

$$Q_{\beta_1,\beta_2} := \text{closure}(\bigcap_{k=0}^{\infty} f^n_{\beta_1,\beta_2}([-1, 1]^2)).$$

In the case $\beta_1 + \beta_2 < 1$ dimensional theoretical properties of the dynamical system $(Q_{\beta_1,\beta_2}, f_{\beta_1,\beta_2})$ are well known:

Theorem 2.2.1.

Let $\beta_1 + \beta_2 < 1$ and d be the unique positive number satisfying $\beta_1^d + \beta_2^d = 1$ then $\dim_B Q_{\beta_1,\beta_2} = \dim_H Q_{\beta_1,\beta_2} = d + 1$ and there is an ergodic measure of full Hausdorff dimension for the system $(Q_{\beta_1,\beta_2}, f_{\beta_1,\beta_2})$.

This result seems to be folklore in the dimension theory of dynamical systems. In fact the attractor in the non-overlapping situation is a product of a standard Cantor set in the line with the interval $[-1, 1]$. The ergodic measure of full dimension is a product of a Cantor measure (a Bernoulli measure on the standard Cantor set) with the normalized Lebesgue measure on $[-1, 1]$. We refer to chapter 23 of [PE2] for theses facts.

We consider in this work the overlapping situation, which means $(\beta_1, \beta_2) \in P^2_{olapp} := \{(\beta_1, \beta_2)|\beta_1 + \beta_2 \geq 1\}$.

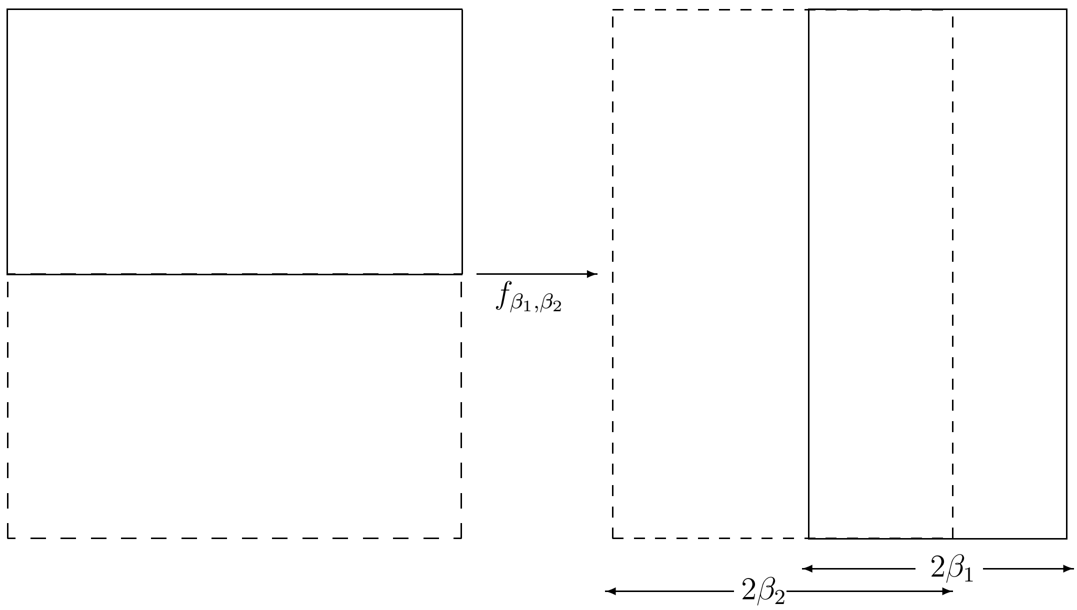

Figure 3: The action of f_{β_1,β_2} on the square $[-1, 1]^2$ where $\beta_1 + \beta_2 > 1$

If $(\beta_1, \beta_2) \in P^2_{olapp}$ the attractor of the map f_{β_1,β_2} is obviously the hole square $[-1, 1]^2$ with Hausdorff and box-counting dimension equal to two. The interesting problem is whether there exist an ergodic measure of full Hausdorff dimension resp. whether the variational principle for Hausdorff dimension holds for $([-1, 1]^2, f_{\beta_1,\beta_2})$ if $\beta_1 + \beta_2 \geq 1$.

New results

From the work of Alexander and Yorke [AY] and a result of Solomyak (see 6.1.1.) it is easy to deduce that for almost all $\beta \in (0.5, 1)$ there is an ergodic measure of full dimension for the Fat Baker's transformation $([-1,1]^2, f_\beta)$. This measure is given by the product of an infinite convolved Bernoulli measure with the normalized Lebesgue measure on $[-1,1]$ (see 7.3.). Our main result about the Fat Baker's transformation is that the variational principle for Hausdorff dimension does not hold for $([-1,1]^2, f_\beta)$ if β is the reciprocal of a PV number (see 10.1.1. (1)). This result is of great interest. Its is the first known example showing that the variational principle for Hausdorff dimension can fail to hold because of number theoretical peculiarities. Beside this we have new results in the asymmetric situation. On the one hand we will show that if $\beta_1\beta_2 < 0.25$ then the variational principle does not hold for the systems $([-1,1], f_{\beta_1,\beta_2})$. On the other hand we will see that for almost all $(\beta_1,\beta_2) \in P^2_{trans} := \{(\beta_1,\beta_2) \in P^2_{olapp} | \beta_2 \leq \beta_1 \leq 0.649\}$ with $\beta_1\beta_2 > 0.25$ there exists an ergodic measure of full Hausdorff dimension for the system $([-1,1], f_{\beta_1,\beta_2})$.

The last class of dynamical systems we study in this work is given by piecewise affine maps in dimension three:

$$\hat{f}_\vartheta : [-1,1]^3 \longmapsto [-1,1]^3$$

$$\hat{f}_\vartheta(x,y,z) = \begin{cases} (\beta_1 x + (1-\beta_1), 2y-1, \tau_1 z + (1-\tau_1)) & \text{if} \quad y \geq 0 \\ (\beta_2 x - (1-\beta_2), 2y+1, \tau_2 z - (1-\tau_2)) & \text{if} \quad y < 0 \end{cases}$$

where $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$.

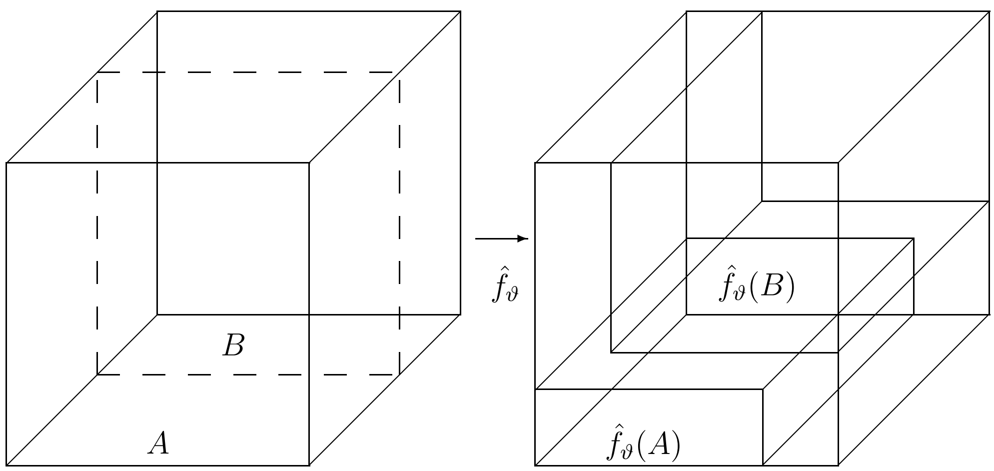

Figure 4: The action of $\hat{f}_\vartheta$ on $[-1,1]^3$.

We see that the projection of $\hat{f}_\vartheta$ onto the (x,y)-plane is f_{β_1,β_2} for $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$. Obviously the maps f_{β_1,β_2} are non invertible if $\beta_1 + \beta_2 \geq 1$ but their lifts $\hat{f}_\vartheta$ are invertible.

Furthermore we notice that the attractor $\hat{\Lambda}_\vartheta$ of $\hat{f}_\vartheta$ is a product of the self-affine set Λ_ϑ in the (x,z)-plane with the interval $[-1,1]$ on the y-axis:

$$\hat{\Lambda}_\vartheta = \text{closure}(\bigcap_{k=0}^{\infty} f^n_\vartheta([-1,1]^3)) = \{(x,y,z)|(x,z) \in \Lambda_\vartheta, \quad y \in [-1,1]\}.$$

New results

By the product structure of the sets $\hat{\Lambda}_\vartheta$ and proposition A5 we have $\dim_{H/B} \hat{\Lambda}_\vartheta = \dim_{H/B} \Lambda_\vartheta + 1$. Thus all our results about Hausdorff dimension and box-counting dimension of the self-affine sets Λ_ϑ have an analogon for $\hat{\Lambda}_\vartheta$ (see 4.1.,7.1.,7.2., 10.2.1.).

Very interesting is the question whether the variational principle for Hausdorff dimension holds. We will show in 7.1. that in the generic situation (for almost all $\vartheta \in P^4_{trans}$) it can only hold for $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ if we have $\log_{\tau_1} \log(2\beta_1) = \log_{\tau_2}(2\beta_2)$. Thus the variational principle for Hausdorff dimension is not generic on the parameter set P^4_{trans} if we consider $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ but it is generic if we consider $(\Lambda_\vartheta, T_\vartheta)$. This phenomenon is related to fact that the map $\hat{f}_\vartheta$ is hyperbolic; it has both a stable and an unstable direction (see also the first remark after 7.2.).
For the symmetric systems $(\hat{\Lambda}_{\beta,\beta,\tau,\tau}, \hat{f}_{\beta,\beta,\tau,\tau})$ the situation is different. There exists an ergodic measure of full dimension for almost all $\beta \in (0.5, 1)$ (see 7.1.). But again there are number theoretical peculiarities. If $\beta \in (0,5)$ is the reciprocal of a PV number and τ is small then variational principle does not hold for $(\hat{\Lambda}_{\beta,\beta,\tau,\tau}, \hat{f}_{\beta,\beta,\tau,\tau})$ (see 10.1.1.(2))

3. Applying symbolic dynamics

3.1. Shift Codings

Let $\Sigma = \{-1,1\}^{\mathbb{Z}}$ and $\Sigma^+ = \{-1,1\}^{\mathbb{N}_0}$. With the product metric defined by

$$d(\underline{s},\underline{t}) = \sum_{k=-\infty \text{ resp. } 0}^{\infty} |s_k - t_k| 2^{-|k|} \text{ for } \underline{s} = (s_k) \text{ and } \underline{t} = (t_k)$$

Σ (resp. Σ^+) becomes a perfect, totally disconnected and compact metric space; see proposition 7.6. of [DGS]. A cylinder set in Σ (resp. Σ^+) is given by

$$[t_0, t_1 \ldots, t_u]_v := \{(s_k) | s_{v+k} = t_k \text{ for } k = 0, \ldots, u\}.$$

The forward shift map σ on Σ (resp. Σ^+) is given by $\sigma((s_k)) = (s_{k+1})$. The backward shift σ^{-1} is defined on Σ and given by $\sigma((s_k)) = (s_{k-1})$.

We will use (Σ, σ) resp. (Σ, σ^{-1}) and (Σ^+, σ) to describe the dynamics of the systems defined in the previous chapter symbolically by coding the points of the invariant set. We begin with the class of self-affine repellers defined in 2.1. .

Given $\underline{s} \in \Sigma^+$ we denote by $\sharp_k(\underline{s})$ the cardinality of $\{s_i | s_i = -1 \quad i = 0 \ldots k\}$. For $\gamma_1, \gamma_2 \in (0,1)$ we define a map $\pi^*_{\gamma_1,\gamma_2} : \Sigma^+ \longrightarrow [\frac{-\gamma_2}{1-\gamma_2} \frac{\gamma_1}{1-\gamma_1}]$ by

$$\pi^*_{\gamma_1,\gamma_2}(\underline{s}) = \sum_{k=0}^{\infty} s_k \gamma_2^{\sharp_k(\underline{s})} \gamma_1^{k-\sharp_k(\underline{s})+1}.$$

We scale this map so that it is into $[-1,1]$. Let L_{γ_1,γ_2} be the affine transformation on the line that maps $\frac{-\gamma_2}{1-\gamma_2}$ to -1 and $\frac{\gamma_1}{1-\gamma_1}$ to 1 and let $\pi_{\gamma_1,\gamma_2} = L_{\gamma_1,\gamma_2} \circ \pi^*_{\gamma_1,\gamma_2}$. For $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$ we set $\pi_\vartheta = (\pi_{\beta_1,\beta_2}, \pi_{\tau_1,\tau_2})$.

Proposition 3.1.1.

The systems (Σ^+, σ) and $(\Lambda_\vartheta, T_\vartheta)$ are homeomorph conjugated via π_ϑ.

Proof

It is obvious that $\hat{\pi}_{\gamma_1,\gamma_2}$ is continuous since

$$d(s,t) \leq \frac{1}{2^n} \Rightarrow s_k = t_k \text{ for } k = 0, \ldots, n \Rightarrow |\hat{\pi}_{\gamma_1,\gamma_2}(s) - \hat{\pi}_{\gamma_1,\gamma_2}(t)| \leq \frac{\gamma_1^{n+1}}{1-\gamma_1} + \frac{\gamma_2^{n+1}}{1-\gamma_2}.$$

hence π_ϑ is continuous. Just looking at the definition of $\hat{\pi}_{\gamma_1,\gamma_2}$ we see that

$$\pi^*_{\gamma_1,\gamma_2}((s_{k+1})) = \{ \begin{array}{ll} \gamma_1^{-1} \pi^*_{\gamma_1,\gamma_2}((s_k)) - 1 & \text{if} \quad s_0 = 1 \\ \gamma_2^{-1} \pi^*_{\gamma_1,\gamma_2}((s_k)) + 1 & \text{if} \quad s_0 = -1 \end{array} .$$

Hence we have

$$\pi_{\gamma_1,\gamma_2}(\sigma(\underline{s})) = L_{\gamma_1,\gamma_2}(\pi^*_{\gamma_1,\gamma_2}((s_{k+1}))) = \{ \begin{array}{ll} L_{\gamma_1,\gamma_2}(\gamma_1^{-1}\pi^*_{\gamma_1,\gamma_2}((s_k)) - 1) & \text{if} \quad s_0 = 1 \\ L_{\gamma_1,\gamma_2}(\gamma_2^{-1}\pi^*_{\gamma_1,\gamma_2}((s_k)) + 1) & \text{if} \quad s_0 = -1 \end{array}$$

$$= \{ \begin{array}{ll} \gamma_1^{-1}\pi_{\gamma_1,\gamma_2}(\underline{s}) + (1-\gamma_1^{-1}) & \text{if} \quad s_0 = 1 \\ \gamma_2^{-1}\pi_{\gamma_1,\gamma_2}(\underline{s}) - (1-\gamma_2^{-1}) & \text{if} \quad s_0 = -1 \end{array} \quad (*).$$

Since $\pi_\vartheta(\underline{s}) \in T_{1,\vartheta}([-1,1]^2)$ if $s_0 = 1$ and $\pi_\vartheta(\underline{s}) \in T_{-1,\vartheta}([-1,1]^2)$ if $s_0 = -1$ this implies

$$\pi_\vartheta(\sigma(\underline{s})) = \{ \begin{array}{ll} T_{1,\vartheta}^{-1}(\pi_\vartheta(\underline{s})) & \text{if} \quad s_0 = 1 \\ T_{-1,\vartheta}^{-1}(\pi_\vartheta(\underline{s})) & \text{if} \quad s_0 = -1 \end{array} = T_\vartheta(\pi_\vartheta(\underline{s})).$$

This means that σ and T_ϑ are conjugated via π_ϑ. Furthermore we see by induction that

$$\pi_\vartheta(\underline{s}) = T_{s_0,\vartheta} \circ \ldots \circ T_{s_{n-1},\vartheta}(\pi_\vartheta(\sigma^n(\underline{s}))) \in T_{s_0,\vartheta} \circ \ldots \circ T_{s_{n-1},\vartheta}([-1,1]^2)$$

and thus

$$\pi_\vartheta(\underline{s}) = \lim_{n\longrightarrow\infty} T_{s_0,\vartheta} \circ \ldots \circ T_{s_{n-1},\vartheta}([-1,1]^2).$$

So π_ϑ is onto Λ_ϑ and invertible since $T_{1,\vartheta}([-1,1]^2) \cap T_{-1,\vartheta}([-1,1]^2) = \emptyset$. The continuity of the inverse map follows from compactness.

□

Now we examine the systems $([-1,1]^2, f_{\beta_1,\beta_2})$ for $(\beta_1,\beta_2) \in P^2_{olapp}$. Define ς from $\Sigma^- := \{-1,1\}^{\mathbb{Z}^-}$ onto [-1,1] by

$$\varsigma(\underline{s}) = \sum_{k=1}^{\infty} s_{-k}2^{-k} \quad \text{where} \quad \underline{s} = (s_k)_{k\in\mathbb{Z}^-} \in \Sigma^-.$$

This function is well known. It maps the signed dyadic expansion of a point in $[-1,1]$ to this point. ς is continuous and one to one restricted to $(\Sigma^-\backslash\{(s_k)|\exists k_0\forall k \leq k_0 : s_k = 1\}) \cup \{(1)\}$. Let $\bar{\Sigma} = (\Sigma\backslash\{(s_k)|\exists k_0\forall k \leq k_0 : s_k = 1\}) \cup \{(1)\}$. For $(\beta_1,\beta_2) \in P^2_{olapp}$ we now define $\bar{\pi}_{\beta_1,\beta_2} : \Sigma \longmapsto [-1,1]^2$ by $\bar{\pi}_{\beta_1,\beta_2}((s_k)) = (\pi_{\beta_1,\beta_2}((s_k)_{k\in\mathbb{N}_0}), \varsigma((s_k)_{k\in\mathbb{Z}^-}))$

Proposition 3.1.2.

$\bar{\pi}_{\beta_1,\beta_2}$ is continuous, surjective and conjugates the backward shift σ^{-1} and f_{β_1,β_2} on $\bar{\Sigma}$.

Proof

It is obvious that the map is continuous and surjective since the components are continuous and onto $[-1,1]$.
Let $\underline{s} = (s_k) \in \bar{\Sigma}$. We have $(s_{k+1})_{k\in\mathbf{Z}^-} \neq (\ldots,1,1,-1)$ and hence

$$\varsigma((s_{k+1})_{k\in\mathbb{Z}^-}) = \sum_{k=1}^{\infty} s_{-k+1}2^{-k} \geq 0 \Leftrightarrow s_0 = 1.$$

Thus

$$f_{\beta_1,\beta_2} \circ \bar{\pi}_{\beta_1,\beta_2}((s_{k+1})) =$$
$$\begin{cases} (\beta_1\pi_{\beta_1,\beta_2}((s_{k+1})_{k\in\mathbb{N}_0}) + (1-\beta_1), 2\varsigma((s_{k+1})_{k\in\mathbb{Z}^-}) - 1) \text{ if } s_0 = +1 \\ (\beta_2\pi_{\beta_1,\beta_2}((s_{k+1})_{k\in\mathbb{N}_0}) - (1-\beta_2), 2\varsigma((s_{k+1})_{k\in\mathbb{Z}^-}) + 1) \text{ if } s_0 = -1 \end{cases}.$$

In view of (*) in the proof of 3.1.1. and the definition of ς we now see that $f_{\beta_1,\beta_2} \circ \bar{\pi}_{\beta_1,\beta_2}((s_{k+1})) = \bar{\pi}_{\beta_1,\beta_2}((s_k))$. σ as a map of Σ is invertible and we get $f_{\beta_1,\beta_2} \circ \bar{\pi}_{\beta_1,\beta_2}(\underline{s}) = \bar{\pi}_{\beta_1,\beta_2}(\sigma^{-1}(\underline{s}))$.

□

Now we have a look at the lifts $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$. For $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2) \in P^4_{all}$ we define $\hat{\pi}_\vartheta : \Sigma \longrightarrow \hat{\Lambda}_\vartheta$ by $\hat{\pi}_{\beta_1,\beta_2}((s_k)) = (\pi_{\beta_1,\beta_2}((s_k)_{k\in\mathbb{N}_0}), \varsigma((s_k)_{k\in\mathbb{Z}^-}), \pi_{\tau_1,\tau_2}((s_k)_{k\in\mathbb{N}_0})))$

Proposition 3.1.3.

$\hat{\pi}_\vartheta$ is continuous and surjective. Moreover it is bijective from $\bar{\Sigma}$ onto $\hat{\Lambda}_\vartheta$ and conjugates the backward shift map σ^{-1} and $\hat{f}_\vartheta$ on $\bar{\Sigma}$.

Proof

It is obvious that $\hat{\pi}_\vartheta$ is continuous. Treating the third component in the same way as the first we see that $\hat{\pi}_\vartheta$ conjugates σ^{-1} and $\hat{f}_\vartheta$ on $\bar{\Sigma}$ using the arguments of the proof of 3.1.2. . That the map is onto $\hat{\Lambda}_\vartheta$ and one to one restricted to $\bar{\Sigma}$ follows from proposition 3.1.1. and the properties of the map ς.

□

Given a shift coding it is easy deduce interesting properties of a dynamical system. We say that a topological dynamical system (A,T) has **strange dynamics**, if it has the following properties:

(1) There are periodic orbits of all periods for T in A
(2) The set of periodic points of T is dense in A
(3) There are orbits of T which are dense in A.

Property (3) is known as topological transitivity of the system (A,T). From the propositions of these section we get the following corollary:

Corollary 3.1.4

The dynamical systems $([-1,1]^2, f_{\beta_1,\beta_2})$, $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ and $(\Lambda_\vartheta, T_\vartheta)$ have strange dynamics.

Proof

It is easy to see that the systems (Σ^+, σ), and $(\bar{\Sigma}, \sigma^{-1})$ have strange dynamics. Since all our coding maps are surjective and continuous it follows that all our systems have properties (2) and (3). Since π_ϑ and the restriction of $\hat{\pi}_\vartheta$ to $\bar{\Sigma}$ are bijective (1) holds for the systems $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ and $(\Lambda_\vartheta, T_\vartheta)$. We observe that the points of a periodic orbit in $\bar{\Sigma}$ have different images under $\bar{\pi}_{\beta_1,\beta_2}$ since the component given by ς is different. Thus (1) holds for $([-1,1]^2, f_{\beta_1,\beta_2})$ as well.

□

3.2. Representation of ergodic measures

Given a compact metric space X we denote by $M(X)$ the set of all Borel probability measures on X. With the weak* topology $M(X)$ becomes a compact, convex and metricable space. If T is a Borel measurable transformation on X we call a measure μ T-invariant if $\mu \circ T^{-1} = \mu$. The set of all invariant measures forms a compact, convex and nonempty subset of $M(X)$. A invariant measure μ is called ergodic if $T^{-1}B = B \Rightarrow \mu(B) \in \{0,1\}$ holds for all Borel sets B in X. $M(X,T) := \{\mu \in M(X) | \mu \;\; T\text{-ergodic}\}$ is compact, convex and nonempty. It consists of the extreme points of the set of invariant measures. By b^p for $p \in (0,1)$ we denote the Bernoulli measure on Σ resp. Σ^+, which is the product of the discrete measure giving 1 the probability p and -1 the probability $(1-p)$. We write b for the equal-weighted Bernoulli measure $b^{0.5}$. The Bernoulli measures are ergodic with respect to forward and backward shifts. For these basic facts in ergodic theory we refer to the book of Denker, Grillenberger and Sigmund [DGS].
We will need one more definition. Given b^p on $\{-1,1\}^{\mathbb{Z}^-}$ we define the corresponding Bernoulli measure ℓ^p on $[-1,1]$ by $\ell^p = b^p \circ \varsigma^{-1}$. $\ell := \ell^{0.5}$ is the normalized Lebesgue measure on [-1,1].

We will now introduce the measures we study in the context of our dynamical systems.
Let $\mu \in M(\Sigma^+, \sigma)$ and $\gamma_1, \gamma_2 \in (0,1)$. We define two Borel probability measures on the real line by

$$\mu^*_{\gamma_1,\gamma_2} = \mu \circ (\pi^*_{\gamma_1,\gamma_2})^{-1} \quad \text{and} \quad \mu_{\gamma_1,\gamma_2} = \mu \circ (\pi_{\gamma_1,\gamma_2})^{-1}.$$

The measure μ_{γ_1,γ_2} is just $\mu^*_{\gamma_1,\gamma_2}$ scaled on the interval $[-1,1]$ by the transformation L_{γ_1,γ_2}:

$$\mu_{\gamma_1,\gamma_2} = \mu^*_{\gamma_1,\gamma_2} \circ L^{-1}_{\gamma_1,\gamma_2}.$$

If $\gamma_1 + \gamma_2 \geq 1$ we say that μ_{γ_1,γ_2} is overlapping, if not, we say that this measure is non-overlapping. For $\gamma \in (0,1)$ we write μ_γ instead of $\mu_{\gamma,\gamma}$ and call this measure symmetric.
$b^p_{\gamma_1,\gamma_2}$ for a Bernoulli measure b^p is a **self-similar measure** in the sense of the following proposition:

Proposition 3.2.1.

For all $p \in (0,1)$ and all $\gamma_1, \gamma_2 \in (0,1)$. the relation $b^p_{\gamma_1,\gamma_2} = p b^p_{\gamma_1,\gamma_2} \circ S_1 + (1-p) b^p_{\gamma_1,\gamma_2} \circ S_2$ holds with $S_1(x) = \gamma_1^{-1} x + (1-\gamma_1^{-1})$ and $S_2(x) = \gamma_2^{-1} x - (1-\gamma_2^{-1})$.

Proof

$b^p_{\gamma_1,\gamma_2}(B) = b^p(\pi^{-1}_{\gamma_1,\gamma_2}(B))$
$= b^p(\{\underline{s} | s_1 = 1 \wedge \pi_{\gamma_1,\gamma_2}(\underline{s}) \in B\}) + b^p(\{\underline{s} | s_1 = -1 \wedge \pi_{\gamma_1,\gamma_2}(\underline{s}) \in B\})$
$= b^p(\{\underline{s} | s_1 = 1 \wedge S_1^{-1} \circ \pi_{\gamma_1,\gamma_2} \circ \sigma(\underline{s}) \in B\}) + b^p(\{\underline{s} | s_1 = -1 \wedge S_2^{-1} \circ \pi_{\gamma_1,\gamma_2} \circ \sigma(\underline{s}) \in B\})$
$= b^p(\{\underline{s} | s_1 = 1 \wedge \sigma(\underline{s}) \in \pi^{-1}_{\gamma_1,\gamma_2}(S_1(B))\}) + b^p(\{\underline{s} | s_1 = -1 \wedge \sigma(\underline{s}) \in \pi^{-1}_{\gamma_1,\gamma_2}(S_2(B))\})$
$= p\ b^p_{\gamma_1,\gamma_2} \circ S_1(B) + (1-p)\ b^p_{\gamma_1,\gamma_2} \circ S_2(B)$ holds for all Borel subsets B of the real line.

□

The symmetric self-similar measures b^p_γ are usually called **infinite convolved Bernoulli measures** because of the following fact:

Proposition 3.2.2.

The measures b^p_γ are given by the infinite convolution of the discrete measures $b^{p,n}_\gamma$, which give $(1-\gamma)\gamma^n$ the probability p and $-(1-\gamma)\gamma^n$ the probability $(1-p)$.

Proof

b^p_γ is obviously the distribution of the random variable $Y^p_\gamma = \sum_{n=0}^\infty X^{p,n}(1-\gamma)\gamma^n$ where $X^{p,n}$ are independent random variables taking the values 1 and -1 with probability p resp. $1-p$. It is well known that the distribution of the sum of independent random variables is the convolution of the distributions of these random variables. But the distribution of $X^{p,n}(1-\gamma)\gamma^n$ is given by the measure $b^{p,n}_\gamma$.

□

We remark that we do not have a convolution structure for the asymmetric measures $b^p_{\beta_1,\beta_2}$. We can not write the measure as a distribution of the sum of independent random variables in this case, because the term that is added randomly at the n'th step depends on the terms that were added before.

In chapter six and nine we will continue with the discussion of the measures defined here.

Now we go to characterize the ergodic measures for the dynamical system $(\Lambda_\vartheta, T_\vartheta)$.

Proposition 3.2.3.

The map $\mu \longrightarrow \mu_\vartheta := \mu \circ \pi_\vartheta^{-1}$ is a affine homeomorphism from $M(\Sigma^+, \sigma)$ onto $M(\Lambda_\vartheta, T_\vartheta)$. If $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2)$ then the projection of μ_ϑ onto the x-axis is μ_{β_1,β_2} and the projection onto the z-axis is μ_{τ_1,τ_2}.

Proof

The first statement follows from proposition 3.1.1 using proposition 3.11. of [DGS] and the remark on page 24 of [DGS]. The second statement is a direct consequence of the product structure of the map π_ϑ and the definition of the involved measures.

□

We now describe all ergodic measures for the dynamical system $([-1,1]^2, f_{\beta_1,\beta_2})$. We will need pr_+, the projection from Σ onto Σ^+.

Proposition 3.2.4.

$\mu \longmapsto \bar{\mu}_{\beta_1,\beta_2} := \mu \circ \bar{\pi}^{-1}_{\beta_1,\beta_2}$ is a continuous affine map from $M(\Sigma, \sigma)$ onto $M([-1,1]^2, f_{\beta_1,\beta_2})$. The projection of $\bar{\mu}_{\beta_1,\beta_2}$ onto the x-axis is the measure $(pr^+\mu)_{\beta_1,\beta_2}$ and $\bar{b}^p_{\beta_1,\beta_2}$ is the product of $b^p_{\beta_1,\beta_2}$ with ℓ^p.

Proof

Since $\bar{\pi}_{\beta_1,\beta_2}$ is surjective and continuous we get from proposition 3.1. of [DGS] that $\mu \longmapsto \bar{\mu}_{\beta_1,\beta_2} := \mu \circ \bar{\pi}^{-1}_{\beta_1,\beta_2}$ is a continuous affine map from $M(\Sigma)$ onto $M([-1,1]^2)$. If μ is shift invariant we obviously have $\mu(\bar{\Sigma}) = 1$. Because we know from proposition 3.1.2. that π_{β_1,β_2} conjugates the backward shift and f_{β_1,β_2} on $\bar{\Sigma}$ we get that $\bar{\mu}_{\beta_1,\beta_2}$ is f_{β_1,β_2} ergodic if μ is shift ergodic.
It remains to show that the map is onto $M([-1,1]^2, f_{\beta_1,\beta_2})$ restricted to $M(\Sigma, \sigma)$. So let us choose an arbitrary measure ξ in $M([-1,1]^2, f_{\beta_1,\beta_2})$.

We first want to show that $\xi(\pi_{\beta_1,\beta_2}(\Sigma\backslash\bar{\Sigma}))=0$. Let D be set of all numbers of the form $k/2^n$ with $n\in I\!N$ and $|k|\le n-1$. A direct calculation shows that:

$$\pi_{\beta_1,\beta_2}(\Sigma\backslash\bar{\Sigma})=(D\times[-1,1])\cup(\{1\}\times[-1,1))=(\bigcup_{k=0}^{\infty}f_{\beta_1,\beta_2}^{-k}(\{0\}\times[1,-1]))\cup(\{1\}\times[-1,1)).$$

Recall that the measure ξ is in particular shift invariant. Hence the measure of the first set in union is zero because it is given by a disjunct infinite union of sets with the same measure. The measure of the second set is zero since $\{1\}\times[-1,1)\subseteq f_{\beta_1,\beta_2}^{-k}(\{1\}\times[1-2\beta_1^k,1))\ \ \forall\ k\ge 0$.
Now take $\mu_{pre}\in M(\Sigma)$ such that $\mu_{pre}\circ\pi_{\beta_1,\beta_2}^{-1}=\xi$. μ_{pre} is not necessary shift invariant so we define a measure μ as a weak* accumulation point of the sequence

$$\mu_n:=\frac{1}{n+1}\sum_{i=0}^{n}\mu_{pre}\circ\sigma^{-n}.$$

From the considerations above we have $\mu_{pre}(\bar{\Sigma})=1$ and hence:

$$\mu_n\circ\pi_{\beta_1,\beta_2}^{-1}=\frac{1}{n+1}\sum_{i=0}^{n}\mu_{pre}\circ\sigma^{-n}\circ\pi_{\beta_1,\beta_2}^{-1}$$

$$=\frac{1}{n+1}\sum_{i=0}^{n}\mu_{pre}\circ\pi_{\beta_1,\beta_2}^{-1}\circ f_{\beta_1,\beta_2}^{-i}=\frac{1}{n+1}\sum_{i=0}^{n}\xi\circ f_{\beta_1,\beta_2}^{-i}=\xi.$$

Thus $\bar{\mu}_{\beta_1,\beta_2}$ is just the measure ξ and μ is shift invariant by definition. We have thus shown that the set $M(\xi):=\{\mu|\mu\ \sigma\text{-invariant and }\mu_{\beta_1,\beta_2}=\xi\}$ of Borel measures on Σ is not empty. Since the map $\mu\longmapsto\bar{\mu}_{\beta_1,\beta_2}$ is continuous and affine on the set of σ-invariant measures we know that $M(\xi)$ is compact and convex. It is a consequence of Krein-Milman theorem that there exists an extremal point μ of $M(\xi)$.
We claim that μ is an extremal point of the set of all σ-invariant Borel measures on Σ and hence ergodic.
If this is not the case then we have $\mu=t\mu_1+(1-t)\mu_2$ where $t\in(0,1)$ and μ_1,μ_2 are two distinct σ-invariant measures. This implies $\xi=t(\mu_1)_{\beta_1,\beta_2}+(1-t)(\mu_2)_{\beta_1,\beta_2}$. Since ξ is ergodic we have $(\mu_1)_{\beta_1,\beta_2}=(\mu_2)_{\beta_1,\beta_2}=\xi$ and hence $\mu_1,\mu_2\in M(\xi)$. This is a contradiction to μ being extremal in $M(\xi)$.

Now we calculate the projection: $pr_X\bar{\mu}_{\beta_1,\beta_2}(B)=\bar{\mu}_{\beta_1,\beta_2}(B\times[-1,1])=\mu(\bar{\pi}_{\beta_1,\beta_2}^{-1}(B\times I))=\mu(\{-1,1\}^{\mathbb{Z}_0^-}\times\pi_{\beta_1,\beta_2}^{-1}(B))=\mu(pr_+^{-1}(\pi_{\beta_1,\beta_2}^{-1}(B)))=pr_+\mu(\pi_{\beta_1,\beta_2}^{-1}(B))=(pr_+\mu)_{\beta_1,\beta_2}(B)$

Since the measure b^p on Σ is the product of b^p on $\{-1,1\}^{\mathbb{Z}_0^-}$ and b^p on Σ^+ and ς maps b^p on $\{-1,1\}^{\mathbb{Z}_0^-}$ to ℓ^p we see that $\bar{b}_{\beta_1,\beta_2}=b_{\beta_1,\beta_2}\times\ell^p$.

□

Let us now give an analysis of ergodic measures for the lifts $(\hat{\Lambda}_\vartheta,\hat{f}_\vartheta)$.

Proposition 3.2.5.

The map $\mu \longmapsto \hat{\mu}_\vartheta := \mu \circ \hat{\pi}_\vartheta^{-1}$ is a affine homeomorphism from $M(\Sigma, \sigma)$ onto $M(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$. The projection of $\hat{\mu}_\vartheta$ on the (x, z)-plane is $(pr_+\mu)_\vartheta$. Moreover $\hat{b}^p_\vartheta$ is a product of b^p_ϑ with ℓ^p. If $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2)$ then the projection of $\hat{\mu}_\vartheta$ on the (x, y)-plane is $\bar{\mu}_{\beta_1,\beta_2}$.

Proof

By the same reasoning we used in the proof of 3.2.3. we can show that the map is continuous, affine and surjective. We will use the fact that $\hat{\pi}_\vartheta$ restricted to $\bar{\Sigma}$ is a bijection to show that the map is invertible.
Given a measure $\xi \in M(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ we define a measure $F\xi \in M(\Sigma)$ by $F\xi(B) = \xi(\hat{\pi}_\vartheta(B))$ for all Borel sets in Σ. Let μ be in $M(\Sigma, \sigma)$ and B be a Borel set in Σ. We have $F\mu_\vartheta(B) = \mu(\pi_\vartheta^{-1}(\pi_\vartheta(B))) = \mu(\bar{\Sigma} \cap \pi_\vartheta^{-1}(\pi_\vartheta(B))) = \mu(\bar{\Sigma} \cap B) = \mu(B)$. Hence F is the inverse map to $\mu \longrightarrow \hat{\mu}_\vartheta$. The continuity of F follows by the compactness of $M(\Sigma, \sigma)$.
The first statement about the projections follows in the same way as our projection result in 3.2.4. . The second statement is is obvious since $pr_{XY}\hat{\pi}_\vartheta = \bar{\pi}_{\beta_1,\beta_2}$.

□

From the proposition of this sections and the propositions of the last section we get a corollary about the metric entropy of the measure theoretical dynamical systems we study. For the definition of the metric entropy $h_\mu(T)$ of a dynamical system (X, T) with an invariant measure μ and a treatment of the properties of this quantity we recommend [WA], [DGS] or [KH]. To get the corollary we cab use for instance proposition 11.14. of [WA].

Corollary 3.2.6.

$h_{\mu_\vartheta}(T_\vartheta) = h_\mu(\sigma)$ holds for all $\mu \in M(\Sigma^+, \sigma)$ and $h_{\mu_\vartheta}(\hat{f}_\vartheta) = h_\mu(\sigma)$ holds for all $\mu \in M(\Sigma, \sigma)$.

We also get the inequality $h_{\bar{\mu}_{\beta_1,\beta_2}}(f_{\beta_1,\beta_2}) \leq h_\mu(\sigma)$ as a corollary of 3.1.2. and 3.2.3. . In fact even the identity $h_{\bar{\mu}_{\beta_1,\beta_2}}(f_{\beta_1,\beta_2}) = h_\mu(\sigma)$ holds. This is easy to see for Bernoulli measures by projecting the systems onto the y-axis using the product structure of $\bar{b}^p_{\beta_1,\beta_2}$ but more difficult if we consider other measures. We will sketch a proof of this identity using conditional measures and dimensions in 5.3.6. .

4. Calculation of box-counting dimension

Before we discuss the dimension of measures and the Hausdorff dimension of sets in the context of our dynamical systems we calculate here the box-counting dimension of the repellers Λ_ϑ and the attractors $\hat{\Lambda}_\vartheta$ defined in chapter two. We refer to appendix A for the definition of the various kinds of dimension and basic facts in dimension theory.

Theorem 4.1.

If $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$ and d is the unique positive number satisfying

$$\beta_1 \tau_1^d + \beta_2 \tau_2^d = 1$$

then

$$\dim_B \Lambda_\vartheta = d + 1 \text{ and } \dim_B \hat{\Lambda}_\vartheta = d + 2.$$

Let us make a few remarks on this theorem:

Remarks

(1) A simple calculation shows that our theorem is consistent with the result of Pollicott and Weiss [PW] in the special case $\beta_1 = \beta_2 =: \beta$ and $\tau_1 = \tau_2 =: \tau$ (see theorem 2.1.3.).

(2) Recall from the classical work of Moran [MO] that the Hausdorff and box-counting dimension of a self-similar Cantor set induced by by transformations with contraction rates τ_1 and τ_2 is given by the solution of $\tau_1^d + \tau_2^d = 1$. There is an analogy to our formula. In our setting of an self-affine set with overlaps in the projections the contraction rates in the second direction induces weights in the dimension formula.

(3) The overlapping condition $\beta_1 + \beta_2 \geq 1$ is necessary for our formula to hold. In the case $\beta_1 + \beta_2 < 1$ the Hausdorff and box-Counting dimension of the self-affine set is given by the bigger solution of the equations $\beta_1^x + \beta_2^x = 1$ and $\tau_1^x + \tau_2^x = 1$. This can be shown by transferring the arguments of Pollicott and Weiss [PW] in the non overlapping symmetric to the non overlapping asymmetric situation .

(4) It may be interesting to notice that it follows from our result and the implicit function theorem that the function

$$\vartheta \longmapsto \dim_B \Lambda_\vartheta$$

is C^∞ on the interior of P^4_{all}.

Proof of 4.1.

Let $f(t) = \beta_1\tau_1^t + \beta_2\tau_2^t$. Since $f(0) = \beta_1 + \beta_2 \geq 1$ and f is strictly monotonous decreasing with $\lim_{t\longrightarrow\infty} f(t) = 0$ there is an unique positive number d with $\beta_1\tau_1^d + \beta_2\tau_2^d = 1$. Fix d.
Given a real number $r > 0$ we define a set of finite sequences by

$$X_r := \{(s_1, \ldots, s_k) | \min\{\tau_1, \tau_2\}r \leq \tau_{s_1}\tau_{s_2}\ldots\tau_{s_k} < r \text{ where } s_j \in \{1, 2\}\ \forall j = 1 \ldots k\}.$$

Notice that the sequences in X_r have not the same length. Let $\bar{k}(r)$ be the maximal length of a sequence in X_r. We observe that for every sequence $(s_j) \in \{1, 2\}^{\bar{k}(r)}$ there is an unique k such that $(s_1, \ldots, s_k) \in X_r$. Thus we get

$$\sum_{(s_1,\ldots,s_k)\in X_r} \beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^d$$

$$= \sum_{(s_1,\ldots,s_k)\in X_r} \beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^d(\beta_1\tau_1^d + \beta_2\tau_2^d)^{\bar{k}(r)-k}$$

$$= \sum_{(s_1,\ldots,s_{\bar{k}(r)})\in\{1,2\}^{\bar{k}(r)}} \beta_{s_1}\beta_{s_2}\ldots\beta_{s_{\bar{k}(r)}}(\tau_{s_1}\tau_{s_2}\ldots\tau_{s_{\bar{k}(r)}})^d = (\beta_1\tau_1^d + \beta_2\tau_2^d)^{\bar{k}(r)} = 1 \qquad (1).$$

Beside equitation (1) we need one more fact. Let v be the unique positive number satisfying $\tau_1^v + \tau_2^v = 1$. Since $\tau_1 + \tau_2 < 1$ we have $v \leq 1 \leq d + 1$. Consequently

$$\sum_{(s_1,\ldots,s_k)\in X_r} (\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^{d+1} \leq \sum_{(s_1,\ldots,s_k)\in X_r} (\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k})^v = 1 \qquad (2).$$

Now we are prepared to begin with the main proof. We define a cover of Λ_ϑ by

$$C_r = \{\pi_\vartheta([\kappa(s_1), \ldots, \kappa(s_k)]_0) | (s_1, \ldots, s_k) \in X_r\}$$

where $\kappa(1) = 1$ and $\kappa(2) = -1$. Since $\{[\kappa(s_1), \ldots, \kappa(s_k)]_0 | (s_1, \ldots, s_k) \in X_r\}$ is a cover of Σ^+ we get from 3.1.1. that C_r is in fact a cover of Λ_ϑ.
An element of C_r is a rectangle parallel to the axis with x-length $2\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}$ and y-length $2\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}$. We cover each of this rectangles by squares parallel to the axis of side length $2\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}$. We choose the squares in a row such that they only intersect in their boundary. So we get for each rectangle a covering by $\lceil\frac{\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}}\rceil$ squares (here $\lceil x\rceil$ denotes the smallest integer bigger than x). In this way we obtain a new cover $\hat{C}_r$ of Λ_ϑ, which consists of squares with side length in $(2\min\{\tau_1, \tau_2\}r, 2r]$. Furthermore the number $\hat{N}(r)$ of elements in $\hat{C}_r$ is given by

$$\hat{N}(r) = \sum_{(s_1,\ldots,s_k)\in X_r} \lceil\frac{\beta_{s_1}\beta_{s_2}\ldots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\ldots\tau_{s_k}}\rceil.$$

Now we have the following upper estimate,

$$\hat{N}(r)r^{d+1} \leq \min\{\tau_1,\tau_2\}^{-(d+1)} \sum_{(s_1,\dots,s_k)\in X_r} \lceil \frac{\beta_{s_1}\beta_{s_2}\dots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\dots\tau_{s_k}} \rceil (\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d+1}$$

$$\leq \min\{\tau_1,\tau_2\}^{-(d+1)} (\sum_{(s_1,\dots,s_k)\in X_r} \beta_{s_1}\beta_{s_2}\dots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d} + \sum_{(s_1,\dots,s_k)\in X_r} (\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d+1})$$

$$\leq^{(1)/(2)} 2\min\{\tau_1,\tau_2\}^{-(d+1)}$$

and the following lower estimate:

$$\hat{N}(r)r^{d+1} \geq \sum_{(s_1,\dots,s_k)\in X_r} \lceil \frac{\beta_{s_1}\beta_{s_2}\dots\beta_{s_k}}{\tau_{s_1}\tau_{s_2}\dots\tau_{s_k}} \rceil (\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d+1}$$

$$\geq \sum_{(s_1,\dots,s_k)\in X_r} \beta_{s_1}\beta_{s_2}\dots\beta_{s_k}(\tau_{s_1}\tau_{s_2}\dots\tau_{s_k})^{d} =^{(1)} 1.$$

Now let $N(r)$ be the minimal cardinality of an arbitrary cover of Λ_ϑ with squares parallel to the axis of side length $2r$. Obviously we have $N(r) \leq \hat{N}(r)$ but we need another argument for an opposite estimate.
Let R be a rectangle in the cover C_r. We see that the projection of $\Lambda_\vartheta \cap C_r$ on the x-axis has the full x-length of the rectangle since we assumed $\beta_1 + \beta_2 \geq 1$. This implies that the intersection of each square in $\hat{C}_r$ with Λ_ϑ is not empty. Thus if we have a cover of Λ_ϑ each element of $\hat{C}_r$ has to be intersected by at least one element of the cover. But one square with side length $2r$ can not intersect more than $9\min\{\tau_1,\tau_2\}^{-2}$ squares in $\hat{C}_r$ because the squares in $\hat{C}_r$ have side length bigger than $2\min\{\tau_1,\tau_2\}r$ and intersect, if at all, only in the boundary. It follows that $N(r) \geq 1/9\min\{\tau_1,\tau_2\}^2\hat{N}(r)$.
Putting our estimates together we obtain

$$\frac{1}{9}\min\{\tau_1,\tau_2\}^2 \leq N(r)r^{d+1} \leq 2\min\{\tau_1,\tau_2\}^{-(d+1)}$$

and hence

$$\dim_B \Lambda_\vartheta = \lim_{r\longrightarrow\infty} \frac{\log N(r)}{\log(2r)^{-1}} = \lim_{r\longrightarrow\infty} \frac{\log N(r)}{\log r^{-1}} = d+1.$$

The formula $\dim_B \hat{\Lambda}_\vartheta = d+2$ follows from the product structure of $\hat{\Lambda}_\vartheta$ and proposition A5 of appendix A. So our proof is complete.

□

An analysis of the Hausdorff dimension of the sets Λ_ϑ and $\hat{\Lambda}_\vartheta$ is very difficult. We we will present our results in chapter seven and ten.

5. Dimension formulas and estimates for ergodic measures

5.1. Lyapunov exponents and charts

In this chapter we want to apply the general dimension theory of ergodic measures that was developed in the last twenty years (see [YO], [LY], [BPS] and references there in) to the systems we study. Our aim is to find formulas and upper bounds for the dimension of ergodic measures for $(\Lambda_\vartheta, T_\vartheta)$, $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ and $([-1,1]^2, f_{\beta_1,\beta_2})$ in terms of Lyapunov exponents, metric entropy and the dimension of the measures μ_{β_1,β_2}. In this section we do some preparations, namely we show the existence of Lyapunov exponents and charts related to a measure $\hat{\mu}_\vartheta$ on $\hat{\Lambda}_\vartheta$ and calculate the exponents.

Lemma 5.1.1.

There is a subset $\Omega_\vartheta \subseteq \hat{\Lambda}_\vartheta$ which has full measure for a all $\hat{\mu}_\vartheta \in M(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ such that $\hat{f}_\vartheta$ is a bijection on Ω_ϑ and $\hat{f}_\vartheta$ is differentiable for all $\mathbf{x} = (x, y, z) \in \Omega_\vartheta$ with

$$D_{\mathbf{x}}\hat{f}_\vartheta = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \tau_{\tau_1} \end{pmatrix} \quad \text{if} \quad y > 0 \qquad D_{\mathbf{x}}\hat{f}_\vartheta = \begin{pmatrix} \beta_2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix} \quad \text{if} \quad y < 0.$$

Proof

Denote by S the singularity $[-1,1] \times \{0\} \times [-1,1]$ of the system and define the set Ω_ϑ by

$$\Omega_\vartheta = \bigcap_{n=-\infty}^{\infty} \hat{f}_\vartheta^n([-1,1]^3 \backslash S).$$

By definition we have $\hat{f}_\vartheta(\Omega_\vartheta) = \Omega_\vartheta$ and since $\hat{f}_\vartheta$ is injective it is in fact a bijection on Ω_ϑ. Moreover if $\mathbf{x} \in \Omega_\vartheta$ then $\mathbf{x} \notin S$ and hence $\hat{f}_\vartheta$ is differentiable and has obviously the derivative that we stated in the lemma. We only have to show now that $\mu_\vartheta(\Omega_\vartheta) = 1$. By elemental calculations we see that

$$\Omega_\vartheta = (\{(x,y,z) \in \hat{\Lambda}_\vartheta | y \neq 1, \quad y \neq -1\} \cup \{(1,1,1), (-1,-1,-1)\}) \backslash \bigcup_{n=0}^{\infty} f^{-n}(S).$$

Since $\hat{\mu}_\vartheta$ is invariant and the union in the expression above is disjoint it has zero measure. It remains to show that $\hat{\mu}_\vartheta([-1,1] \times \{1\} \times [-1,1]) = \hat{\mu}_\vartheta(\{(1,1,1)\}$ and $\hat{\mu}_\vartheta([-1,1] \times \{-1\} \times [-1,1]) = \hat{\mu}_\vartheta(\{(-1,-1,-1)\}$. But this is obvious since $\hat{f}_\vartheta$ is just a contraction with fixed point $(1,1,1)$ resp. $(-1,-1,-1)$ on the sets $[-1,1] \times \{1\} \times [-1,1]$ resp. $[-1,1] \times \{1\} \times [-1,1]$.

Now define linear subspaces of $I\!R^3$ by

$$E^u =< \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} > \quad E^s =< \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} >$$

$$E^{tau} =< \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} > \quad E^{beta} =< \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} > .$$

Given a Borel measure μ on Σ and $\gamma_1, \gamma_2 \in (0,1)$ we set

$$\Xi^{\mu}_{\gamma_1,\gamma_2} = \mu([1]_0)\log\gamma_1 + \mu([-1]_0)\log\gamma_2.$$

Proposition 5.1.2.

Given $\mu \in M(\Sigma,\sigma)$ and $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2) \in P^4_{all}$ we have for $\hat{\mu}_\vartheta$-almost all $\mathbf{x} \in \hat{\Lambda}_\vartheta$.

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log ||D_{\mathbf{x}} f^n_\vartheta v|| = \log 2 \qquad \forall v \in E^u$$

$$\text{If} \quad \Xi^{\mu}_{\beta_1,\beta_2} \geq \Xi^{\mu}_{\tau_1,\tau_2} \ : \ \lim_{n\longrightarrow\infty} \frac{1}{n} \log ||D_{\mathbf{x}} f^n_\vartheta v|| = \{ \begin{array}{lll} \Xi^{\mu}_{\beta_1,\beta_2} & \text{if} & v \in E^s \backslash E^{tau} \\ \Xi^{\mu}_{\tau_1,\tau_2} & \text{if} & v \in E^{tau} \end{array}$$

$$\text{If} \quad \Xi^{\mu}_{\beta_1,\beta_2} \leq \Xi^{\mu}_{\tau_1,\tau_2} \ : \ \lim_{n\longrightarrow\infty} \frac{1}{n} \log ||D_{\mathbf{x}} f^n_\vartheta v|| = \{ \begin{array}{lll} \Xi^{\mu}_{\tau_1,\tau_2} & \text{if} & v \in E^s \backslash E^{beta} \\ \Xi^{\mu}_{\beta_1,\beta_2} & \text{if} & v \in E^{beta} \end{array}$$

Proof

By lemma 5.1.1. we have for $\hat{\mu}_\vartheta$-almost all $\mathbf{x} \in \hat{\Lambda}_\vartheta$

$$\log ||D_{\mathbf{x}} f^n_\vartheta(\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix})|| = n \log 2 + \log y \qquad \forall n \geq 0.$$

This implies our claim about E^u. Now we look at E^s. By lemma 5.1.1. and proposition 3.1.3. and 3.2.5. we have for $\hat{\mu}_\vartheta$-almost all $\mathbf{x} \in \hat{\Lambda}_\vartheta$

$$\log ||D_{\mathbf{x}} f^n_\vartheta(\begin{pmatrix} x \\ 0 \\ z \end{pmatrix})|| = \log \sqrt{(x\beta_1^{n-\sharp_n(\underline{s})+1}\beta_2^{\sharp_n(\underline{s})})^2 + (z\tau_1^{n-\sharp_n(\underline{s})+1}\tau_2^{\sharp_n(\underline{s})})^2} \qquad \forall n \geq 0$$

where $\underline{s} = (s_k) = \hat{\pi}_\vartheta^{-1}(\mathbf{x})$ and $\sharp_n(\underline{s})$ counts the number of entrys in the set $\{s_0, s_{-1}, \ldots, s_{-n}\}$ that are -1.

We now have to determine the limit of this expression for μ- almost all $\underline{s} \in \Sigma$. By Birkhoffs ergodic theorem (see 4.1.2. of [KH]) we have:

$$\lim_{n\longrightarrow\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(\sigma^{-k}(\underline{s})) = \int f d\mu \qquad \mu - \text{a.e.}$$

for all L^1 functions f on Σ with respect to μ. Applying this to the functions

$$f_{beta}(\underline{s}) = \begin{array}{ll} \log\beta_1 & \text{if} \quad s_0 = 1 \\ \log\beta_2 & \text{if} \quad s_0 = -1 \end{array} \qquad f_{tau}(\underline{s}) = \begin{array}{ll} \log\tau_1 & \text{if} \quad s_0 = 1 \\ \log\tau_2 & \text{if} \quad s_0 = -1 \end{array}$$

we obtain

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log \beta_1^{n-\bar{\sharp}_n(\underline{s})+1} \beta_2^{\bar{\sharp}_n(\underline{s})} = \Xi^{\mu}_{\beta_1,\beta_2} \text{ and } \lim_{n\longrightarrow\infty} \frac{1}{n} \log \tau_1^{n-\bar{\sharp}_n(\underline{s})+1} \tau_2^{\bar{\sharp}_n(\underline{s})} = \Xi^{\mu}_{\tau_1,\tau_2} \quad \mu - \text{ a.e.}$$

and from this by elemental calculus

$$\lim_{n\longrightarrow\infty} \frac{1}{n} \log \sqrt{(x\beta_1^{n-\bar{\sharp}_n(\underline{s})+1} \beta_2^{\bar{\sharp}_n(\underline{s})})^2 + (z\tau_1^{n-\bar{\sharp}_n(\underline{s})+1} \tau_2^{\bar{\sharp}_n(\underline{s})})^2} = \max\{\Xi^{\mu}_{\beta_1,\beta_2}, \Xi^{\mu}_{\tau_1,\tau_2}\} \quad \mu - \text{ a.e.}$$

if $x \neq 0$ and $y \neq 0$. This implies our claims about the stable directions.

□

This proposition means that Lyapunov exponents exists almost everywhere for the systems $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta, \hat{\mu}_\vartheta)$ if μ is ergodic. E^u is the unstable direction with Lyapunov exponent $\log 2$ and E^s is the stable direction with exponent $\Xi^{\mu}_{\beta_1,\beta_2}$ or $\Xi^{\mu}_{\tau_1,\tau_2}$ depending on which quantity is bigger. Accordingly E^{tau} or E^{beta} is the strong stable direction with Lyapunov exponent $\Xi^{\mu}_{\tau_1,\tau_2}$ resp. $\Xi^{\mu}_{\beta_1,\beta_2}$.

In order to guarantee the existence of Lyapunov charts associated with the Lyapunov exponents we have to show that the set of points that does not approach the singularity $S := [-1,1] \times \{0\} \times [-1,1]$ with exponential rate has full measure. Precisely we have:

Lemma 5.1.3.

Given $\mu \in M(\Sigma,\sigma)$ and $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2) \in P^4_{all}$ we have for all $\epsilon > 0$

$$\hat{\mu}_\vartheta(\{\mathbf{x} \in \hat{\Lambda}_\vartheta | \exists l > 0 \ \forall n > 0 \ \ d(f^n(\mathbf{x}), S) > (1/l)e^{-\epsilon n}\}) = 1,$$

Proof

Fix $\epsilon > 0$. First note that it is sufficient if we show

$$\hat{\mu}_\vartheta(\{\mathbf{x} \in \hat{\Lambda}_\vartheta | \exists (n_k)_{k\in I\!N} \longrightarrow \infty \ \forall k > 0 \ d(f^{n_k}(\mathbf{x}), S) \leq e^{-\epsilon n_k}\}) = 0$$

because if we have for a point $\mathbf{x}$ that $\exists n_0 \forall n > n_0 \ d(f^n(\mathbf{x}), S) > e^{-\epsilon n}$ then there exists $l > 0$ such that $d(f^n(\mathbf{x}), S) > (1/l)e^{-\epsilon n} \ \forall n > 0$.
By 3.1.3. and the definition of the measure $\hat{\mu}_\vartheta$ this assertion is equivalent to the following statement about the symbolic system $(\Sigma, \sigma^{-1}, \mu)$:

$$\mu(N) = 0 \text{ where } N := \{\underline{s} \in \hat{\Sigma} | \exists (n_k)_{k\in I\!N} \longrightarrow \infty \ \forall k > 0 \ d(\sigma^{-n_k}(\underline{s}), \tilde{S}) \leq e^{-\epsilon n_k}\}$$

and $\tilde{S} = \{\underline{s} \in \Sigma | s_{-1} = 1 \text{ and } s_k = -1 \ \forall k < -1\}$. We will now prove this.
If $\underline{s} \in N$ we have $d(\sigma^{-n_k}(\underline{s}), \tilde{S}) \leq e^{-\epsilon n_k} \forall k > 0$ By the definition of the metric d this implies

$$\sigma^{-n_k}(\underline{s}) \in [\underbrace{-1, -1, \ldots, -1, 1}_{\lceil c\epsilon n_k \rceil}]_{-\lceil c\epsilon n_k \rceil - 1} \quad \forall k > 0$$

where the constant c is independent of ϵ, n_k and $\underline{s}$. This gives us:

$$\sigma^i(\underline{s}) \notin [1]_{-2} \quad i = n_k, \ldots, n_k + \lceil c\epsilon n_k \rceil - 1 \quad \forall k > 0.$$

Thus we have:

$$N \subseteq \{\underline{s} | \exists (n_k)_{k\in I\!N} \longrightarrow \infty \ \forall k > 0 \ : \ \sigma^i(\underline{s}) \notin [1]_{-2} \ \ i = n_k, \ldots, n_k + \lceil c\epsilon n_k \rceil - 1\}.$$

Applying lemma 7.1. of [ST2] for the ergodic system (Σ, σ, μ) (with $Y = [1]_{-2}$) we obtain $\mu(N) = 0$.

□

By this lemma the systems $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta, \hat{\mu}_\vartheta)$ fall into the class of generalized hyperbolic attractors in the sense of Schmeling and Troubetzkoy [ST1,2.1.]. From [ST1,3] it follows that our systems have appropriate Lyapunov charts almost everywhere with respect to the exponents given in 5.1.2. . For the definition and the constructed of these Lyapunov charts we reefer to [KS].

5.2. Exact dimensionality and Ledrappier Young formula

Usually the general theory for the dimension of ergodic measures is stated in the context of C^2-diffeomorphisms in order to guarantee the existence of Lyapunov exponents and charts. But invertibility and the existence of Lyapunov exponents and charts almost everywhere is enough to apply this theory. We refer to section 4 of [ST1] for this fact. This is of great importance for us. For the systems $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta, \hat{\mu}_\vartheta)$

we have shown invertibility and the existence of Lyapunov exponents and charts almost everywhere in last section. We are thus allowed to apply the general results found in [BPS], [LY] and [YO] in our context.
To this end first define partitions W^s and W^u of $[-1,1]^3$ in the stable and in the unstable directions of $\hat{f}_\vartheta$ by the partition elements

$$W^s(\mathbf{x}) = [-1,1] \times \{y\} \times [-1,1] \qquad W^u(\mathbf{x}) = \{x\} \times [-1,1] \times \{z\}$$

where $\mathbf{x} = (x,y,z) \in \hat{\Lambda}_\vartheta$. Given $\hat{\mu}_\vartheta \in M(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ we have conditional measures $\hat{\mu}^s_\vartheta(\mathbf{x})$ on W^s and $\hat{\mu}^u_\vartheta(\mathbf{x})$ on W^u. These measures are unique $\hat{\mu}_\vartheta$-almost everywhere fulfilling the relations:

$$\hat{\mu}_\vartheta(B) = \int \hat{\mu}^s_\vartheta(\mathbf{x})(B \cap W^s(\mathbf{x}))d\hat{\mu}_\vartheta(\mathbf{x}) \quad \text{resp.} \quad \hat{\mu}_\vartheta(B) = \int \hat{\mu}^u_\vartheta(\mathbf{x})(B \cap W^u(\mathbf{x}))d\hat{\mu}_\vartheta(\mathbf{x})$$

for all Borel sets B in $[-1,1]^3$. We refer to [LY] and [RO] for informations about conditional measures on measurable partitions.

Let us define balls in the elements of the partitions by

$$B^s_r((x,y,z)) = \{(\bar{x},\bar{y},\bar{z})|\bar{y} = y \text{ and } (\bar{x},\bar{z}) \in B_r(x,z)\},$$

$$B^u_r((x,y,z)) = \{(\bar{x},\bar{y},\bar{z})|\bar{x} = x \quad \bar{z} = z \text{ and } \bar{y} \in B_r(y)\}.$$

Now applying the results of Barreira, Schmeling and Pesin [BPS] to the system $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta, \hat{\mu}_\vartheta)$ we obtain:

Proposition 5.2.1.

Let $\mu \in M(\Sigma,\sigma)$, $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2) \in P^4_{all}$ and let $\hat{\mu}^s_\vartheta(\mathbf{x})$ be conditional measures on W^s and $\hat{\mu}^u_\vartheta(\mathbf{x})$ conditional measures on W^u with respect to $\hat{\mu}_\vartheta$. We have:

$$\overline{\underline{d^s}}(\mathbf{x},\hat{\mu}^s_\vartheta(\mathbf{x})) := \overline{\underline{\lim}}_{r\longrightarrow\infty} \frac{\log \hat{\mu}^s_\vartheta(\mathbf{x})(B^s_r(\mathbf{x}))}{\log r} = const. =: \dim \hat{\mu}^s_\vartheta \quad \hat{\mu}_\vartheta - a.e.$$

$$\overline{\underline{d^u}}(\mathbf{x},\hat{\mu}^u_\vartheta(\mathbf{x})) := \overline{\underline{\lim}}_{r\longrightarrow\infty} \frac{\log \hat{\mu}^u_\vartheta(\mathbf{x})(B^u_r(\mathbf{x}))}{\log r} = const. =: \dim \hat{\mu}^u_\vartheta \quad \hat{\mu}_\vartheta - a.e.$$

$$\overline{\underline{d}}(\mathbf{x},\hat{\mu}_\vartheta) := \overline{\underline{\lim}}_{r\longrightarrow\infty} \frac{\log \hat{\mu}_\vartheta(\mathbf{x})(B_r(\mathbf{x}))}{\log r} = \dim \hat{\mu}^u_\vartheta + \dim \hat{\mu}^s_\vartheta =: \dim \hat{\mu}_\vartheta \quad \hat{\mu}_\vartheta - a.e.$$

An introduction to the local dimension, which is used here, can be found in appendix A. The proposition means that the measure $\hat{\mu}_\vartheta$ is exact dimensional and that the dimension is given by the sum of the unstable and stable dimension resp. the local dimension of conditional measures on partitions in stable and unstable directions, which is almost everywhere constant.

Now we want to have some information about the quantities $\dim \hat{\mu}_\vartheta^u$ and $\dim \hat{\mu}_\vartheta^s$. This is easy for the unstable dimension, because this direction is one dimensional. The next proposition follows from Ledrappier and Young [LY] or from the work of Young [YO]:

Proposition 5.2.2.

Under the assumptions of 5.2.1. we have $\dim \hat{\mu}_\vartheta^u = h_{\hat{\mu}_\vartheta}(\hat{f}_\vartheta)/\log 2$.

An analysis of $\dim \hat{\mu}_\vartheta^s$ is more difficult because we have two unstable directions with different expansion rates. If $\Xi^\mu_{\beta_1,\beta_2} \geq \Xi^\mu_{\tau_1,\tau_2}$ we have a partition W^{ss} in the strong stable direction given by the partition elements

$$W^{ss}(\mathbf{x}) = \{x\} \times \{y\} \times [-1,1] \qquad \text{where} \qquad \mathbf{x} = (x,y,z) \in \hat{\Lambda}_\vartheta.$$

Given $\hat{\mu}_\vartheta \in M(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ we have conditional measures $\hat{\mu}_\vartheta^{ss}(\mathbf{x})$ on W^{ss}. These measures are unique $\hat{\mu}_\vartheta$-almost everywhere fulfilling the relation:

$$\hat{\mu}_\vartheta(B) = \int \hat{\mu}_\vartheta^{ss}(\mathbf{x})(B \cap W^{ss}(\mathbf{x}))d\hat{\mu}_\vartheta(\mathbf{x})$$

for all Borel sets B in $[-1,1]^3$. From the uniqueness of the conditional measures we have for $\hat{\mu}_\vartheta$-almost all $\mathbf{x} = (x,y,z)$

$$\hat{\mu}^s(\mathbf{x})(B) = \int \hat{\mu}_\vartheta^{ss}(\bar{x},y,z)(B \cap W^{ss}(\bar{x},y,z))dpr_X\hat{\mu}_\vartheta^s(\mathbf{x})(\bar{x})$$

for all Borel sets B in $W^s(\mathbf{x})$. This statement means that the transversal measures in the sense of [LY] of the nested partitions W^s and W^{ss} are in our context given by $pr_X\hat{\mu}_\vartheta^s(\mathbf{x})$.
Now let:

$$B_r^{ss}((x,y,z)) = \{(\bar{x},\bar{y},\bar{z}) | \bar{y} = y \ , \ \bar{x} = x \ \text{ and } \bar{z} \in B_r(z)\} \text{ and}$$
$$B^{trans}((x,y,z)) = \{(\bar{x},\bar{y},0) |\ \bar{x} = x \text{ and } \ \bar{y} \in B_r(y)\}.$$

Applying the results of [LY] about the local dimensions of conditional measures in the context of dynamical systems we obtain:

Proposition 5.2.3.

Let $\mu \in M(\Sigma,\sigma)$ and $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2) \in P^4_{all}$ with $\Xi^\mu_{\beta_1,\beta_2} \geq \Xi^\mu_{\tau_1,\tau_2}$. Let $\hat{\mu}_\vartheta^s(\mathbf{x})$ be conditional measures on W^s and $\hat{\mu}_\vartheta^{ss}(\mathbf{x})$ conditional measures on W^{ss} with respect to $\hat{\mu}_\vartheta$. We have

$$\overline{\underline{d^{ss}}}(\mathbf{x}, \hat{\mu}_\vartheta^{ss}(\mathbf{x})) := \overline{\underline{\lim}}_{r \longrightarrow \infty} \frac{\log \hat{\mu}_\vartheta^{ss}(\mathbf{x})(B_r^{ss}(\mathbf{x}))}{\log r} = const. =: \dim \hat{\mu}_\vartheta^{ss} \quad \hat{\mu}_\vartheta - a.e.$$

$$\overline{d^{trans}}(\mathbf{x}, pr_X \hat{\mu}^s_\vartheta(\mathbf{x})) := \overline{\lim}_{r \longrightarrow \infty} \frac{\log pr_X \hat{\mu}^s_\vartheta(\mathbf{x})(B^{trans}_r(\mathbf{x}))}{\log r}$$
$$= \dim \hat{\mu}^s_\vartheta - \dim \mu^{ss}_\vartheta =: \dim \hat{\mu}^{trans}_\vartheta \quad \hat{\mu}_\vartheta - a.e.$$
$$\dim \hat{\mu}^s_\vartheta = \frac{h_{\hat{\mu}_\vartheta}(\hat{f}_\vartheta)}{-\Xi^\mu_{\tau_1,\tau_2}} + (1 - \frac{\Xi^\mu_{\beta_1,\beta_2}}{\Xi^\mu_{\tau_1,\tau_2}}) \dim \hat{\mu}^{trans}_\vartheta$$

The last equitation is known in dimension theory of dynamical systems as **Ledrappier-Young formula**.

5.3. Some consequences

We will find here some interesting consequences of the general results of the last section. First we have an upper bound on the dimension of the measures $\hat{\mu}_\vartheta$.

Proposition 5.3.1.

Let $\mu \in M(\Sigma, \sigma)$, $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$ and let d the unique positive number satisfying $\beta_1 \tau_1^d + \beta_2 \tau_2^d = 1$. We have:

$$\dim \hat{\mu}_\vartheta \leq \frac{h_\mu(\sigma)}{\log 2} + d + 1.$$

Proof

Combining 5.2.1. and 5.2.2. with 3.2.6. we have:

$$\dim \hat{\mu}_\vartheta = \frac{h_\mu(\sigma)}{\log 2} + \dim \hat{\mu}^s_\vartheta.$$

Since $\hat{\mu}_\vartheta$ is a measure on $\hat{\Lambda}_\vartheta$ the measures $\hat{\mu}^s_\vartheta(x, y, z)$ are by definition concentrated on the set $\{(\bar{x}, \bar{y}, \bar{z}) | y = \bar{y} \quad (x, z) \in \Lambda_\vartheta\}$. Hence we have

$$\dim_H \hat{\mu}^s_\vartheta(x, y, z) \leq \dim_H \Lambda_\vartheta \leq \dim_B \Lambda_\vartheta \quad \forall (x, y, z) \in \hat{\Lambda}_\vartheta.$$

Using theorem A2 we now get $\dim \hat{\mu}^s_\vartheta \leq \dim_B \Lambda_\vartheta$. But from 4.1. we know $\dim_B \Lambda_\vartheta = d + 1$, which competes the proof.

□

It is well known in the theory of dynamical systems that the equal weighted Bernoulli measure is the unique ergodic Borel measure of maximal entropy $\log 2$ for the system (Σ, σ); see 8.9. of [WA]. Thus the last proposition shows that the only ergodic measures for the attractor $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ that can have full box-counting dimension is the equal weighted Bernoulli measure $\hat{b}_\vartheta$.
We now present an other upper bound on the dimension of $\hat{\mu}_\vartheta$, which is in terms of the dimension of the measures $(pr^+\mu)_{\beta_1,\beta_2}$ where pr^+ as usual denotes the projection from Σ onto Σ^+.

Proposition 5.3.2.

Under the assumption of proposition 5.2.3. we have:

$$\dim \hat{\mu}_\vartheta \leq \frac{h_\mu(\sigma)}{\log 2} + \frac{h_\mu(\sigma)}{-\Xi^\mu_{\tau_1,\tau_2}} + (1 - \frac{\Xi^\mu_{\beta_1,\beta_2}}{\Xi^\mu_{\tau_1,\tau_2}}) \dim_H (pr^+\mu)_{\beta_1,\beta_2}.$$

Proof

The result follows immediately combining 5.2.1. with 5.2.2. and 5.2.3. if we show the inequality

$$\dim \hat{\mu}^{trans}_\vartheta \leq \dim_H (pr^+\mu)_{\beta_1,\beta_2}.$$

To see this choose a Borel set B in the line with $(pr^+\mu)_{\beta_1,\beta_2}(B) = 1$. Because we know from 3.2.3. that $pr_X \nu_\vartheta = \nu_{\beta_1,\beta_2}$ $\forall \nu \in M(\Sigma^+, \sigma)$ we have $(pr_+\mu)_\vartheta(B \times [-1,1]) = 1$. From 3.2.5. we know $pr_{XZ} \hat{\mu}_\vartheta = (pr_+\mu)_\vartheta$. Hence we get $\hat{\mu}_\vartheta(B \times [-1,1] \times [-1,1]) = 1$. By the definition of the conditional measures $\hat{\mu}^s_\vartheta(\mathbf{x})$ we get $\hat{\mu}^s_\vartheta(x,y,z)(B \times \{y\} \times [-1,1]) = 1$ $\hat{\mu}_\vartheta$-almost everywhere. This implies $pr_X \hat{\mu}^s_\vartheta(x,y,z)(B \times \{y\}) = 1$ and hence $\dim_H pr_X \hat{\mu}^s_\vartheta(x,y,z) \leq \dim_H B$ $\hat{\mu}_\vartheta$-almost everywhere. With 5.2.3. and A2 we now get $\dim \hat{\mu}^{trans}_\vartheta \leq \dim_H B$. This implies the desired inequality since B was an arbitrary Borel set with $(pr^+\mu)_{\beta_1,\beta_2}(B) = 1$.

□

For the Bernoulli measures $\hat{b}^p_\vartheta$ and b^p_ϑ we get explicit dimension formulas in terms of the dimension of the measures self-similar measures $b^p_{\beta_1,\beta_2}$

Proposition 5.3.3.

For all $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$ and $p \in (0,1)$ with $p \log \beta_1 + (1-p) \log \beta_2 \geq p \log \tau_1 + (1-p) \log \tau_2$ we have:

$$\dim \hat{b}^p_\vartheta = \frac{-p \log p - (1-p) \log(1-p)}{\log 2} + \frac{p \log p + (1-p) \log(1-p)}{p \log \tau_1 + (1-p) \log \tau_2}$$

$$+(1 - \frac{p \log \beta_1 + (1-p) \log \beta_2}{p \log \tau_1 + (1-p) \log \tau_2}) \dim_H b^p_{\beta_1,\beta_2}$$

and

$$\dim b^p_\vartheta = \frac{p \log p + (1-p) \log(1-p)}{p \log \tau_1 + (1-p) \log \tau_2} + (1 - \frac{p \log \beta_1 + (1-p) \log \beta_2}{p \log \tau_1 + (1-p) \log \tau_2}) \dim_H b^p_{\beta_1,\beta_2}.$$

Proof

We know from 3.2.5. that the measure $\hat{b}^p_\vartheta$ is the product of the measure b^p_ϑ in the (x,z)-plane with the measure ℓ^p on the y-axis. From this follows, that the conditional measures $(\hat{b}^p_\vartheta)^s(\mathbf{x})$ are given by the measure b^p_ϑ for $\hat{b}^p$-almost all $\mathbf{x}$. Furthermore the transversal measures $pr_X(\hat{b}^p_\vartheta)^s(\mathbf{x})$ are given by $b^p_{\beta_1,\beta_2}$ because we have from 3.2.3. $pr_X\hat{b}^p_\vartheta = b_{\beta_1,\beta_2}$. The dimension formulas are now just a consequence of the propositions of section 5.2. and the following explicit formulas:

$$h_{b^p}(\sigma) = -p\log p - (1-p)\log(1-p) \qquad \text{and } \Xi^{\gamma_1,\gamma_2}_{b^p} = p\log\gamma_1 + (1-p)\log\gamma_1.$$

For the first formula see for instance 12.4. of [DGS]. The second one is obvious.

□

We have the following upper bound on the Hausdorff dimension of the ergodic measure for the projected system $([-1,1]^2, f_{\beta_1,\beta_2})$:

Proposition 5.3.4.

Let $\mu \in M(\Sigma,\sigma)$ and $(\beta_1,\beta_2) \in P^2_{olapp}$. We have:

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \le \frac{h_\mu(\sigma)}{\log 2} + 1.$$

Proof

Fix $(\beta_1,\beta_2) \in P^2_{olapp}$ and choose $\tau \in (0,0.5)$. Let $\vartheta = (\beta_1,\beta_2,\tau,\tau)$. Applying 5.3.2. we have:

$$\dim \hat{\mu}_\vartheta \le \frac{h_\mu(\sigma)}{\log 2} + \frac{\log(\beta_1+\beta_2)}{\log \tau^{-1}} + 1.$$

From proposition 3.2.5. we know $pr_{XY}\hat{\mu}_\vartheta = \bar{\mu}_{\beta_1,\beta_2}$, which obviously implies $\dim_H \bar{\mu}_{\beta_1,\beta_2} \ge \dim_H \hat{\mu}_\vartheta$. Hence

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \le \frac{h_\mu(\sigma)}{\log 2} + \frac{\log(\beta_1+\beta_2)}{\log \tau^{-1}} + 1 \quad \forall \tau \in (0,0.5).$$

Letting $\tau \longrightarrow 0$ we get our result.

□

From this proposition and 3.2.4. it follows, that the only ergodic measures for the attractor $([-1,1]^2, f_{\beta_1,\beta_2})$ that can have full Hausdorff dimension is the equal weighted Bernoulli measure $\bar{b}_{\beta_1,\beta_2}$.

We like to include here an upper bound on the Hausdorff dimension of the measures $\bar{\mu}_{\beta_1,\beta_2}$, which can be proved elementary without the results of 5.2. .

Proposition 5.3.5.

Let $\mu \in M(\Sigma, \sigma)$ and $(\beta_1, \beta_2) \in P^2_{olapp}$. We have $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \dim_H (pr^+\mu)_{\beta_1,\beta_2} + 1$

Proof

Let B be a Borel set with $(pr^+\mu)_{\beta_1,\beta_2}(B) = 1$. By 3.2.4. we have $pr_X \bar{\mu}_{\beta_1,\beta_2} = (pr^+\mu)_{\beta_1,\beta_2}$. Hence $\bar{\mu}_{\beta_1,\beta_2}(B \times [-1,1]) = 1$. Thus $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \dim_H(B \times [-1,1])$ and by proposition A5 $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \dim_H(B) + 1$. Since B was an arbitrary Borel set with full $(pr^+\mu)_{\beta_1,\beta_2}$ measure we get our result.

□

Our results here show that the study of the measures ν_{β_1,β_2} and especially the self-similar measures $b^p_{\beta_1,\beta_2}$ is essential for us. This discussion occupies the next chapter. But before we state a result about the entropy of the measures $\bar{\mu}_{\beta_1,\beta_2}$.

Proposition 5.3.6.

If $\mu \in M(\Sigma, \sigma)$ and $(\beta_1, \beta_2) \in P^2_{olapp}$ we have $h_{\bar{\mu}_{\beta_1,\beta_2}}(f_{\beta_1,\beta_2}) = h_\mu(\sigma)$.

This fact is not trivial because the system $([-1,1]^2, f_{\beta_1,\beta_2}, \bar{\mu}_{\beta_1,\beta_2})$ is only a measure theoretical factor of (Σ, σ, μ). We know only a long and quite complicated proof of this proposition using conditional measures and dimensions. Because we do not need this proposition in the main line of our argumentation we think it is enough if we give a sketch of our proof; the details can be found in [NE].

Sketch of proof

We first define a partition $\bar{W}^u$ of $[-1,1]^2$ and a partition W^{su} of $[-1,1]^3$ by

$$\bar{W}^u(x,y) = \{x\} \times [-1,1] \qquad W^{su}(x,y,z) = \{x\} \times [-1,1]^2.$$

Given $\bar{\mu}_{\beta_1,\beta_2}$ we have conditional measures $\bar{\mu}^u_{\beta_1,\beta_2}(x,y)$ on the elements of $\bar{W}^u$ and given $\hat{\mu}_\vartheta$ we have conditional measures $\hat{\mu}^{su}_\vartheta(x,y,z)$ on the elements of $W^{su}(x,z,z)$. Using properties and uniqueness of conditional measures it is possible to show that the following relations hold for $\hat{\mu}_\vartheta$-almost all $(x,y,z) \in \hat{\Lambda}_\vartheta$:

$$(1) \qquad pr_{XY} \hat{\mu}^{su}_\vartheta(x,y,z) = \bar{\mu}^u_{\beta_1,\beta_2}(x,y)$$

$$(2) \qquad \hat{\mu}^{su}_\vartheta(x,y,z)(B \cap W^{su}(x,y,z)) = \int \hat{\mu}^u_\vartheta(x,\bar{y},\bar{z})(B \cap W^u(x,\bar{y},\bar{z})) d\hat{\mu}^{su}_\vartheta(x,y,z)(\bar{y},\bar{z}).$$

From 5.2.1. and A2 it follows that $\dim_H \hat{\mu}^u_\vartheta(x,y,z) = \dim \hat{\mu}^u_\vartheta$ holds $\hat{\mu}_\vartheta$-almost everywhere. But this implies for $\hat{\mu}_\vartheta$-almost all $(x,y,z) \in \hat{\Lambda}_\vartheta$

$$(3) \qquad \dim_H \hat{\mu}^u_\vartheta(x,\bar{y},\bar{z}) = \dim \hat{\mu}^u_\vartheta \text{ for } \hat{\mu}^{su}_\vartheta(x,y,z)\text{-almost all } (x,\bar{y},\bar{z}) \in W^{su}(x,y,z).$$

Let G be the set of all (x,y,z) such that (1),(2) and (3) hold. Fix $(x,y,z) \in G$. Let B be an arbitrary Borel set such that $\bar{\mu}^u_{\beta_1,\beta_2}(x,y)(\{x\} \times B) = 1$. With (1) it follows that $\hat{\mu}^{su}_\vartheta(x,y,z)(\{x\} \times B \times [-1,1]) = 1$ and with (2) we get from this $\hat{\mu}^u_\vartheta(x,\bar{y},\bar{z})(\{x\} \times B \times \{\bar{z}\}) = 1$ for $\hat{\mu}^{su}_\vartheta(x,y,z)$ -almost all $(x,\bar{y},\bar{z}) \in W^{su}(x,y,z)$. Hence $\dim_H \hat{\mu}^u_\vartheta(x,\bar{y},\bar{z}) \le \dim_H B$ for $\hat{\mu}^{su}_\vartheta(x,y,z)$ -almost all $(x,\bar{y},\bar{z}) \in W^{su}(x,y,z)$ and with (3) $\dim \hat{\mu}^u_\vartheta \le \dim_H B$. Since B was arbitrary, G has full $\hat{\mu}_\vartheta$ measure and $\hat{\mu}_\vartheta$ projects to $\bar{\mu}_{\beta_1,\beta_2}$ this shows:

$$\dim \hat{\mu}^u_\vartheta \le \dim_H \bar{\mu}^u_{\beta_1,\beta_2}(x,y) \qquad \bar{\mu}_{\beta_1,\beta_2}\text{-a.e.}$$

Now let us look at the entropy. On the one hand we know $\dim \hat{\mu}^u_\vartheta = h_\mu(\sigma)/\log 2$ from 5.2.2. . On the other hand it is by means of [MN] not difficult to see in rather direct way that $\dim_H \bar{\mu}^u_{\beta_1,\beta_2}(x,y) \le h_{\bar{\mu}_{\beta_1,\beta_2}}(f_{\beta_1,\beta_2})/\log 2$ holds $\bar{\mu}_{\beta_1,\beta_2}$-a.e. (see [NE]). Hence $h_\mu(\sigma) \le h_{\bar{\mu}_{\beta_1,\beta_2}}(f_{\beta_1,\beta_2})$. For the opposite inequality see the remark after 3.2.6 .

□

It seems to be plausible that a more direct proof of the last proposition should be possible only working with the entropy of conditional measures and without using dimensions at all. But we have not elaborated this.

6. Overlapping self-similar measures

6.1. Main results

In this section we begin to study the self-similar measure $b^p_{\gamma_1,\gamma_2}$ defined in 3.2. for $\gamma_1, \gamma_2 \in (0,1)$ and $p \in (0,1)$. If $\gamma_1 + \gamma_2 < 1$ the measure is concentrated on a Cantor set and hence singular. We will discuss here the overlapping case and thus assume $\gamma_1 + \gamma_2 \geq 1$. The overlapping symmetric self-similar measures b^p_γ are usually called infinitely convolved Bernoulli measures. They raised great interest in the literature. Using Fourier transformation techniques Winter [WI] showed in 1935 that b_γ is absolutely continuous if $\gamma = \frac{1}{\sqrt[n]{2}}$ with $n \geq 0$ and Erdös [ER2] showed in 1940 that the measure is absolute continuous for almost all γ in a small neighborhood of one. Recently one mayor progress was achieved by Solomyak:

Theorem 6.1.1. [SO1]

The measure b_γ is absolutely continuous with square integrable density for almost all $\gamma \in (0.5, 1)$.

We like to inform the reader here that there are parameter values γ with special number theoretical properties such that b_γ is singular. We will discuss this issue in detail in chapter nine.

Peres and Solomyak [PS1] found a considerably simplified proof of theorem 6.1.1. . Moreover they extended the technique used in this proof to the measures b^p_β, which have different weights. They proved:

Theorem 6.1.2. [PS2]

Let $p \in (0,1)$. The measures b^p_γ are absolutely continuous for almost all $\gamma \in (p^p(1-p)^{1-p}, 0.649)$ and singular if $\gamma < p^p(1-p)^{1-p}$. If $p \in [1/3, 2/3]$ then the bound 0.649 in this statement can be replaced by 1.

As far as we know the overlapping asymmetric self-similar measures $b^p_{\gamma_1,\gamma_2}$ have not been studied jet. This will be our task here. We will prove an analogon of 6.1.2. in the asymmetric situation. Let us first define a subset of the parameters set

$$P^2_{olapp} := \{(\gamma_1, \gamma_2) \in (0,1)^2 | \gamma_1 + \gamma_2 \geq 1\}$$

by

$$P^2_{trans} := \{(\gamma_1, \gamma_2) \in P^2_{olapp} | \gamma_2 \leq \gamma_1 \leq 0.649\}.$$

Now we formulate our result:

Theorem 6.1.3.

Let $p \in (0,1)$ and $P^2_{abs} := \{(\gamma_1,\gamma_2) \in P^2_{trans} | (\gamma_2 p)^p(\gamma_1(1-p))^{1-p} \leq \gamma_1\gamma_2\}$. The measures $b^p_{\gamma_1,\gamma_2}$ are absolutely continuous for almost all $(\gamma_1,\gamma_2) \in P^2_{abs}$ in the sense of two dimensional Lebesgue measure and singular if $(\gamma_2 p)^p(\gamma_1(1-p))^{1-p} > \gamma_1\gamma_2$.

The first part of this theorem follows from corollary 6.2.2. of the next section using the theorem of Fubini. The singularity assertion is stated in corollary 6.3.2. and follows from a more general upper bound on the box-counting dimension of the measures μ_{γ_1,γ_2} we will prove in 6.3.1. .

We think that it is necessary to make a few remarks on our main result:

Remarks

(1) Fist note that by the symmetry of the measures in question the assumption of $\gamma_2 \leq \gamma_1$ in the definition of P^2_{trans} means no loss of generality.

(2) We have to say a few word about the bound 0.649 that appears in 6.1.3. (and also in 6.1.2.). On the first sight this bound seems to be somewhat crude. In the proof we will see that it is due to a certain **transversality condition** that we need. In fact the bound is given by the infimum of all double zeros of power series with absolute value of the coefficients less equal to one and first coefficient equal to one. 0.649 is an approximation of this quantity. We refer to step 4 of the proof of proposition 6.2.1. for this issue.

(3) Peres and Solomyak [PS2] used some additional arguments concerning Fourier transformations to improve the bound to 1 in the symmetric situation if $p \in [1/3, 2/3]$. These arguments do not work if $p < 1/3$. We have not been able to improve the bound in the asymmetric situation but we do not believe that this bound is really essential.

6.2. Absolute continuity

Let us first recall some definitions from chapter three. The measures $b^{p*}_{\gamma_1,\gamma_2}$ are given by $(\pi^*_{\gamma_1,\gamma_2})^{-1} \circ b^p$. b^p is the Bernoulli measure on $\Sigma^+ = \{-1,1\}^{\mathbb{N}_0}$ with probability distribution $(p, 1-p)$ on $\{1,-1\}$ and the map $\pi^*_{\gamma_1,\gamma_2}$ is given by

$$\pi^*_{\gamma_1,\gamma_2}(\underline{s}) = \sum_{k=0}^{\infty} s_k \gamma_2^{\sharp_k(\underline{s})} \gamma_1^{k-\sharp_k(\underline{s})+1}.$$

The quantity $\sharp_k(\underline{s})$ counts how often -1 appears in $\{s_0,\ldots,s_k\}$. The measures $b^p_{\gamma_1,\gamma_2}$ are just the measures $b^{p*}_{\gamma_1,\gamma_2}$ scaled by the affine transformation that maps $\frac{-\gamma_2}{1-\gamma_2}$ to

-1 and $\frac{\gamma_1}{1-\gamma_1}$ to 1. Now we state our result on absolute continuity and density of the measures at hand.

Proposition 6.2.1.

Let $p \in (0,1)$, $q \in (1,2]$ and $c \in (0,1]$. The density of the measures $b^p_{\gamma,c\gamma}$ is in L^q for almost all $\gamma \in [\gamma_0(c,q,p), 0.649]$ where $\gamma_0(c,q,p) = (p^q + c^{1-q}(1-p)^q)^{\frac{1}{q-1}}$.

The technique we will use in the following proof is similar to argumentations that have been developed in [PS 1/2].

Proof

Obviously it is enough if we show that the proposition holds for the unscaled measures $b^{p*}_{\gamma,c\gamma}$.
Fix p, q and c during the proof.

1. Step: An integral condition for the measures to have density in L^q

We define the (lower) local density of a measure μ on the real line by

$$\underline{D}(\mu, x) = \underline{\lim}_{r\longrightarrow 0} \frac{\mu(B_r(x))}{2r}.$$

If we have

$$\int (\underline{D}(\mu, x))^{q-1} d\mu(x) < \infty$$

then μ is absolute continuous and has density in L^q. This follows from Mattila [MA,2.12]. Thus it is sufficient for us to show that

$$\Im(\gamma_0) := \int_{\gamma_0}^{0.649} \int \ (D(b^{p*}_{\gamma,c\gamma}, x))^{q-1} db^{p*}_{\gamma,c\gamma}(x) \ d\gamma < \infty$$

holds for all $\gamma_0 > \gamma_0(c,q,p)$.

2. Step: Some estimates on the integral

By applying Fatou's lemma then changing variables using the definition of the measures $b^{p*}_{\gamma,c\gamma}$ and reversing the order of integration we obtain:

$$\Im(\gamma_0) \leq \underline{\lim}_{r\longrightarrow 0} \frac{1}{(2r)^{q-1}} \int_{\gamma_0}^{0.649} \int \ (b^{*p}_{\gamma,c\gamma}(B_r(x)))^{q-1} \ db^{*p}_{\gamma,c\gamma}(x) d\gamma$$

$$= \underline{\lim}_{r\longrightarrow 0} \frac{1}{(2r)^{q-1}} \int_{\gamma_0}^{0.649} \int_{\Sigma^+} (b^{*p}_{\gamma,c\gamma}(B_r(\pi^*_{\gamma,c\gamma}(\underline{s}))))^{q-1} \ db^p(\underline{s}) d\gamma$$

$$= \underline{\lim}_{r\longrightarrow 0} \frac{1}{(2r)^{q-1}} \int_{\Sigma^+} \int_{\gamma_0}^{0.649} (b^{*p}_{\gamma,c\gamma}(B_r(\pi^*_{\gamma,c\gamma}(\underline{s}))))^{q-1} \; d\gamma db^p(\underline{s}).$$

Applying Hölder's inequality, $\int f^\alpha \leq C_1(\int f)^\alpha$ where $\alpha \in (0,1]$ and $f \geq 0$, we get

$$\Im(\gamma_0) \leq C_1 \underline{\lim}_{r\longrightarrow 0} \frac{1}{(2r)^{q-1}} \int_{\Sigma^+} (\int_{\gamma_0}^{0.649} b^{*p}_{\gamma,c\gamma}(B_r(\pi^*_{\gamma,c\gamma}(\underline{s})) \; d\gamma)^{q-1} db^p(\underline{s}).$$

Now note that

$$\int_{\gamma_0}^{0.649} b^{*p}_{\gamma,c\gamma}(B_r(\pi^*_{\gamma,c\gamma}(\underline{s}))) \; d\gamma = \int_{\gamma_0}^{0.649} \int \mathbf{1}_{B_r(\pi^*_{\gamma,c\gamma}(\underline{s}))}(x) \; db^{*p}_{\gamma,c\gamma}(x) d\gamma$$

$$= \int_{\gamma_0}^{0.649} \int_{\Sigma^+} \mathbf{1}_{\{\underline{t}| \; |\pi^*_{\gamma,c\gamma}(\underline{s}) - \pi^*_{\gamma,c\gamma}(\underline{t})| \leq r\}} db^p(\underline{t}) d\gamma$$

$$= \int_{\Sigma^+} \ell(\{\gamma \in [\gamma_0, 0.649]| \; |\pi^*_{\gamma,c\gamma}(\underline{s}) - \pi^*_{\gamma,c\gamma}(\underline{t})| \leq r\}) \; db^p(\underline{t}).$$

Thus $\Im(\gamma_0)$ is bounded from above by

$$C_1 \; \underline{\lim}_{r\longrightarrow 0} \frac{1}{(2r)^{q-1}} \int_{\Sigma^+} (\int_{\Sigma^+} \ell(\{\gamma \in [\gamma_0, 0.649]| \; |\pi^*_{\gamma,c\gamma}(\underline{s}) - \pi^*_{\gamma,c\gamma}(\underline{t})| \leq r\}) \; db^p(\underline{t}))^{q-1} db^p(\underline{s}).$$

3. Step: Using the structure of the map π

For $\underline{s} = (s_k)$ and $\underline{t} = (t_k)$ in Σ^+ let $|\underline{s} \wedge \underline{t}| = \min\{k | s_k \neq t_k\}$. We have:

$$\phi_{\underline{s},\underline{t}}(\gamma) := \pi_{\gamma,c\gamma}(\underline{s}) - \pi_{\gamma,c\gamma}(\underline{t}) = \sum_{k=0}^{\infty} (s_k c^{\sharp_k(\underline{s})} - t_k c^{\sharp_k(\underline{t})}) \gamma^{k+1}$$

$$= \gamma^{|\underline{s}\wedge\underline{t}|+1} \sum_{k=0}^{\infty} (s_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{t})}) \gamma^k$$

$$= \gamma^{|\underline{s}\wedge\underline{t}|+1} (s_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{t})}) (1 + \sum_{k=1}^{\infty} \underbrace{\frac{s_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{k+|\underline{s}\wedge\underline{t}|} c^{\sharp_{k+|\underline{s}\wedge\underline{t}|}(\underline{t})}}{s_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{s})} - t_{|\underline{s}\wedge\underline{t}|} c^{\sharp_{|\underline{s}\wedge\underline{t}|}(\underline{t})}}}_{:=a_k(\underline{s},\underline{t})} \gamma^k)$$

$$= 1/2(s_{|\underline{s}\wedge\underline{t}|} - t_{|\underline{s}\wedge\underline{t}|})(1+c)\gamma^{|\underline{s}\wedge\underline{t}|+1} c^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} (1 + \sum_{k=1}^{\infty} a_k(\underline{s},\underline{t})\gamma^k).$$

For the last equitation we used the fact that $\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s}) = \sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{t})$[1]. Now setting $g_{\underline{s},\underline{t}}(\gamma) = 1 + \sum_{k=1}^{\infty} a_k(\underline{s},\underline{t})\gamma^k$ and $C_2 = 1/2(s_{|\underline{s}\wedge\underline{t}|} - t_{|\underline{s}\wedge\underline{t}|})(1+c)$ we have the formula

$$\phi_{\underline{s},\underline{t}}(\gamma) = C_2 \gamma^{|\underline{s}\wedge\underline{t}|+1} c^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} g_{\underline{s},\underline{t}}(\gamma).$$

[1] We use the convention that $\sharp_n(\underline{s}) = 0$ if $n < 0$

Here the absolute value of C_2 does not depend on $\underline{s}$ and $\underline{t}$. We now claim that the absolute value of the coefficients of the polynomials $g_{\underline{s},\underline{t}}$ is less or equal to one:

$$|a_k(\underline{s},\underline{t})| \leq 1 \qquad \forall\ k > 0 \text{ and } \underline{s},\underline{t} \in \Sigma^+.$$

Since $\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s}) = \sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{t})$ we can write:

$$|a_k(\underline{s},\underline{t})| = \frac{(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s})} - (\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t})}}{1+c}.$$

But we have

$$|(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s})} - (\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))_k c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t})}| \leq |c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s})}| + |c^{\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t})}| \leq 1 + c$$

by the definition of $|\underline{s} \wedge \underline{t}|$, which proves our claim.

4. Step: The transversality condition

We say that the ρ-**transversality** condition holds for a C^1 function g on a closed interval I if $g(x) < \rho \Rightarrow g^{'}(x) > \rho\ \forall x \in I$. This means that the graph of the function g crosses all horizontal lines that it meets below height λ transversally with slope at most $-\rho$. Obviously the transversality condition holds for some ρ on an interval I if and only if g has no double zero on the interval I.

If we have the ρ-transversality condition for g on I then

$$\ell\{x \in I||g(x)| \leq r\} \leq 2r\rho^{-1} \qquad \forall r > 0.$$

This is easy to see. If $r \geq \rho$ then the claim is obvious. If $r < \rho$ then g is monotonous decreasing with $g^{'} < -\rho$ on the set $\{x \in I||g(x)| \leq r\}$ by ρ-transversality. But this immediately yields the assertion.

From lemma 2 of [PS2] we know that:

$$O := \inf\{x|x \text{ is a double zero of a power series } f = 1 + \sum_{k=1}^{\infty} a_k x^k \text{ with } |a_k| \leq 1\}$$
$$\approx 0.649138.$$

It follows that there is a ρ such that the ρ-transversality condition holds for all polynomials $f = 1 + \sum_{k=1}^{\infty} a_k x^k$ with $|a_k| \leq 1$ on the Interval $[0, O]$. Especially ρ transversality holds for all polynomials $g_{\underline{s},\underline{t}}$ defined in the third step of our proof on $[0, 0.649]$. Thus we get:

$$\ell\{\gamma \in [\gamma_0, 0.649] |\ |\phi_{\underline{s},\underline{t}}(\gamma)| \leq r\}$$

$$\leq \ell\{\gamma \in [\gamma_0, 0.649] |\ |g_{\underline{s},\underline{t}}(\gamma)| \leq r|C_2|^{-1}\gamma^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\}$$

$$\leq 2\rho^{-1}r|C_2|^{-1}\gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} = C_3 r\gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} \text{ with } C_3 = 2\rho^{-1}|C_2|^{-1}.$$

5. Step: Integrating

We put our estimates of step two and four together and obtain

$$\Im(\gamma_0) \leq C_4 \int_{\Sigma^+}(\int_{\Sigma^+} \gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\ db^p(\underline{t}))^{q-1}db^p(\underline{s})$$

where $C_4 = C_1 C_3^{q-1}2^{1-q}$. Now we integrate:

$$\int_{\Sigma^+} \gamma_0^{-|\underline{s}\wedge\underline{t}|-1}c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\ db^p(\underline{t}) = \sum_{n=0}^{\infty} \gamma_0^{-n-1}c^{\sharp_{n-1}(\underline{s})}b^p(\{\underline{t} \in \Sigma^+ |\ |\underline{s}\wedge\underline{t}| = n\}$$

$$= \sum_{n=0}^{\infty} \gamma_0^{-n-1}c^{-\sharp_{n-1}(\underline{s})}p^{n-\sharp_{n-1}(\underline{s})}(1-p)^{\sharp_{n-1}(\underline{s})}(s_n(1/2-p)+1/2)$$

Using the inequality $(\sum x_i)^\alpha \leq \sum x_i^\alpha$ for $\alpha = q-1 \leq 1$ we continue with:

$$\Im(\gamma_0) \leq C_4 \sum_{n=0}^{\infty}\int_{\Sigma^+}(\gamma_0^{-n-1}c^{-\sharp_{n-1}(\underline{s})}p^{n-\sharp_{n-1}(\underline{s})}(1-p)^{\sharp_{n-1}(\underline{s})}(s_n(1/2-p)+1/2))^{q-1}db^p(\underline{s})$$

$$= C_4\sum_{n=0}^{\infty}\gamma_0^{(-n-1)(q-1)}((1-p)^{q-1}p+p^{q-1}(1-p))\sum_{k=0}^{n}(c^{-k}(1-p)^k p^{n-k})^{q-1}b^p\{\underline{s} \in \Sigma^+ | \sharp_{n-1}(\underline{s}) = k\}$$

$$= C_4((1-p)^{q-1}p + p^{q-1}(1-p))\sum_{n=0}^{\infty}\gamma_0^{(-n-1)(q-1)}\sum_{k=0}^{n}\binom{n}{k}((1-p)^q c^{1-q})^k p^{q(n-k)}$$

$$= C_4((1-p)^{q-1}p + p^{q-1}(1-p))\gamma_0^{1-q}\sum_{n=0}^{\infty}(\gamma_0^{-(q-1)}((1-p)^q c^{1-q} + p^q)^n.$$

The sum in the last expression converges exactly if $\gamma_0 > \gamma_0(c,q,p) = (p^q + c^{1-q}(1-p)^q)^{\frac{1}{q-1}}$. So $\Im(\gamma_0) < \infty$ holds for all $\gamma_0 > \gamma_0(c,q,p)$ and our proof is complete.

□

Proposition 6.2.1. has the following corollary:

Corollary 6.2.2.

Let $c \in (0,1]$ and $p \in (0,1)$. The measures $b^p_{\gamma,c\gamma}$ are absolutely continuous for almost all $\gamma \in [\gamma_0(c,p), 0.649]$ where $\gamma_0(c,p) = p^p((1-p)/c)^{1-p}$.

Proof

One only has to show that $\lim_{q\longrightarrow 1}\gamma_0(c,q,p)=\gamma_0(c,p)$. But this is easy to see by taking logarithm and using the rule of l'Hospital.

$\square$

6.3. An upper bound the on dimension

We will prove here an general upper bound on the box-counting dimension of the measures μ_{γ_1,γ_2} defined in 3.2. where $\gamma_1,\gamma_2\in(0,1)$ are arbitrary and μ is a shift ergodic measures. Applying this general upper bound to the overlapping self-similar measures $b^p_{\gamma_1,\gamma_2}$ implies the singularity assertion of our main theorem 6.1.3. . Recall that $\Xi^\mu_{\gamma_1,\gamma_2}=\mu([1]_0)\log\gamma_1+\mu([-1]_0)\log\gamma_2$.

Proposition 6.3.1.

If $\mu\in M(\Sigma^+,\sigma)$ and $\gamma_1,\gamma_2\in(0,1)$ we have:

$$\overline{\dim}_B\mu_{\gamma_1,\gamma_2}\leq\min\{1,\frac{h_\mu(\sigma)}{-\Xi^\mu_{\gamma_1,\gamma_2}}\}.$$

Proof

Fix γ_1,γ_2 and p. First note that it is trivial that the box-counting dimension of the measure in question is less or equal to one since it is defined on the real line. We now define a metric $\delta^{\gamma_1,\gamma_2}$ on Σ^+ by

$$\delta^{\gamma_1,\gamma_2}(\underline{s},\underline{t})=\gamma_1^{|\underline{s}\wedge\underline{t}|-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\gamma_2^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}.$$

We first claim that

$$d^{\gamma_1,\gamma_2}(\underline{s},b^p):=\lim_{\epsilon\longrightarrow 0}\frac{\log B^{\gamma_1,\gamma_2}_\epsilon(\underline{s})}{\log\epsilon}=\frac{h_\mu(\sigma)}{-\Xi^\mu_{\gamma_1,\gamma_2}}\qquad\mu\text{-almost everywhere.}$$

Here d^{γ_1,γ_2} is the local dimension of the measure b^p with respect to metric $\delta^{\gamma_1,\gamma_2}$ and accordingly $B^{\gamma_1,\gamma_2}_\epsilon$ is a ball of radius ϵ with respect to this metric. Applying Birkhoffs ergodic theorem (see 4.1.2. of [KH]) to (Σ^+,σ,μ) with the function

$$h(\underline{s})=\{\begin{array}{lll}\log\gamma_1 & \text{if} & s_0=1\\ \log\gamma_2 & \text{if} & s_0=-1\end{array}$$

we see that:

$$\lim_{n\longrightarrow\infty}\frac{1}{n+1}\log\operatorname{diam}_{\gamma_1,\gamma_2}([s_0,\dots,s_n]_0)=\lim_{n\longrightarrow\infty}\frac{1}{n+1}\sum_{k=0}^{n+1}h(\sigma^k(\underline{s}))=\int h\,d\mu(\underline{s})=\Xi^\mu_{\gamma_1,\gamma_2}$$

μ-almost everywhere. By Shannon-McMillan-Breiman theorem (see [DGS] 13.4.) we have:

$$\lim_{n\longrightarrow\infty} -\frac{1}{n+1}\log\mu([s_0,\ldots,s_n]_0) = h_\mu(\sigma) \qquad \mu\text{-almost everywhere.}$$

Thus we see:

$$\lim_{\epsilon\longrightarrow 0}\frac{\log B_\epsilon^{\gamma_1,\gamma_2}(\underline{s})}{\log\epsilon} = \lim_{n\longrightarrow\infty}\frac{\log\mu([s_0,\ldots,s_n]_0)}{\mathrm{diam}_{\gamma_1,\gamma_2}([s_0,\ldots,s_n]_0)} = \frac{h_\mu(\sigma)}{-\Xi^\mu_{\gamma_1,\gamma_2}}$$

Of course we can define the box-counting dimension of the measure μ with respect to the metric $\delta^{\gamma_1,\gamma_2}$ on Σ in exactly the same way as we define the box-counting dimension of a measure on $I\!R^q$ in appendix A. Furthermore it is not difficult to see that an analogon of A2 holds for Borel probability measures on the metric space $(\Sigma^+,\delta^{\gamma_1,\gamma_2})$. Thus we have

$$\dim_B^{\gamma_1,\gamma_2}\mu = \frac{h_\mu(\sigma)}{-\Xi^\mu_{\gamma_1,\gamma_2}}$$

where the box-counting dimension $\dim_B^{\gamma_1,\gamma_2}$ has to be calculated using $\delta^{\gamma_1,\gamma_2}$.

Now we claim that the map $\pi^*_{\gamma_1,\gamma_2}$ is Lipschitz with respect to the metric $\delta^{\gamma_1,\gamma_2}$:

$$|\pi^*_{\gamma_1,\gamma_2}(\underline{s}) - \pi^*_{\gamma_1,\gamma_2}(\underline{t})| \le \sum_{k=|\underline{s}\wedge\underline{t}|}^{\infty} |s_k\gamma_1^{k-\sharp_k(\underline{s})+1}\gamma_2^{\sharp_k(\underline{s})} - t_k\gamma_1^{k-\sharp_k(\underline{t})+1}\gamma_2^{\sharp_k(\underline{t})}|$$

$$= \gamma_1^{|\underline{s}\wedge\underline{t}|-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}\gamma_2^{\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})}$$

$$\sum_{k=0}^{\infty}|s_{k+|\underline{s}\wedge\underline{t}|}\gamma_1^{k-\sharp_k(\sigma^{|\underline{s}\wedge-\underline{t}|}(\underline{s}))+1}\gamma_2^{\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{s}))} - t_{k+|\underline{s}\wedge\underline{t}|}\gamma_1^{k-\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))+1}\gamma_2^{\sharp_k(\sigma^{|\underline{s}\wedge\underline{t}|}(\underline{t}))}|$$

$$\le \delta^{\gamma_1,\gamma_2}(\underline{s},\underline{t})\frac{2}{1-\max\{\gamma_1,\gamma_2\}}.$$

But the map π_{γ_1,γ_2} is just $\pi^*_{\gamma_1,\gamma_2}$ scaled on $[-1,1]$ and hence Lipschitz with respect $\delta^{\gamma_1,\gamma_2}$ as well. Since applying a Lipschitz map to the measures μ does obvious not increase its box-counting dimension, the proof is complete.

□

Let us remark that it is well known that the Hausdorff and box-counting dimension of of μ_{γ_1,γ_2} equals $-h_\mu(\sigma)/\Xi^\mu_{\gamma_1,\gamma_2}$ in the case that $\gamma_1+\gamma_2<1$; see for instance 13.1. of [PE2]. In our work we are more interested in the overlapping case $\gamma_1+\gamma_2\ge 1$.

From 6.3.1. we get the following corollary about the self-similar measures $b^p_{\gamma_1,\gamma_2}$:

Corollary 6.3.2.

Let $\gamma_1, \gamma_2, p \in (0,1)$. We have

$$\overline{\dim}_B b^p_{\gamma_1,\gamma_1} \leq \min\{1, \frac{p\log p + (1-p)\log(1-p)}{p\log\gamma_1 + (1-p)\log\gamma_2}\}.$$

Moreover $b^p_{\gamma_1,\gamma_2}$ is singular if $(\gamma_2 p)^p(\gamma_1(1-p))^{1-p} > \gamma_1\gamma_2$.

Proof

To see the upper bound just recall that $h_{b^p}(\sigma) = -p\log p - (1-p)\log(1-p)$ and $\Xi_{\gamma_1,\gamma_2}(b^p) = p\log\gamma_1 + (1-p)\log\gamma_2$. From the upper bound we have $\overline{\dim}_B b^p_{\gamma_1,\gamma_2} < 1$ if $(\gamma_2 p)^p(\gamma_1(1-p))^{1-p} > \gamma_1\gamma_2$, which clearly implies our singularity assertion.

□

7. Generic dimensional theoretical properties of the systems

We will now formulate our main generic results about the dimensional theoretical properties of the class of repellers $(\Lambda_\vartheta, T_\vartheta)$ and the class of attractors $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$. The term "generic" has to be understood in a special sense referring to the Lebesgue measure on certain subspaces of the parameter space P^4_{all}. The restrictions of our generic results depend on the transversality condition; see chapter six. Recall from this chapter that

$$P^2_{trans} = \{(\beta_1, \beta_2) | \beta_1 + \beta_2 \geq 1 \text{ and } 0 < \beta_2 \leq \beta_1 \leq 0.649\}.$$

Theorem 7.1.

General case
For all $p \in (0,1)$ and almost all $(\beta_1, \beta_2) \in P^2_{trans}$ and all $\tau_1, \tau_2 > 0$ with $\tau_1 + \tau_2 < 1$ and $\log \tau_2 \log p/\beta_1 = \log \tau_1 \log(1-p)/\beta_2$ we have:

$$\dim_H b^p_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = \frac{\log p/\beta_1}{\log \tau_1} + 1 \quad \text{and}$$

$$\dim_H \hat{\Lambda}_\vartheta = \dim_B \hat{\Lambda}_\vartheta = \frac{\log p/\beta_1}{\log \tau_1} + 2$$

where $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2)$. Moreover if $p = 0.5$ then $\hat{b}^p_\vartheta$ has full dimension on $\hat{\Lambda}_\vartheta$ and if $p \neq 0.5$ then the variational principle for Hausdorff dimension does not hold for $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$.

Special case $\beta_1 = \beta_2 = \beta$
For $p \in (0,1)$ set $I = (0.5, 1)$ if $p \in (1/3, 2/3)$ and $I = (0.5, 0.649)$ if not. We have for almost all $\beta \in I$ and all $\tau_1, \tau_2 > 0$ with $\tau_1 + \tau_2 < 1$ and $\log \tau_2 \log p/\beta = \log \tau_1 \log(1-p)/\beta$:

$$\dim_H b^p_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = \frac{\log p/\beta}{\log \tau_1} + 1 \text{ and } \dim_H \hat{\Lambda}_\vartheta = \dim_B \hat{\Lambda}_\vartheta = \frac{\log p/\beta}{\log \tau_1} + 2$$

where $\vartheta = (\beta, \beta, \tau_1, \tau_2)$. Moreover if $p \neq 0.5$ (which means $\tau_1 \neq \tau_2$) then the variational principle for Hausdorff dimension does not hold for $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$.

Special case $\tau_1 = \tau_2 = \tau$
For almost all $(\beta_1, \beta_2) \in P^2_{trans}$ and all $\tau \in (0, 0.5)$ we have:

$$\dim_H b^p_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = \frac{\log(\beta_1 + \beta_2)}{\log \tau^{-1}} + 1$$

$$\text{and} \qquad \dim_H \hat{\Lambda}_\vartheta = \dim_B \hat{\Lambda}_\vartheta = \frac{\log(\beta_1+\beta_2)}{\log \tau^{-1}} + 2$$

where $\vartheta = (\beta_1, \beta_2, \tau, \tau)$ and $p = \beta_1/(\beta_1+\beta_2)$. Moreover if $\beta_1 \neq \beta_2$ then the variational principle for Hausdorff dimension does not hold for $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$.

Special case $\tau_1 = \tau_2 = \tau$ and $\beta_1 = \beta_2 = \beta$
For almost all $\beta \in [0,5,1]$ and all $\tau \in (0, 0.5)$ we have

$$\dim_H b^{0.5}_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = \frac{\log 2\beta}{\log \tau^{-1}} + 1 \qquad \text{and}$$

$$\dim_H \hat{b}^{0.5}_\vartheta = \dim_H \hat{\Lambda}_\vartheta = \dim_B \hat{\Lambda}_\vartheta = \frac{\log 2\beta}{\log \tau^{-1}} + 2$$

where $\vartheta = (\beta, \beta, \tau, \tau)$.

We include a corollary, which states the main general result of the last theorem in a weaker but more straightforward way. Recall that:

$$P^4_{trans} = \{(\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all} | (\beta_1, \beta_2) \in P^2_{trans}\}.$$

Corollary 7.2.

For almost all $\vartheta \in P^4_{trans}$ we have:

$$\dim_H b^p_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = d+1 \text{ and } \dim_H \hat{\Lambda}_\vartheta = \dim_B \hat{\Lambda}_\vartheta = d+2$$

where $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2)$ and d is the solution of $\beta_1 \tau_1^x + \beta_2 \tau_2^x = 1$ and $p = \beta_1 \tau_1^d$.

Let us discuss our results:

Remarks

(1) Corollary 7.2. shows that on the set of parameters P^4_{trans} we have generically the identity of Hausdorff and box-counting dimension for the repellers $(\Lambda_\vartheta, T_\vartheta)$ and the attractors $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$.

(2) The existence of a measure of full dimension is only a generic property of the repellers. Not even the variational principle for Hausdorff dimension holds generically with respect to the Lebesgue measure on P^4_{trans} for the attractors. It holds for $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ only if we have $\log_{\tau_1} \log(2\beta_1) = \log_{\tau_2}(2\beta_2)$. In the proof we will see that this phenomenon is due to the fact that one can not maximize the stable and the unstable dimension (resp. the dimension of the corresponding conditional measures) at the same time. In the context of Axiom A diffeomorphisms exactly this

was observed by Manning and McCluskey [MM]. It allowed them to show that the variational principle is not generic for Axiom A systems in the topological sense; it only holds on a nowhere dense set of Axiom A systems.

(3) Now we comment on the special case $\beta_1 = \beta_2$ and $\tau_1 = \tau_2$. The condition on β we need to get the identity for Hausdorff and box-counting dimension in this case is $\dim_H b_\beta = 1$. By Solomyak's theorem (6.1.1.) this condition holds almost everywhere in $(0,5,1)$. Pollicott and Weiss used the number theoretical condition that β is a Garsia-Erdös number to get this identity (see 2.1.3.). The property to be a Garsia-Erdös number can been shown to be equivalent to the absolute continuity of b_β with uniformly bounded density (see 2.1.3. and 5. of [PW]). This condition seems to be stronger than $\dim b_\beta = 1$. But in fact we do not know if there are numbers such that $\dim b_\beta = 1$ and β is not Garsia-Erdös.

We will now formulate our result about the generic dimensional theoretical properties of the projected systems $([-1,1]^2, f_{\beta_1,\beta_2})$ including the Fat Baker's transformations f_β.

Theorem 7.3.

General case
For almost all $(\beta_1,\beta_2) \in P^2_{trans}$ with $\beta_1\beta_2 \geq 0.25$ the measure $\bar{b}_{\beta_1,\beta_2} = b_{\beta_1,\beta_2} \times \ell$ is a measure full dimension for $([-1,1]^2, f_{\beta_1,\beta_2})$. But if $\beta_1\beta_2 < 0.25$ then the variational principle for Hausdorff dimension does not hold for the system $([-1,1]^2, f_{\beta_1,\beta_2})$.

Special case $\beta_1 = \beta_2$
For almost all $\beta \in (0.5,1)$ $\bar{b}_\beta = b_\beta \times \ell$ is a measure full dimension for $([-1,1]^2, f_\beta)$

The claim about the Fat Baker's transformation f_β in 7.3. is in fact just a simple consequence of Solomyak's theorem [SO1] and the work of Alexander and Yorke [AY].

Before we begin with the proofs we remark that we have number theoretical exceptions to our generic results in the symmetric situation $\beta_1 = \beta_2$. These results are formulated in chapter ten. Let us now go into the proofs.

Proof of 7.1.

General case:

Fix $p \in (0.5,1)$. We first claim that for almost all $(\beta_1,\beta_2) \in P^2_{trans}$ and $\tau_1,\tau_2 \in (0,1)$ with $\tau_1+\tau_2 < 1$ and $\log\tau_2 \log p/\beta_1 = \log\tau_1 \log(1-p)/\beta_2$ the identity $\dim b^p_{\beta_1,\beta_2} = 1$

holds.
If we are given $(\beta_1, \beta_2) \in P^2_{trans}$ and there exists $\tau_1 + \tau_2 < 1$ with $\frac{\log p/\beta_1}{\log \tau_1} = \frac{\log(1-p)/\beta_2}{\log \tau_2} =: d$ we have:

$$(p\beta_1)^p((1-p)\beta_2)^{1-p} = (\beta_1\beta_2\tau_1^d)^p(\beta_1\beta_2\tau_2^d)^{1-p} = \beta_1\beta_2\tau_1^{dp}\tau_2^{d(1-p)} < \beta_1\beta_2.$$

Now we see that our claim follows from theorem 6.1.3. with the help of A3.
Fix $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2)$ with the properties of our claim and let d be defined as above.
We have:

$$\beta_1\tau_1^d + \beta_2\tau_2^d = \beta_1\tau_1^{\frac{\log p/\beta_1}{\tau_1}}\beta_2\tau_2^{\frac{(1-p)\beta_2}{\tau_2}} = p + (1-p) = 1.$$

From 4.1. we thus get:

$$\dim_B \Lambda_\vartheta = d + 1 \qquad \text{and} \qquad \dim_B \Lambda_\vartheta = d + 2$$

Moreover from 5.3.3. we get:

$$\dim_H b^p_\vartheta = \frac{p\log p + (1-p)\log(1-p)}{p\log\tau_1 + (1-p)\log\tau_2} + (1 - \frac{p\log\beta_1 + (1-p)\log\beta_2}{p\log\tau_1 + (1-p)\log\tau_2})$$

$$= 1 + \frac{\beta_1\tau_1^d\log\beta_1\tau_1^d + \beta_2\tau_2^d\log\beta_2\tau_2^d - (\beta_1\tau_1^d\log\beta_1 + \beta_2\tau_2^d\log\beta_2)}{\beta_1\tau_1^d\log\tau_1 + \beta_2\tau_2^d\log\tau_2}$$

$$= 1 + \frac{\beta_1\tau_1^d\log\tau_1^d + \beta_2\tau_2^d\log\tau_2^d}{\beta_1\tau_1^d\log\tau_1 + \beta_2\tau_2^d\log\tau_2} = d + 1.$$

Just by definition we have $\dim_H b^p_\vartheta \leq \dim_H \Lambda_\vartheta \leq \dim_B \Lambda_\vartheta$. Thus we get

$$\dim_H b^p_\vartheta = \dim_H \Lambda_\vartheta = \dim_B \Lambda_\vartheta = d + 1$$

and with the help of A5

$$\dim_H \hat{\Lambda}_\vartheta = \dim_B \hat{\Lambda}_\vartheta = d + 2.$$

Our first statement in the general situation is proved.
Consider the special case $p = 0.5$. We get from 5.3.3.

$$\dim \hat{b}_\vartheta = \frac{h_b(\sigma)}{\log 2} + d + 1 = d + 2.$$

This means that $\hat{b}_\vartheta$ is a measure of full dimension.
Now consider the opposite case $p \neq 0.5$. Assume that the variational principle for Hausdorff dimension holds for $(\Lambda_\vartheta, \hat{f}_\vartheta)$. Then by 3.2.5. there is a sequence of measures $\mu_n \in M(\Sigma, \sigma)$, such that

$$\dim_H(\hat{\mu}_n)_\vartheta \longrightarrow d + 2$$

Recall that the equal-weighted Bernoulli measure b is the unique measure in $M(\Sigma, \sigma)$ which maximizes the metric entropy with $h_b(\sigma) = \log 2$ and that the metric entropy is upper-semi-continuous on $M(\Sigma, \sigma)$. By this facts and 5.3.1. we necessarily have

$$\mu_n \longrightarrow b.$$

From 5.3.2. we have the inequality

$$\dim_H \hat{\mu}_\vartheta \leq 1 + \frac{h_\mu(\sigma)}{\log 2} - \frac{h_\mu(\sigma) + \Xi^\mu_{\beta_1,\beta_2}}{\Xi^\mu_{\tau_1,\tau_2}}$$

for all $\mu \in M(\Sigma, \sigma)$. With the help of upper semi-continuity of $h_\mu(\sigma)$ we thus get:

$$\overline{\lim}_{n\longrightarrow\infty} \dim_H(\hat{\mu}_n)_\vartheta \leq 2 - \frac{\log 2 + 0.5 \log \beta_1 + 0.5 \log \beta_2}{0.5 \log \tau_1 + 0.5 \log \tau_2}.$$

We have:

$$-\frac{\log 2 + 0.5 \log \beta_1 + 0.5 \log \beta_2}{0.5 \log \tau_1 + 0.5 \log \tau_2} = -\frac{2 \log 2 + \log p - d \log \tau_1 + \log(1-p) - d \log \tau_2}{\log \tau_1 + \log \tau_2}$$

$$= d - \frac{2 \log 2 + \log p + \log(1-p)}{\log \tau_1 + \log \tau_2} < d,$$

which implies $\overline{\lim}_{n\longrightarrow\infty} \dim_H(\hat{\mu}_n)_\vartheta < d + 2$. This is a contradiction and the variational principle for Hausdorff dimension does not hold for $(\Lambda_\vartheta, \hat{f}_\vartheta)$.

Special case $\beta_1 = \beta_2 = \beta$:

One proves the result by exactly the same arguments that we used in the general situation. The only difference is that one uses the theorem of Peres and Solomyak (6.1.2) for the symmetric self-similar measures instead of theorem 6.1.3. for the asymmetric ones.

Special case $\tau_1 = \tau_2 = \tau$:

Setting $p = \frac{1}{1+c}$ in 6.2.2. we have for all $c \in (0, 1]$ $\dim_H b^{1/(1+c)}_{\beta,c\beta} = 1$ for almost all $\beta \in [\frac{1}{1+c}, 0.649]$. Using the theorem of Fubini we get from this $\dim_H b^p_{\beta_1,\beta_2} = 1$ with $p = \frac{\beta_1}{\beta_1+\beta_2}$ for almost all $(\beta_1, \beta_2) \in P^2_{trans}$.
Now from 4.1. and 5.3.3. the dimension formula for the $(\Lambda_\vartheta, T_\vartheta)$ and with help A5 the dimension formula for $\hat{\Lambda}_\vartheta$ follows.
If $\beta_1 \neq \beta_2$ our result about the variational principle can be proved by the same arguments that we used in the general situation if $p \neq 0.5$.

Special case $\beta_1 = \beta_2 = \beta$ and $\tau_1 = \tau_2 = \tau$:

The statement is just an obvious consequence of 4.1., Solomyak's theorem 6.1.1. and the dimension formula in 5.3.3. .

□

Proof of 7.2.

It follows directly from the general case in 7.1. that for all $p \in (0,1)$ there exists a set $A(p) \subseteq P^2_{trans}$ with $\ell^2(A(p)) = \ell^2(P^2_{trans})$ such that for all $(\beta_1, \beta_2) \in A(p)$ and all $\tau_1, \tau_2 > 0$ with $\tau_1 + \tau_2 < 1$ and $\log\tau_2 \log p/\beta_1 = \log\tau_1 \log(1-p)/\beta_2$ our statement about the dimensions holds. Let $G(\tau_1)$ be given by the following union:

$$\bigcup_{p\in(0,1)} \{(\beta_1, \beta_2, \tau_2) | (\beta_1, \beta_2) \in A(p)\ ,\ \tau_1+\tau_2 < 1\ ,\ \log\tau_2 \log p/\beta_1 = \log\tau_1 \log(1-p)/\beta_2\}.$$

It is easy to see that the union

$$\bigcup_{p\in(0,1)} \{(\beta_1, \beta_2, \tau_2) | (\beta_1, \beta_2) \in P^2_{trans}\ ,\ \tau_1+\tau_2 < 1\ ,\ \log\tau_2 \log p/\beta_1 = \log\tau_1 \log(1-p)/\beta_2\}$$

equals the set $\{(\beta_1, \beta_2, \tau_2) | (\beta_1, \beta_2) \in P^2_{trans}\ ,\ \tau_1 + \tau_2 < 1\}$. By the theorem of Fubini we thus have $\ell^3(G(\tau_1)) = \ell^3\{(\beta_1, \beta_2, \tau_2) | (\beta_1, \beta_2) \in P^2_{trans}\ ,\ \tau_1 + \tau_2 < 1\}$. Now let

$$G = \bigcup_{\tau_1\in(0,1)} \{(\beta_1, \beta_2, \tau_1, \tau_2) | (\beta_1, \beta_2, \tau_2) \in G(\tau_1)\}.$$

Note that we have $G \subseteq P^4_{trans}$ and $\ell^4(G) = \ell^4(P^4_{trans})$. But by definition our dimension formulas hold for all $\vartheta \in G$. This competes the proof.

□

Proof of 7.3.

General case:

First recall from 3.2.4. that $\bar{b}_{\beta_1,\beta_2} = b_{\beta_1,\beta_2} \times \ell$ is an ergodic measure for the system $([-1,1]^2, f_{\beta_1,\beta_2})$. It follows directly from 6.1.3. that the measure b_{β_1,β_2} is absolutely continuous for almost all $(\beta_1, \beta_2) \in P^2_{trans}$ with $\beta_1\beta_2 \geq 0.25$. For these (β_1, β_2) the measure $b_{\beta_1,\beta_2} \times \ell$ is absolutely continuous as well and thus has dimension two (see A3). This proves our first statement.

If $\beta_1\beta_2 < 0.25$ then we have $h_b(\sigma) < -\Xi^b_{\beta_1,\beta_2}$. By upper semi continuity of the metric entropy there is a weak* neighborhood U of b in $M(\Sigma^+, \sigma)$ such that

$-h_\mu(\sigma)/\Xi^\mu_{\beta_1,\beta_2} \leq c_1 < 1$ holds for all $\mu \in U$. Now note that we have from 5.3.5. and 6.3.1.

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq 1 + \dim_H (pr^+\mu)_{\beta_1,\beta_2} \leq 1 + \frac{-h_{pr^+\mu}(\sigma)}{\Xi^{pr^+\mu}_{\beta_1,\beta_2}}.$$

From these facts we get $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq c + 1 < 2$ for all $\mu \in \tilde{U} = (pr^+)^{-1}(U)$. Obviously $\tilde{U}$ is a neighborhood of b in $M(\Sigma, \sigma)$. Furthermore we have by 5.3.4.

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \frac{h_\mu(\sigma)}{\log 2} + 1.$$

Again by upper semi continuity of metric entropy it follows that $\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq c_2 + 1 < 2$ for all $\mu \in M(\Sigma,\sigma)\backslash\bar{U}$. Putting these facts together we get:

$$\dim_H \bar{\mu}_{\beta_1,\beta_2} \leq \max\{c_1, c_2\} + 1 < 2 = \dim[-1,1]^2 \qquad \forall \mu \in M(\Sigma,\sigma).$$

This proves our second statement.

Special case $\beta_1 = \beta_2$:

Recall from 3.2.4. that $\bar{b}_{\beta = b_\beta \times \ell}$ is an ergodic measure for the system $([-1,1]^2, f_\beta)$. From 6.1.1. we know that the measure b_β is absolutely continuous for almost all $\beta \in (0.5, 1)$. For these β we know that $b_\beta \times \ell$ is absolutely continuous and thus has dimension two (see A3).

□

8. Extension of some results to Markov chains

This chapter forms a kind of supplement to last four chapters. We will extend some of our main general and generic results to invariant sets for the maps T_ϑ and $\hat{f}_\vartheta$ that correspond to special Markov chains.

Let $A = \begin{pmatrix} a_{1,1} & a_{-1,1} \\ a_{1,-1} & a_{-1,-1} \end{pmatrix}$ be a $(2,2)$-matrix with entrys a_{ij} in $\{0,1\}$. By Σ_A (resp. Σ_A^+) we denote the subset of Σ (resp. Σ^+) given by $\{(s_k)|a_{s_k s_{k+1}} = 1\}$. These sets are obviously invariant under the shift map σ (resp. σ^{-1}). The systems (Σ_A^+, σ) and (Σ^A, σ^{-1}) are called (1-step) Markov chains (see [KH]). Now we define subsets Λ_A^ϑ and $\hat{\Lambda}_A^\vartheta$ of $[-1,1]^2$ by

$$\Lambda_A^\vartheta = \pi_\vartheta(\Sigma_A^+) \qquad \text{and} \qquad \hat{\Lambda}_A^\vartheta = \hat{\pi}_\vartheta(\Sigma_A)$$

for $\vartheta \in P_{all}^4$. By 3.1.1. the set Λ_A^ϑ is invariant under the map T_ϑ and by 3.1.3. the set $\hat{\Lambda}_A^\vartheta$ is invariant under the map $\hat{f}_\vartheta$.

If the matrix A is not in $\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\}$ then the sets Σ_A (resp. Σ_A^+) and consequently the sets $\hat{\Lambda}_\vartheta^A$ (resp. Λ_ϑ^A) are at most countable. Dimensional theoretical properties are trivial in this case. By symmetry we may restrict our attention to the case $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Fix this matrix for the rest of this chapter. We remark that the dynamical system (Σ_A^+, σ) is known as **goldenshift** (see [SV]).

For $p \in (0,1)$ define Markov measures on Σ_A (resp. Σ_A^+) in the following way: Consider the stochastic matrix $P = \begin{pmatrix} p_{1,1} & p_{-1,1} \\ p_{1,-1} & p_{-1,-1} \end{pmatrix} := \begin{pmatrix} p & 1 \\ 1-p & 0 \end{pmatrix}$ and the stochastic vector $(p_1, p_{-1}) = (1/(2-p), (1-p)/(2-p))$. Define a measure on the cylinder sets in Σ_A (resp. Σ_A^+) by

$$m^p([t_0, t_1 \dots, t_u]_v) = p_{t_0} \prod_{i=0}^{u-1} p_{t_i t_{i+1}}.$$

Now extend this measure to a Borel probability measure m^p on Σ_A (resp. Σ_A^+). It is well known that m_p is ergodic with respect to the shift map (see [DGS]). Define measures m_ϑ^p and $\hat{m}_\vartheta^p$ by

$$m_\vartheta^p = m^p \circ \pi_\vartheta^{-1} \qquad \text{and} \qquad \hat{m}_\vartheta^p = m^p \circ \hat{\pi}_\vartheta^{-1}.$$

By 3.2.3. m_ϑ^p is an ergodic measure for the system $(\Lambda_\vartheta^A, T_\vartheta)$ and by 3.2.5. $\hat{m}^p$ is an ergodic measure for the systems $(\hat{\Lambda}_\vartheta^A, \hat{f}_\vartheta)$.

Our main result in this chapter is nothing but an extension of 4.1. and 7.1.2. to the invariant sets Λ_ϑ^A and $\hat{\Lambda}_\vartheta$. Let

$$P_A^4 = \{(\beta_1, \beta_2, \tau_1, \tau_2) | \beta_1 + \beta_1\beta_2 \geq 1, \tau_1 + \tau_2 < 1\}$$

$$\text{and} \qquad P_{A-trans}^4 = \{(\beta_1, \beta_2, \tau_1, \tau_2) \in P_A^4 | \beta_1 \leq \beta_2 \leq 0.649\}.$$

Theorem 8.1.1.

(1) For all $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P_A^4$ we have

$$\dim_B \Lambda_\vartheta^A = d + 1 \quad \text{and} \quad \dim_B \hat{\Lambda}_\vartheta^A = d + 1 + \frac{\log((\sqrt{5}+1)/2)}{\log 2}$$

where d is unique positive number satisfying $\beta_1\tau_1^d + \beta_1\beta_2(\tau_1\tau_2)^d = 1$.

(2) For almost all $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P_{A-trans}^4$ (in the sense of four dimensional Lebesgue measure) we have

$$\dim_H m_\vartheta^p = \dim_H \Lambda_\vartheta^A = \dim_B \Lambda_\vartheta^A \quad \text{and} \quad \dim_H \hat{\Lambda}_\vartheta^A = \dim_B \hat{\Lambda}_\vartheta^A$$

where $p = \beta_1\tau_1^d$ and d is as in (1).

Remarks

(1) The condition $\beta_1 + \beta_1\beta_2 \geq 1$ is necessary. It means that the projection of Λ_A^ϑ onto the first component has positive length. This fact is essential for our proof (see also remark (3) in chapter four).

(2) Note that we can write our dimension formula in the symmetric situation $\vartheta = (\beta, \beta, \tau, \tau) \in P_A^4$ using the topological entropy $h_{top}(\sigma_{|\Sigma_A})$:

$$\dim_B \Lambda_\vartheta^A = 1 + \frac{\log\beta + h_{top}(\sigma_{|\Sigma_A})}{\log\tau} \qquad \text{and}$$

$$\dim_B \hat{\Lambda}_\vartheta^A = 1 + \frac{\log\beta + h_{top}(\sigma_{|\Sigma_A})}{\log\tau} + \frac{\log\beta + h_{top}(\sigma_{|\Sigma_A})}{\log 2}$$

(3) Of course the reader will ask the question if there are generalizations of 8.1.1. to n-step Markov chains (see [KH] for definition). Let us first discuss the box-counting dimension. We were not able to prove an analogon of 8.1.1.(1) for all n-step Markov chains. But under certain assumption such a generalization is in fact possible using our methods. Let us discuss this in detail. We say that a Markov chain Σ_{Markov}

(resp. Σ^+_{Markov}) has block form if there is a set of finite sequences B=$\{b_1,\dots,b_k\}$, where b_i have entrys in $\{1,-1\}$, such that each element of Σ_{Markov} (resp. Σ^+_{Markov}) can be written as a sequence of elements in B. If Σ_{Markov} has block form and in addition $pr_X\pi_\vartheta(\Sigma_{Markov})$ has positive length we get

$$\dim_B \pi_\vartheta(\Sigma^+_{Markov}) = d+1 \quad \text{and} \quad \dim_B \hat{\pi}_\vartheta(\Sigma_{Markov}) = d+1+\frac{h_{top}(\sigma_{|\Sigma_{Markov}})}{\log 2}.$$

Here d is the solution of

$$\sum_{i=0}^{k} \tau_1^{\sharp_1(b_i)}\tau_2^{\sharp_{-1}(b_i)}(\beta_1^{\sharp_1(b_i)}\beta_2^{\sharp_{-1}(b_i)})^x = 1$$

where $\sharp_1$ counts the number of entrys that are 1 and $\sharp_{-1}$ counts the number of entrys that are -1 in an element b of B and $h_{top}(\sigma_{|\Sigma_{Markov}})$ denotes the topological entropy of the Markov chain. The proof of this statement differs from the proof of 8.1.1.(1) only in technical respects; no new idea is needed. We have thus decided not to write down the proof of this assertion.
We remark that some but not all Markov chains have block form. For instance the blocks (1) and $(-1,1,1)$ induces a 2-step Markov chain. But the 2-step Markov chain which is is given by excluding only the block $(2,1,2)$ does not have block form.

(4) A generalization of 8.1.1.(2) fails because we need the transversality condition to treat the Hausdorff dimension (see chapter six). To see this again consider the Markov chain induced by the blocks (1) and $(-1,1,1)$. The condition for overlapping projections is $\beta_1+\beta_1^2\beta_2 \geq 1$, which implies $\beta_1 \geq 0.65$ or $\beta_2 \geq 0.65$. This contradicts the transversality condition $\beta_1 \leq \beta_2 \leq 0.649$.

Now we want to give a comprehensive proof of 8.1.1., only elaborating the details that are different from what was done in the last chapters.

Proof of 8.1.1.

Fix $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2) \in P^4_A$ and the number d.

1. Step: Calculation of box-counting dimension

Let $\tau_3 = \tau_1\tau_2$ and $\beta_3 = \beta_1\beta_2$. Given $r>0$ we define a set of finite sequences by

$$X_r := \{(s_1,\dots,s_k)|\min\{\tau_1,\tau_3\}r \leq \tau_{s_1}\tau_{s_2}\dots\tau_{s_k} < r \text{ where } s_j \in \{1,3\}\ \forall j=1\dots k\}.$$

Let

$$C_r = \{\pi_\vartheta([\kappa(s_1),\dots,\kappa(s_k)]_0)|(s_1,\dots,s_k)\in X_r\}$$

where $\kappa(1)=1$ and $\kappa(3)=(-1,1)$. Since $\{[\kappa(s_1),\dots,\kappa(s_k)]_0)|(s_1,\dots,s_k)\in X_r\}$ is a cover of Σ_A^+ we have that C_r is a cover of Λ_ϑ^A. An element of C_r is a rectangle parallel to the axis with x-length $2\beta_{s_1}\beta_{s_2}\dots\beta_{s_k}$ and y-length $2\tau_{s_1}\tau_{s_2}\dots\tau_{s_k}$. We cover each of this rectangles by squares parallel to the axis of side length $2\tau_{s_1}\tau_{s_2}\dots\tau_{s_k}$. We choose the squares in a row such that they only intersect in their boundary. In this way we obtain a new cover $\hat{C}_r$ of Λ_ϑ^A, which consists of squares with length in $(2\min\{\tau_1,\tau_3\}r,2r]$. By exactly the same arguments we used in the proof of 4.1. we see that we have the following estimates for the number of elements $\hat{N}_r$ in the cover $\hat{C}_r$:

$$1\le r^{d+1}\hat{N}(r)\le 2\min\{\tau_1,\tau_3\}^{-(d+1)}.$$

Now we want to analyze the sets $\Lambda_\vartheta^A\cap R$ where R is a rectangle in C_r. First note that the projection of Λ_ϑ^A onto the x-axis is given by the set I fulfilling the relation $I=L_1(I)\cup L_1\circ L_2(I)$ where $L_1(x)=\beta_1x+(1-\beta_1)$ and $L_2(x)=\beta_2x+(1-\beta_2)$. Using the fact that $\beta_1+\beta_1\beta_2\ge 1$ a direct calculation shows that I is the interval $[\frac{\beta_1\beta_2-2\beta_1+1}{1-\beta_1\beta_2},1]$. Let l be the length of this interval. We now see that $\ell(pr_X\Lambda_\vartheta^A)=l$. But this implies $\ell(pr_X(R\cap\Lambda_\vartheta^A))=l\ell(pr_X(R))$. Thus the number of those squares in $\hat{C}_r$ that have nonempty intersection with Λ_ϑ^A is bigger than $l\hat{N}(r)$. One square with side length $2r$ parallel to the axis can not intersect more than $9\min\{\tau_1,\tau_3\}^{-2}$ squares in $\hat{C}_r$ because the squares in $\hat{C}_r$ have side length bigger than $2\min\{\tau_1,\tau_3\}r$ and intersect, if at all, only in the boundary. Thus if we have a cover of $\hat{\Lambda}_\vartheta^A$ with square of side length $2r$ parallel to the axis, this cover has at least $1/9\min\{\tau_1,\tau_3\}^2l\hat{N}(r)$ elements. Let $N(r)$ be the minimal cardinality of an cover of Λ_ϑ^A with square of side length $2r$ parallel to the axis. Putting our estimates together we obtain

$$1/9\min\{\tau_1,\tau_3\}^2l\le 1/9\min\{\tau_1,\tau_3\}^2l\hat{N}(r)r^{d+1}\le N(r)r^{d+1}l\le\min\{\tau_1,\tau_3\}^{-(d+1)}.$$

This shows $\dim_B\Lambda_\vartheta^A=d+1$. It remains to deduce the dimension formula for $\hat{\Lambda}_\vartheta^A$. By the product structure of the map $\hat{\pi}_\vartheta$ we get:

$$\hat{\Lambda}_\vartheta^A=\{(x,y,z)|(x,z)\in\Lambda_\vartheta^A \text{ and } y\in F\}$$

where $F=\iota(pr^-(\Sigma_A))$. Here ι is defined in 3.1. and pr^- is the projection from Σ onto Σ^-. Define $\bar{\iota}$ by

$$\bar{\iota}((s_k)_{k\in\mathbb{N}_0})=\sum_{k=0}^{\infty}s_k2^{-k-1}.$$

It is easy to see that $F=\bar{\iota}(\Sigma_A^+)$. But this set is well known in dimension theory and we get from [FU]

$$\dim_H F=\dim_B F=\frac{h_{top}(\sigma_{|\Sigma_A^+})}{\log 2}=\frac{\log((\sqrt{5}+1)/2)}{\log 2}.$$

Using the definition of the box-counting dimension with δ-mesh cubes (see 3.1. of [FA]) we see $\dim_B \hat{\Lambda}^A_\vartheta = \dim_B \Lambda^A_\vartheta + \dim_B F$. This gives us the dimension formula for $\hat{\Lambda}^A_\vartheta$. Now the first part of theorem 8.1.1. is proved. **2. Step:** The dimension of the Markov measures m^p_ϑ

If we assume $\Xi^{m_p}_{\tau_1,\tau_2} \leq \Xi^{m_p}_{\beta_1,\beta_2}$ we get from the Ledrappier-Young formula (see 5.2.3. and 5.3.3.) the following formula for the Markov measures m^p_ϑ:

$$\dim m^p_\vartheta = \frac{h_{m_p}(\sigma)}{-\Xi^{m_p}_{\tau_1,\tau_2}} + (1 - \frac{\Xi^{m_p}_{\beta_1,\beta_2}}{\Xi^{m_p}_{\tau_1,\tau_2}}) \dim m^p_{\beta_1,\beta_1}.$$

Here the measure $m^p_{\beta_1,\beta_2}$ is given by $m^p_{\beta_1,\beta_2} = m^p \circ \pi^{-1}_{\beta_1,\beta_2}$.

Just by definition we have

$$\Xi^{m_p}_{\beta_1,\beta_2} = \frac{1}{2-p}\log\beta_1 + \frac{1-p}{2-p}\log\beta_2 \quad \text{and} \quad \Xi^{m_p}_{\tau_1,\tau_2} = \frac{1}{2-p}\log\tau_1 + \frac{1-p}{2-p}\log\tau_2.$$

Furthermore we know from 4.4. of [KH],

$$h_{m^p}(\sigma) = -(\frac{p}{2-p}\log p + \frac{1-p}{2-p}\log(1-p)).$$

This gives us the formula:

$$\dim m^p_\vartheta = \frac{p\log p + (1-p)\log(1-p)}{\log\tau_1 + (1-p)\log\tau_2} + (1 - \frac{\log\beta_1 + (1-p)\log\beta_2}{\log\tau_1 + (1-p)\log\tau_2}) \dim m^p_{\beta_1,\beta_1}.$$

3. Step: Absolute continuity of the measures $m^p_{\beta_1,\beta_2}$

We claim that an analogon of proposition 6.2.1. holds for the Markov measures $m^p_{\beta_1,\beta_2}$ on the real line:

Claim: Let $p \in (0,1)$ and $c \in (0,1]$. The measures $m^p_{\beta,c\beta}$ are absolutely continuous with density in L^2 for almost all $\beta \in [0, 0.649]$ with $p^2/\beta + (1-p)^2/(c\beta^2) \leq 1$.

Using the arguments of the first four steps in the proof of 6.2.1. we see that it is enough if we show

$$\Im(\beta_0) = \int_{\Sigma^+}\int_{\Sigma^+} \beta_0^{-|\underline{s}\wedge\underline{t}|-1} c^{-\sharp_{|\underline{s}\wedge\underline{t}|-1}(\underline{s})} dm^p(\underline{t}) dm^p(\underline{s}) < \infty$$

for all β_0 with $p^2/\beta_0 + (1-p)^2/(c\beta_0^2) \leq 1$. Here all notations are the same as in the proof of 6.2.1. . We integrate:

$$\Im(\beta_0) = \sum_{n=0}^{\infty} \int_{\Sigma^+} \sum_{n=0}^{\infty} \gamma_0^{-n-1} c^{-\sharp_{n-1}(\underline{s})} m^p(\{\underline{t} \in \Sigma^+ |\ |\underline{s}\wedge\underline{t}| = n)\} dm^p(\underline{s})$$

$$\leq \max\{p,1-p\} \sum_{n=1}^{\infty} \int_{\Sigma^+} \sum_{n=0}^{\infty} \gamma_0^{-n-1} c^{-\sharp_{n-1}(\underline{s})} m^p([s_0,\dots s_{n-1}]_0) dm^p(\underline{s})$$

$$= \max\{p,1-p\} \beta_0^{-1} \sum_{n=1}^{\infty} \sum_{\underline{s}\in\{-1,1\}^n} \beta_0^{-n} c^{-\sharp_{n-1}(\underline{s})} m^p([s_0,\dots s_{n-1}]_0)^2.$$

For $\underline{t} \in \{1,(-1,1)\}^n$ we denote by $\sharp_1(\underline{t})$ the number of entrys in $\underline{t}$ that are 1 and by $\sharp_{-1,1}(\underline{t})$ the number of entrys that are $(-1,1)$. With this notations we have

$$\Im(\beta_0) \leq \frac{\max\{p,1-p\}}{2-p} \beta_0^{-1} \sum_{n=1}^{\infty} \sum_{\underline{t}\in\{1,(-1,1)\}^n} \gamma_0^{-\sharp_1(\underline{t})-2\sharp_{-1,1}(\underline{t})} c^{-\sharp_{-1,1}(\underline{t})} p^{2\sharp_1(\underline{t})} (1-p)^{2\sharp_{-1,1}(\underline{t})}$$

$$= \frac{\max\{p,1-p\}}{2-p} \beta_0^{-1} \sum_{n=1}^{\infty} (\frac{p^2}{\beta_0})^{\sharp_{-1}(\underline{t})} (\frac{(1-p)^2}{c\beta_0^2})^{\sharp_{-1,1}(\underline{t})}$$

$$= \frac{\max\{p,1-p\}}{2-p} \beta_0^{-1} \sum_{n=1}^{\infty} (\frac{p^2}{\beta_0} + \frac{(1-p)^2}{\beta_0^2 c})^n.$$

Now we see that our claim holds.

4. Step: Conclusion of the proof

We will prove the following statement:

Claim: For all $p \in (0,1)$ and almost all $(\beta_1,\beta_2) \in \{(\beta_1,\beta_2) \in P^2_{trans} | \beta_1 + \beta_1\beta_2 \leq 1\}$ and all τ_1,τ_2 with $\tau_1+\tau_2<1$ and $\frac{\log(p/\beta_1)}{\log \tau_1} = \frac{\log((1-p)/(\beta_1\beta_2))}{\log(\tau_1\tau_2)}$ we have

$$\dim_H m^p_\vartheta = \dim_H \Lambda^A_\vartheta = \dim_B \Lambda^A_\vartheta \quad \text{and} \quad \dim_H \hat{\Lambda}^A_\vartheta = \dim_B \hat{\Lambda}^A_\vartheta$$

where $\vartheta = (\beta_1,\beta_2,\tau_1,\tau_2)$.

Using the argumentations in the proof of 7.1.2. this claim implies the second part of our theorem. Therefore we now gone prove this claim:

With the help of Fubini's theorem we get from our claim in the third step of this proof:

For all $p \in (0,1)$ and almost all $(\beta_1,\beta_2) \in P^2_{trans}$ with $p^2/\beta_1 + (1-p)^2/(\beta_1\beta_2) \leq 1$ the identity $\dim m^p_{\beta_1,\beta_2} = 1$ holds .

If we are given $p \in (0,1)$ and $(\beta_1,\beta_2) \in \{(\beta_1,\beta_2) \in P^2_{trans} | \beta_1 + \beta_1\beta_2 \leq 1\}$ and τ_1,τ_2 with $\tau_1+\tau_2<1$ and $\frac{\log p/\beta_1}{\log \tau_1} = \frac{\log(1-p)/(\beta_1\beta_2)}{\log(\tau_1\tau_2)} =: d$ then we have $p = \beta_1\tau_1^d$, $(1-p) = \beta_1\beta_2(\tau_1\tau_2)^d$ and

$$p^2\beta_1 + (1-p)^2/(\beta_1\beta_2) = \beta_1\tau_1^{2d} + \beta_1\beta_2(\tau_1\tau_2)^{2d} < 1.$$

Hence for all $p \in (0,1)$ and almost all $(\beta_1, \beta_2) \in \{(\beta_1, \beta_2) \in P^2_{trans} | \beta_1 + \beta_1\beta_2 \leq 1\}$ and all τ_1, τ_2 with $\tau_1 + \tau_2 < 1$ and $\frac{\log p/\beta_1}{\log \tau_1} = \frac{\log(1-p)/(\beta_1\beta_2)}{\log(\tau_1\tau_2)}$ we have $\dim m^p_{\beta_1,\beta_2} = 1$ and with the help of the dimension formula from the second step of our proof:

$$\dim m^p_\vartheta = \frac{p\log p + (1-p)\log(1-p)}{\log\tau_1 + (1-p)\log\tau_2} + (1 - \frac{\log\beta_1 + (1-p)\log\beta_2}{\log\tau_1 + (1-p)\log\tau_2})$$

$$= 1 + \frac{\beta_1\tau_1^d \log\beta_1\beta_2(\tau_1\tau_2)^d + \beta_1\beta_2(\tau_1\tau_2)^d \log(\beta_1\beta_2(\tau_1\tau_2)^d - \log\beta_1 - \beta_1\beta_2(\tau_1\tau_2)^d \log\beta_2}{\log\tau_1 + \beta_1\beta_2(\tau_1\tau_2)^d \log\tau_2}$$

$$= 1 + \frac{\log\tau_1^d + \beta_1\beta_2(\tau_1\tau_2)^d \log\tau_2^d}{\log\tau_1 + \beta_1\beta_2(\tau_1\tau_2)^d \log\tau_2} = d + 1.$$

We know from the first step of the proof that $\dim_B \Lambda^A_\vartheta = d + 1$. Hence we get:

$$\dim m^p_\vartheta = \dim_H \Lambda^A_\vartheta = \dim_B \Lambda^A_\vartheta.$$

Moreover we know from the first step of the proof that $\dim_B \hat{\Lambda}^A_\vartheta = \dim_B \Lambda^A_\vartheta + \dim_B F$. On the other hand by the product formula of Falconer [FA1, 7.4.] we have $\dim_H \hat{\Lambda}^A_\vartheta = \dim_H \Lambda^A_\vartheta + \dim_H F$. Thus $\dim_H \hat{\Lambda}^A_\vartheta = \dim_B \hat{\Lambda}^A_\vartheta$ holds. This completes the proof of our claim.

$\square$

9. Erdös measures

9.1. Singularity

In this chapter we usually assume that $\beta \in (0,5,1)$ is the reciprocal **Pisot-Vijayarghavan number** (short: PV number). We refer to appendix B for definition, examples and properties of these algebraic integers. Erdös [ER1] showed in 1939 that the equal-weighted infinitely convolved Bernoulli measure b_β is singular if $\beta \in (0,5,1)$ is the reciprocal PV number. Furthermore it follows from a work of Alexander and Yorke [AY] that $\dim_H b_\beta < 1$ in case that β is the reciprocal PV number. This was observed by Przytycki and Urbanski [PU] who also gave their own proof of this fact. It is not known (and perhaps a difficult problem) whether there are other parameters β than reciprocals of PV numbers such that infinitely convolved measures Bernoulli b_β measure is singular. Some information how big the set of exceptions can be maximal follows from a very resent result of Peres and Schlag [PSch]. They have shown the relation $\dim_H\{\beta \in (0.5,1) | \dim_H b_\beta < 1\} < 1$ for infinitely convolved measures Bernoulli b_β.

Our objects here are all symmetric overlapping measures μ_β where $\beta \in (0,5,1)$ is the reciprocal PV number and $\mu \in M(\Sigma^+,\sigma)$. We call such a measure μ_β an **Erdös measures**.

Our main theorem extends the results mentioned above:

Theorem 9.1.1.

Let $\beta \in (0,5,1)$ be the reciprocal PV number and $\mu \in M(\Sigma^+,\sigma)$. $\dim_H \mu_\beta < 1$ holds if and only if μ_β is singular. Moreover the set $\{\mu \in M(\Sigma^+,\sigma) | \mu_\beta \text{ is singular}\}$ is open in the weak* topology on $M(\Sigma^+,\sigma)$ and contains all Bernoulli measures.

This theorem will follow from several different propositions we prove in this chapter (see the end of 9.3.). In view of 7.1. it is natural to ask whether there are measures $\mu \in M(\Sigma^+,\sigma)$ with $\dim_H \mu_\beta = 1$ at all. We will answer this question in 9.4. . We remark that the technique, we will develop here, can not be extended to the asymmetric measures μ_{β_1,β_2}. We will see where the problems come from.

We first state the generalization of Erdös result to all infinitely convolved Bernoulli measures. The proof we give is nothing but an obvious extension of Erdös original argument.

Proposition 9.1.2.

If $\beta \in (0,5,1)$ is the reciprocal PV number then the measures b^p_β are singular for all $p \in (0,1)$.

Proof

By [JW] we know that the Fourier transformation of a convolution is the product of the Fourier transformation of the convolved measures. Consequently by 3.2.1. the Fourier transformation ϕ of b^p_γ is given by:

$$\phi(b^p_\beta, \omega) = \prod_{n=0}^{\infty} (\cos((1-\beta)\beta^n\omega) + (2p-1)\sin((1-\beta)\beta^n\omega)).$$

We see that:

$$|\phi(b^p_\beta, \omega)| = \prod_{n=0}^{\infty} |(\cos((1-\beta)\beta^n\omega) + (2p-1)\sin((1-\beta)\beta^n\omega))| \geq \prod_{n=0}^{\infty} |\cos((1-\beta)\beta^n\omega)|.$$

Now let $\omega_k = 2\pi\beta^{-k}/(1-\beta)$. We have:

$$|\phi(b^p_\beta, \omega_k)| \geq \prod_{n=0}^{\infty} |\cos(2\pi\beta^{n-k})| = \prod_{n=0}^{k} |\cos(2\pi\beta^{n-k})| \prod_{n=k+1}^{\infty} |\cos(2\pi\beta^{n-k})|$$

$$= C \prod_{n=0}^{k} |\cos(2\pi\beta^{-n})|$$

where C is a constant independent of k and not zero. Now let β be the reciprocal of a PV number. From proposition B1 of appendix B we know that there is a constant $0 < \theta < 1$ such that: $||\beta^{-n}||_{\mathbb{Z}} \leq \theta^n \; \forall n \geq 0$ where $||.||_{\mathbb{Z}}$ denotes the distance to the nearest integer. This implies $|\phi(b^p_\beta, \omega_k)| \geq \hat{C} > 0$ for all $k > 0$. Thus we have that $|\phi(b^p_\beta, \omega)|$ does not tend to zero with $\omega \longrightarrow \infty$. Hence by Riemann-Lebesgue lemma b^p_β can not be absolutely continuous if β is the reciprocal of a PV number. But it follows from the theory of infinity convolutions developed by Jessen and Winter [JW] that b^p_β is either absolutely continuous or singular. This completes the proof.

□

We remark that we have no nice product formula for the Fourier transformation of the $b^{p*}_{\beta_1,\beta_2}$ in the asymmetric situation. In fact we have:

$$\phi(b^{p*}_{\beta_1,\beta_2}, \omega) = \lim_{v \longrightarrow \infty} \sum_{\underline{s} \in \{-1,1\}^v} p^{v-\sharp_v(\underline{s})}(1-p)^{\sharp_v(\underline{s})} \prod_{n=1}^{v} e^{s_n \omega \beta_1^{n-\sharp n(\underline{s})} \beta_2^{\sharp n(\underline{s})}}.$$

and the Fourier transformation of $b^p_{\beta_1,\beta_2}$ is just this function scaled on $[-1,1]$. We have not been able to apply the idea used in the proof of 9.1.2. to this Fourier transformation and have thus not been able to find $\beta_1 \neq \beta_2$ with $\beta_1 + \beta_2 \geq 1$ such that $b^p_{\beta_1,\beta_2}$ is singular.

9.2. Garcia entropy

Garcia [G1/G2] introduced a kind of entropy related to the equal weighted infinitely convolved Bernoulli measures. We will generalize his account here. In contrast to Garsia we will work on the hole shift space Σ^+ and consider all measures μ_β for $\mu \in M(\Sigma^+, \sigma)$.

Let $\sim_{n,\beta}$ be the equivalence relation on Σ^+ given by

$$\underline{i} \sim_{n,\beta} \underline{j} \Leftrightarrow \sum_{k=0}^{n-1} i_k \beta^k = \sum_{k=0}^{n-1} j_k \beta^k$$

and define a partition $\Pi_{n,\beta}$ of Σ^+ by $\Pi_{n,\beta} = \Sigma^+ / \sim_{n,\beta}$. Recall that entropy of a partition Π with respect to a Borel probability measure μ on Σ^+ is

$$H_\mu(\Pi) = -\sum_{P \in \Pi} \mu(P) \log \mu(P).$$

We denote the join of two partitions Π_1 and Π_2 by $\Pi_1 \vee \Pi_2$. This is the partition consisting of all sections $A \cap B$ for $A \in \Pi_1$ and $B \in \Pi_2$.

The following lemma is easy to proof but essential for us.

Lemma 9.2.1.

The partition $\Pi_{n,\beta} \vee \sigma^{-n}(\Pi_{m,\beta})$ is finer than the partition $\Pi_{n+m,\beta}$ and the sequence $H_\mu(\Pi_{n,\beta})$ is sub-additive for a shift invariant measure μ on Σ^+.

Proof

We have that $\sigma^{-n}(\Pi_{m,\beta}) = \Sigma^+ / \asymp_{n,m}$ where $\asymp_{n,m}$ is given by

$$\underline{i} \asymp_{n,m} \underline{j} \Leftrightarrow \sum_{k=n}^{n+m-1} i_k \beta^k = \sum_{k=n}^{n+m-1} j_k \beta^k.$$

and hence $\Pi_{n,\beta} \vee \sigma^{-n}(\Pi_{m,\beta}) = \Sigma^+ / \approx_{n,m}$ where

$$\underline{i} \approx_{n,m} \underline{j} \Leftrightarrow \sum_{k=0}^{n-1} i_k \beta^k = \sum_{k=0}^{n-1} j_k \beta^k \text{ and } \sum_{k=n}^{n+m-1} i_k \beta^k = \sum_{k=n}^{n+m-1} j_k \beta^k.$$

Now obviously $\underline{i} \approx_{n,m} \underline{j} \Rightarrow \underline{i} \sim_{n+m,\beta} \underline{j}$. Thus $\Pi_{n,\beta} \vee \sigma^{-n}(\Pi_{m,\beta})$ is finer than the partition $\Pi_{n+m,\beta}$.
Now let $\mu \in M(\Sigma^+, \sigma)$. By well known properties of H_μ (see 10.13. of [DGS]) we have:

$$H_\mu(\Pi_{n+m,\beta}) \leq H_\mu(\Pi_{n,\beta} \vee \sigma^{-n}(\Pi_{m,\beta}))$$
$$\leq H_\mu(\Pi_{n,\beta}) + H_\mu(\sigma^{-n}(\Pi_{m,\beta})) = H_\mu(\Pi_{n,\beta}) + H_\mu(\Pi_{m,\beta})$$

.

We can now define the **Garsia entropy** $G_\beta(\mu)$ for a shift invariant Borel probability measure μ on Σ^+:

$$G_\beta(\mu) := \lim_{n \longrightarrow \infty} \frac{H_\mu(\Pi_{n,\beta})}{n} = \inf_n \frac{H_\mu(\Pi_{n,\beta})}{n}.$$

The limit in the definition exists and equals the infimum because the sequence $H_\mu(\Pi_{n,\beta})$ is sub-additive. If we have a σ invariant measure μ on the full shift space $\Sigma = \{-1, 1\}^{\mathbb{Z}}$, we define the Garsia entropy of μ as $G_\beta(\mu) := G_\beta(pr_+\mu)$, where pr_+ is the projection of Σ onto Σ^+. $G_\beta(pr_+\mu)$ is well defined because $pr_+\mu$ is σ invariant if μ is σ invariant.

Let us think a moment about the asymmetric case. We could define partitions of Σ^+ in the same way as in the symmetric case. But an analogon of Lemma 9.2.1. does not hold and the limit in the definition of the Garsia entropy does not have to exist.

The Garsia entropy $G_\beta(\mu)$ is less equal to the usual metric entropy $h_\mu(\sigma)$ for a σ invariant measure μ, since the partition of Σ^+ into cylinder sets of length n is finer than $\Pi_{n,\beta}$. If β is not the solution of an algebraic equation with coefficients in $\{-1, 0, 1\}$, these partitions are identical and $G_\beta(\mu) = h_\mu(\sigma)$ holds. Also the partitions $\Pi_{n,\beta}$ are not in general generated by a transformation we can show that the Garsia entropy G, interpreted as a function on the space of σ invariant Borel probability measures on Σ^+, has typical properties of an entropy.

Proposition 9.2.2.

The function $\mu \longmapsto G_\beta(\mu)$ is upper-semi-continuous and affine on the space of σ invariant Borel probability measures on Σ^+.

Proof

We first prove that the function is upper-semi-continuous. Let μ, μ_n be σ invariant Borel probability measures with $\mu_n \to \mu$. Fix $\epsilon > 0$.
From the definition of the Garsia entropy we know there exists an k such that,

$$G_\beta(\mu) \geq \frac{H_\mu(\Pi_{k,\beta})}{k} - \frac{\epsilon}{2}.$$

The elements of the partition $\Pi_{k,\beta}$ are finite unions of cylinder sets in Σ^+ and hence open and closed. Thus we know that, $\lim_{n \longrightarrow \infty} \mu_n(P) = \mu(P) \ \ \forall P \in \Pi_{k,\beta}$.
Hence there exists an n_0 such that for all $n \geq n_0$

$$\frac{1}{k}|H_\mu(\Pi_{k,\beta}) - H_{\mu_n}(\Pi_{k,\beta})| \leq \frac{\epsilon}{2}.$$

Using both inequalitys and 9.2.1. we have:

$$G_\beta(\mu) \geq \frac{1}{k} H_{\mu_n}(\Pi_{k,\beta}) - \epsilon \geq G_\beta(\mu_n) - \epsilon.$$

This proves upper-semi-continuity.

Now let μ_1, μ_2 be σ invariant and $\mu = p\mu_1 + (1-p)\mu_2$ with $p \in (0,1)$. For all partitions Π the inequality

$$0 \leq -pH_{\mu_1}(\Pi) - (1-p)H_{\mu_2}(\Pi) + H_\mu(\Pi) \leq \log 2$$

holds (see proposition 10.13. of [DGS]). Thus by the definition of the Garsia entropy we have $G_\beta(\mu) = pG_\beta(\mu_1) + (1-p)G_\beta(\mu_2)$. But this means that G is affine.

□

The next proposition shows the significants of the Garsia entropy in our discussion.

Proposition 9.2.3.

Let $\beta \in (0.5, 1)$ be the reciprocal of a PV number and μ be a shift invariant Borel probability measure on Σ^+. We have $G_\beta(\mu) \leq \log \beta^{-1}$. Moreover if μ_β is singular then $G_\beta(\mu) < \log \beta^{-1}$ holds.

Garsia sketched in [GA1] a proof of the inequality $G_\beta(b) < \log \beta^{-1}$ using a slightly different notion of Garsia entropy. We will adopt some of his ideas in our proof.

Proof

Fix β. Define π_n from Σ^+ to $[-1,1]$ by $\pi_n((s_k)) = \sum_{k=0}^{n-1} s_k(1-\beta)\beta^k$ and let $\mu_n = \mu \circ \pi_n^{-1}$. Let $\sharp(n)$ be the number of distinct points of the form $\sum_{k=0}^{n-1} \pm(1-\beta)\beta^k$ and $\omega(n)$ be the minimal distance between two of those points. Furthermore denote the points by $x_i^n \;\; i = 1 \ldots \sharp(n)$ and let m_i^n be the μ measure of the corresponding elements in $\Pi_{n,\beta}$, which means $m_i^n = \mu_n(x_i^n)$.

We first state a property of PV numbers we will have to use here, see proposition B2 of appendix B:

$$\beta^{-1} \text{ is PV number } \Rightarrow \exists \;\; \bar{c} : \omega(n) \geq \bar{c}\beta^n.$$

Since $(\sharp(n) - 1)\omega(n) \leq 2$ we get $\sharp(n) \leq 4\omega(n)^{-1} \leq c\beta^{-n}$ with $c := 4\bar{c}^{-1}$.

From this we have that $H_\mu(\Pi_{n,\beta}) \leq \log \sharp(n) \leq \log c + n \log \beta^{-1}$ and hence $G_\beta(\mu) \leq \log \beta^{-1}$.

Now we assume that μ_β is singular. It follows that there exists a constant C such that:
$\forall \epsilon > 0\ \exists$ disjoint intervals $(a_1, b_1), \ldots, (a_u, b_u)$ with

$$\sum_{l=1}^{u}(b_l - a_l) < \epsilon \quad \text{and} \quad \mu_\beta(O) > C \ \text{ where } \ O := \bigcup_{l=0}^{u}(a_l, b_l).$$

With out loss of generality we may assume $\mu_\beta(a_l) = \mu_\beta(b_l) = 0$ for $l = 1 \ldots u$. It is obvious that the discreet distribution μ_n converges weakly to μ_β. Thus we have: $\exists n_1(\epsilon)\ \ \forall n > n_1(\epsilon)\ :\ \mu_n(O) > C$. We now expand the intervals a little bit, so that their length is a multiple of $\omega(n)$.

$$k_{l,n} := \max\{k \mid k\omega(n) \le a_l\} \quad a_{l,n} := k_{l,n}\omega(n)$$

$$\bar{k}_{l,n} := \min\{k \mid b_l \le k\omega(n)\} \quad b_{l,n} := \bar{k}_{l,n}\omega(n)$$

Since $\omega(n) \longrightarrow 0$ we have:
$\exists n_2(\epsilon) > n_1(\epsilon)\ \ \forall n > n_2(\epsilon):\ \ (a_{l,n}, b_{l,n})$ disjunct for $l = 1 \ldots u$ and

$$\sum_{l=1}^{u}(b_{l,n} - a_{l,n}) < \epsilon \quad \text{and} \quad \mu_n(\bar{O}) > C \quad \text{where} \quad \bar{O} = \bigcup_{l=0}^{u}(a_{l,n}, b_{l,n}).$$

Let $\hat{\sharp}(n)$ be the number of distinct points x_i^n in $\bar{O}$. Since in one interval $(a_{l,n}, b_{l,n})$ there are at most $\bar{k}_{l,n} - k_{l,n}$ points x_i^n we have $\omega(n)\hat{\sharp}(n) \le \epsilon$ and hence $\hat{\sharp}(n) \le \epsilon c\beta^{-n}$. For all $n > n_2(\epsilon)$ we can now estimate:

$$
\begin{aligned}
H_\mu(\Pi_{n,\beta}) &= -\sum_{i=1}^{\sharp(n)} m_i^n \log m_i^n = -\sum_{x_i^n \in \bar{O}} m_i^n \log m_i^n - \sum_{x_i^n \notin \bar{O}} m_i^n \log m_i^n \\
&\le \mu_n(\bar{O}) \log \frac{\hat{\sharp}(n)}{\mu_n(\bar{O})} + (1 - \mu_n(\bar{O})) \log \frac{\sharp(n) - \hat{\sharp}(n)}{1 - \mu_n(\bar{O})} \\
&\le \mu_n(\bar{O}) \log \hat{\sharp}(n) + (1 - \mu_n(\bar{O})) \log \sharp(n) + \log 2 \\
&\le \mu_n(\bar{O}) \log \epsilon c\beta^{-n} + (1 - \mu_n(\bar{O})) \log c\beta^{-n} + \log 2 \\
&\le n \log \beta^{-1} + C \log \epsilon + \log c + \log 2.
\end{aligned}
$$

If ϵ is small enough we have $H_\mu(\Pi_{n,\beta})/n < \log \beta^{-1}$ for all $n \ge n_2(\epsilon)$. With the sub-additivity of $H_\mu(\Pi_{n,\beta})$ the desired result follows.

□

In the special case that β is the golden ratio there exists an explicit formula for the Garsia entropy of the equal-weighted Bernoulli measure found by Alexander and Zagier:

Theorem 9.2.4. [AZ]

$$G_{\frac{\sqrt{5}-1}{2}}(b) = \log 2 - \frac{1}{18}\sum_{n=1}^{\infty}\frac{k_n}{4^n} \approx (\log\frac{2}{\sqrt{5}-1}) \cdot 0.995714\ldots$$

$$\text{where} \qquad k_n = \sum_{\substack{a_1,\ldots,a_t \in \mathbb{N} \\ a_1+\ldots+a_t = n \\ p/q = \langle a_1,\ldots,a_t\rangle}} (p+q)\log(p+q)$$

and $\langle a_1, \ldots, a_t\rangle$ denotes the continuous fraction.

This theorem was also proved by Sidorov and Vershik [SV]. One needs quit delicate combinatorial considerations to find this formula and it seems to be very hard to prove a formula for other reciprocals of PV numbers. We do not want to include the prove of 9.2.4. here but we like to present a very nice interpretation of the Garsia entropy of Bernoulli measures in the light of the articles [AZ] and [SV].

If β is the reciprocal of a PV number we define an infinite binary graph associated with β. We label the edges of the graph with -1 each left and $+1$ each right. The vertices at the n'th level of the graph are supposed to correspond to the points x of the form $x = \sum_{k=0}^{n-1} s_k\beta^k$ with $s_k \in \{-1, 1\}$ and paths are the sequences $(s_0, \ldots s_{n-1})$ treated as the representations of these points. If $\beta = \frac{\sqrt{5}-1}{2}$ this graph is called the Fibonacci graph.
Now we may think of a random walk on such a graph where we go left with probability $(1-p)$ and right with the probability p. The probability to reach a vertex x at the n-level of the graph is in our notation just $b_p(P)$ where P is the element of the partition $\Pi_{n,\beta}$ corresponding to x. The entropy of the random walk, we described, is the Garsia entropy $G_\beta(b^p)$.

What has been done in [AZ] and [SV] is (in some sense) to count the number of paths in the Fibonacci graph that reach a vertex x at the n'th level of the graph. This allows to calculate $b(P)$ for $P \in \Pi_{n,\frac{\sqrt{5}-1}{2}}$ and hence the Garsia entropy $G_{\frac{\sqrt{5}-1}{2}}(b)$ resp. the entropy of the random walk with transition probabilities $(1/2, 1/2)$ on the Fibonacci graph.
We remark that this approach is not strong enough to calculate $G_{\frac{\sqrt{5}-1}{2}}(b^p)$ if $p \neq 0.5$. One would have to know not only how many paths reach a vertex, but also how many of them have a given number of steps, say, to the right. It seems awkward to count these quantities.

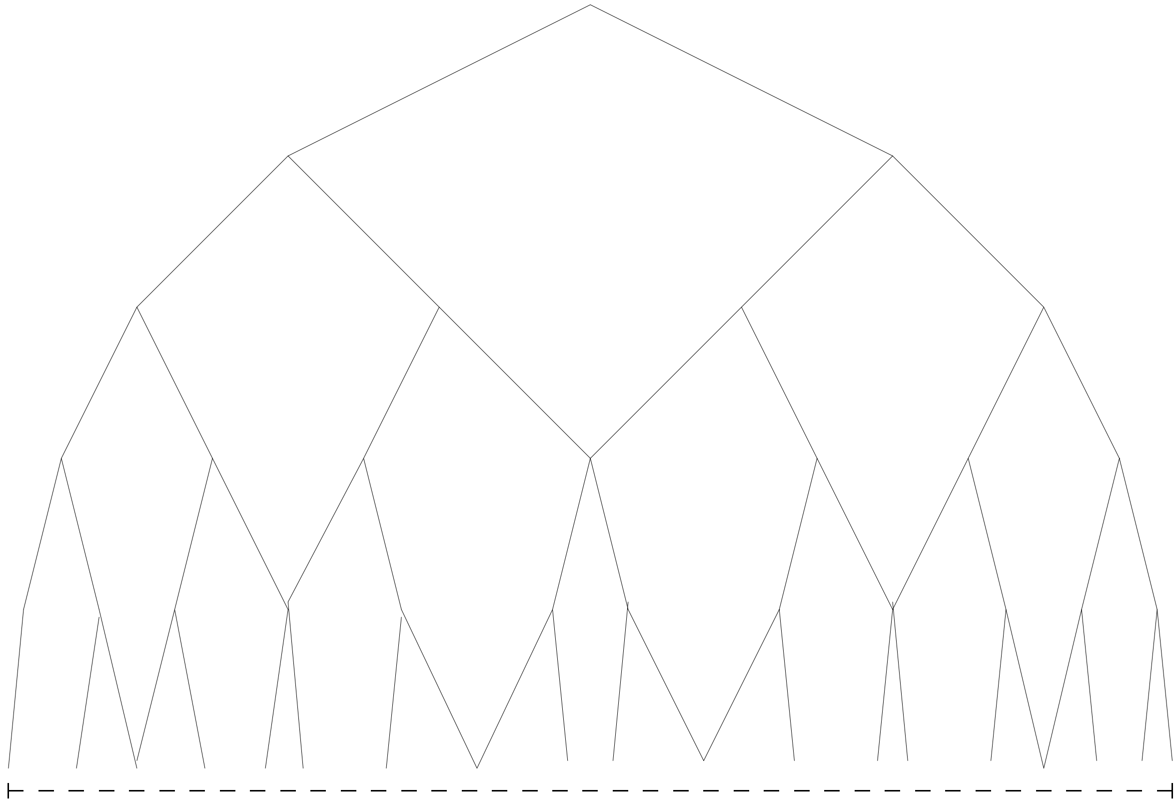

Figure 5: The Fibonacci graph

We end these section with a conjecture about $G_\beta(b^p)$ as a function in p.

Conjecture 9.2.5.

$G_\beta(b^p)$ is a continuous unimodal function in p with maximum at $p = 0.5$.

In the case that $\beta = \frac{\sqrt{5}-1}{2}$ we have some (vague) numerical evidence for this conjecture. Moreover we think that it gets intuitive plausible, if we look at symmetries of the Fibonacci graph.

9.3. An upper bound on the dimension of Erdös measures

We will here prove an upper bound on Hausdorff dimension of all measures μ_β in terms of the Garsia entropy. In view of our result about the Garsia entropy in 9.2.3., if β is the reciprocal of a PV number, this bound is of special interest if we consider Erdös measures.

Because we will operate with Rényi dimension $\dim_R$ (see appendix A) we are interested in an upper bound on the quantity

$$h_\mu(\epsilon) = \inf\{H_\mu(\Pi) | \Pi \text{ a partition with } \mathrm{diam}\Pi \le \epsilon\}$$

by the entropy of the partitions $\Pi_{n,\beta}$ of Σ^+. The following lemma plays the crucial role in our argumentation.

Lemma 9.3.1.

$$h_{\mu_\beta}(2\beta^n) \le H_\mu(\Pi_{n,\beta})$$

Proof

Fix $\beta \in (0.5, 1)$, $\tau \in (0, 0.5)$, a measure μ on Σ^+ and $n \in I\!N$.
We define a partition of $\Lambda_{\beta,\tau}$ by $\wp_n = \pi_{\beta,\tau}(\Pi_{n,\beta})$. By definition we have

$$H_\mu(\Pi_{n,\beta}) = H_{\mu_{\beta,\tau}}(\wp_n).$$

We should say something about the structure of $\wp_n$. The image of a cylinder set $[i_0, \dots, i_{n-1}]_0$ in Σ^+ under $\pi_{\beta,\tau}$ is the part of $\Lambda_{\beta,\tau}$ lying in the rectangle $T_{i_{n-1}} \circ \dots \circ T_{i_0}(Q)$ of x-length $2\beta^n$. It is not difficult to check that two cylinder sets lie in the same element of $\Pi_{n,\beta}$ if and only if the corresponding rectangles lie above each other. So the projection of an element in $\wp_n$ onto the x-axis has length $2\beta^n$.
The projection onto the x-axis of two elements in $\wp_n$ may overlap. Starting with $\wp_n$, we want to construct inductively a partition $\bar{\wp}_n$ of $\Lambda_{\beta,\tau}$ with non-overlapping projections, in a way that does neither increase length of the projections nor entropy. Let $N(\wp)$ be the number of pairs of elements in a partition $\wp$ that do have overlapping projections onto the x-axis. We now construct a finite sequence $\wp_n^k$ of partitions. First let $\wp_n^0 = \wp_n$. Now let $\wp_n^k$ be constructed and $N(\wp_n^k) > 0$.
Let P_1 and P_2 be two elements of $\wp_n^k$ with overlapping projections. Without loss of generality we may assume $\mu_{\beta,\tau}(P_1) \ge \mu_{\beta,\tau}(P_2)$ and define:

$$\hat{P}_1 = P_1 \cup (P_2 \cap (pr_X P_1 \times [-1, 1])) \qquad \hat{P}_2 = P_2 \backslash (pr_X P_1 \times [-1, 1]).$$

We have $\hat{P}_1 \dot\cup \hat{P}_2 = P_1 \dot\cup P_2$, $P_1 \subseteq \hat{P}_1$ and $\hat{P}_2 \subseteq P_2$.
Thus we know: $\mu_{\beta,\tau}(P_1) + \mu_{\beta,\tau}(P_2) = \mu_{\beta,\tau}(\hat{P}_1) + \mu_{\beta,\tau}(\hat{P}_2)$ and
$\mu_{\beta,\tau}(\hat{P}_1) \ge \mu_{\beta,\tau}(P_1) \ge \mu_{\beta,\tau}(P_2) \ge \mu_{\beta,\tau}(\hat{P}_2)$.
Since the function $-x \log x$ is concave, this implies:

$$-(\mu_{\beta,\tau}(\hat{P}_1) \log \mu_{\beta,\tau}(\hat{P}_1) + \mu_{\beta,\tau}(\hat{P}_2) \log \mu_{\beta,\tau}(\hat{P}_2)) \le$$

$$-(\mu_{\beta,\tau}(P_1) \log \mu_{\beta,\tau}(P_1) + \mu_{\beta,\tau}(P_2) \log \mu_{\beta,\tau}(P_2)).$$

Hence if we substitute $\hat{P}_1$, $\hat{P}_2$ for P_1, P_2, we get a partition $\wp_n^{k+1}$ of $\Lambda_{\beta,\tau}$ with non-increased entropy.
From the definition of $\hat{P}_1$ and $\hat{P}_2$ we see that $pr_X\hat{P}_1 = pr_X P_1$, $pr_X\hat{P}_2 \subseteq pr_X P_2$ and that the projections of $\hat{P}_1$ and $\hat{P}_2$ onto the x-axis do not overlap. So the length of the projections are obviously not increased. Furthermore we observe that there cannot be any new overlaps of the projections of $\hat{P}_1$ or $\hat{P}_2$ with the projections of other elements in $\wp_n^k$, that do not appear, when we consider P_1 or P_2. Hence $N(\wp_n^{k+1}) < N(\wp_n^k)$.
So after a finite number of steps we get a partition $\bar{\wp}_n$ with

$$H_{\mu_{\beta,\tau}}(\wp_n) \geq H_{\mu_{\beta,\tau}}(\bar{\wp}_n),$$

non-overlapping projections onto the x-axis and diam $pr_X\bar{\wp}_n \leq 2\beta^n$.
$pr_X\bar{\wp}_n$ is a partition of the interval $[-1,1]$ and we have

$$H_{\mu_\beta}(pr_X\bar{\wp}_n) = H_{\mu_{\beta,\tau}}(\bar{\wp}_n),$$

since the measure μ_β is the projection of $\mu_{\beta,\tau}$ onto the x-axis. Now the proof is complete:

$$h_{\mu_\beta}(2\beta^n) \leq H_{\mu_\beta}(pr_X\bar{\wp}_n) = H_{\mu_{\beta,\tau}}(\bar{\wp}_n) \leq H_{\mu_{\beta,\tau}}(\wp_n) = H_\mu(\Pi_{n,\beta}).$$

□

The idea of cutting up overlaps we used here appeared in an other form in the work of Alexander and Yorke [AY]. From our lemma it is easy for us to deduce the following proposition:

Proposition 9.3.2.

If μ is a shift ergodic Borel probability measure on Σ^+ we have:

$$\dim_H \mu_\beta \leq G_\beta(\mu)/\log\beta^{-1}.$$

Proof

First we estimate the Rényi dimension:

$$\overline{\dim}_R\mu_\beta = \overline{\lim}_{\epsilon\longrightarrow\infty}\frac{h_{\mu_\beta}(\epsilon)}{\log\epsilon^{-1}} = \overline{\lim}_{n\longrightarrow\infty}\frac{h_{\mu_\beta}(2\beta^n)}{\log 0.5\beta^{-n}} = \overline{\lim}_{n\longrightarrow\infty}\frac{h_{\mu_\beta}(2\beta^n)}{n\log\beta^{-1}}$$

$$\leq \lim_{n\longrightarrow\infty}\frac{H_\mu(\Pi_{n,\beta})}{n\log\beta^{-1}} = \frac{G_\beta(\mu)}{\log\beta^{-1}}.$$

Using part (2) of theorem A2 from appendix A we get:

$\forall\delta > 0\ \exists X$ with $\mu_\beta(X) > 0$ and $\underline{d}(x, \mu_\beta) \leq G_\beta(\mu)/\log\beta^{-1} + \delta \quad \forall x \in X$.
But the measure μ_β is exact dimensional, because it is the transversal measure in the context of the ergodic dynamical system $(\Lambda_{\beta,\beta,\tau,\tau}, T_{\beta,\beta,\tau,\tau}, \mu_{\beta,\beta,\tau,\tau})$. This fact was observed by Ledrappier and Porzio, see [LP]. So our estimate must hold μ_β-almost everywhere and by part (3) of theorem A2 we get $\dim_H \mu_\beta \leq G_\beta(\mu)/\log\beta^{-1} + \delta$ for all $\delta > 0$. This proves the proposition.

□

Let us here again remark that Alexander and Yorke [AY] have proved the identity $\dim_R b_\beta = G_\beta(b)/\log\beta^{-1}$ for the equal-weighted infinitely convolved Bernoulli measure b_β. In their proof they used the self-similarity of this measure. Our proof of 9.3.2. shows that appealing to self-similarity is not necessary for the upper bound.

At the beginning of this section we have formulated our main result in theorem 9.1.1. . We are now able to give the proof of this result.

Proof of 9.1.1.

Let $\beta \in (0,5,1)$ be the reciprocal of a PV number. We have for all $\mu \in M(\Sigma^+, \sigma)$:

$$\mu_\beta \text{ is singular} \Rightarrow^{9.2.3.} G_\beta(\mu) < \log\beta^{-1} \Rightarrow^{9.3.2.} \dim_H \mu_\beta < 1 \Rightarrow \mu_\beta \text{ is singular.}$$

These implications prove the first statement of our theorem. Now choose an Erdös measure ξ_β with $\dim\xi_\beta < 1$. We have $G_\beta(\xi) < \log\beta^{-1}$. By upper-semi-continuity of G (9.2.2.) $G_\beta(\mu) < \log\beta^{-1}$ and hence $\dim\mu_\beta < 1$ holds for all μ in a hole weak* neighborhood of ξ in $M(\Sigma^+, \sigma)$. Thus the set $\{\mu \in M(\Sigma, \sigma) | \mu_\beta \text{ is singular}\}$ is open in the weak* topology on $M(\Sigma, \sigma)$. The set contains the Bernoulli measure by proposition 9.1.2. .

□

9.4. Construction of an Erdös measure with full dimension

Let β^{-1} be a PV number as usual. We will construct here a measure $m \in M(\Sigma^+, \sigma)$ (depending on β) such that the Erdös measure m_β has Hausdorff dimension one. From the proof of our main theorem 9.1.1. we know that it is sufficient to find a $m \in M(\Sigma^+, \sigma)$ of full Garsia entropy, which means $G_\beta(m) = \log\beta^{-1}$. In ergodic theory there is a quite natural construction of an invariant measure with maximal metric entropy, see 4.5. of [KH] or 18 of [DGS]. We will use a similar construction for the Garsia entropy.

Proposition 9.4.1.

Let $\beta \in (0.5, 1)$ be the reciprocal of PV number. There exists a measure $m \in M(\Sigma^+, \sigma)$, such that $G_\beta(m) = \log \beta^{-1}$ and hence $\dim_H m_\beta = 1$.

Proof

In this prove we will omit the subscript β. We first construct a shift invariant measure m with $G_\beta(m) = \log \beta^{-1}$ and afterwards prove the existence of an ergodic one.

Recall that $\sharp_\beta(n)$ denotes the number of elements of the partition $\Pi_{\beta,n}$. Now choose measures $m_n \in M(\Sigma^+)$ such that

$$m_n(P) = 1/\sharp_\beta(n) \qquad \forall P \in \Pi_{\beta,n}$$

and let m be a weak* accumulation point of the sequence

$$\bar{m}_n = \frac{1}{n}\sum_{i=0}^{n-1} m_n \circ \sigma^{-i}.$$

By this construction we immediately have that m is invariant under σ.

Given two partitions $\wp_1$ and $\wp_2$ on Σ^+ we write $\wp_1 \preceq \wp_2$ if $\wp_2$ is finer than $\wp_1$. Note that $\wp_1 \preceq \wp_2 \Rightarrow \sigma^{-k}(\wp_1) \preceq \sigma^{-k}(\wp_2)$ and $\sigma^{-k}(\wp_1 \vee \wp_2) \preceq \sigma^{-k}(\wp_1) \vee \sigma^{-k}(\wp_2)$ where $\vee$ denotes the join as usual. Recall that we know from lemma 9.2.1. $\Pi_{n+m} \preceq \Pi_n \vee \sigma^{-n}(\Pi_m)$. From these facts we get by induction $\Pi_{aq} \preceq \bigvee_{i=0}^{a-1} \sigma^{-iq}(P_q)$. Let $\lfloor x \rfloor$ be the integer part of x. Given n and q and k with $0 < q < n$ and $0 \le k < q$ we set $a(k) = \lfloor (n-k)/q \rfloor$ and write $n-k$ in the form $a(k)q + r$ with $0 \le r < q$. We get

$$\begin{aligned}\Sigma_n &\preceq \sigma^{-k}(\Sigma_{n-k}) \vee \Sigma_k \preceq \sigma^{-k}(\Sigma_{a(k)q}) \vee \sigma^{-(a(k)q+k)}(\Sigma_r) \vee \Sigma_k \\ &\preceq \bigvee_{i=0}^{a(k)-1} \sigma^{-iq+r}(\Sigma_q) \vee \sigma^{-(a(k)q+k)}(\Sigma_r) \vee \Sigma_k\end{aligned}$$

and hence

$$\begin{aligned}H_{m_n}(\Sigma_n) &\le \sum_{i=0}^{a(k)-1} H_{m_n}(\sigma^{-iq+k}(\Sigma_q)) + H_{m_n}(\sigma^{-(a(k)q+k)}(\Sigma_r)) + H_{m_n}(\Sigma_k) \\ &\le \sum_{i=0}^{a(k)-1} H_{m_n}(\sigma^{-iq+k}(\Sigma_q)) + 2q\log 2.\end{aligned}$$

The last inequality follows from the fact, that the partitions Σ_q and Σ_r have less than 2^q elements. Now summing over k gives

$$qH_{m_n}(\Sigma_n) \le \sum_{k=0}^{q-1}\sum_{i=0}^{a(k)-1} H_{m_n}(\sigma^{-iq+k}(\Sigma_q)) + 2q^2\log 2 \le nH_{\bar{m}_n}(\Sigma_q) + 2q^2\log 2.$$

This implies

$$\frac{H_{m_n}(\Sigma_n)}{n} \leq \frac{H_{\bar{m}_n}(\Sigma_q)}{q} + 2\log 2\frac{q}{n}.$$

By the definition of the measures m_n we have $H_{m_n}(\Sigma_n) = \log \sharp(n)$ and from proposition B2 of appendix B we know $\sharp(n) \geq \bar{C}\beta^{-n}$. This gives us:

$$\log \beta^{-1} + \frac{\log \bar{C}}{n} \leq \frac{H_{\bar{m}_n}(\Sigma_q)}{q} + 2\log 2\frac{q}{n}.$$

By the definition of m we thus have $\log \beta^{-1} \leq H_m(\Sigma_q)/q$, which implies $\log \beta^{-1} \leq G(m)$. The opposite inequality has been proved in 9.2.3. .

We have shown up to this point that the set $M := \{\mu|\mu\ \sigma\text{-invariant and } G(\mu) = \log \beta^{-1}\}$ of Borel measures on Σ^+ is not empty. We know from 9.2.2. that G is upper semi-continuous and affine, which implies that M is compact and convex with respect to the weak* topology. By Krein-Milman theorem there exist an extremal point m of M.
We show that m is extremal in the space of σ invariant Borel probability measure and hence σ ergodic.
If this is not the case we have $m = p\mu_1 + (1-p)\mu_2$ for two distinct σ invariant measures μ_1 and μ_2 and $p \in (0,1)$. Since μ is extremal in M we have that μ_1 or μ_2 is not in M. From 9.2.3. it follows that $G(\mu_1) < \log \beta^{-1}$ or $G(\mu_2) < \log \beta^{-1}$. This implies $G(m) < \log \beta^{-1}$ because G is affine. This is a contradiction to $m \in M$.

□

We remark here that we do not know if the measure of full dimension in the last proposition is unique. The construction describe is not unique. Obviously one can choose the measures m_n in different ways. But it is not clear, if this induces different Erdös measures of full dimension.

10. Number theoretical peculiarities

10.1. Ergodic Measures

Now we study the ergodic measures for the systems $(\Lambda_\vartheta, T_\vartheta)$, $(\Lambda_\vartheta, \hat{f}_\vartheta)$ and $([-1,1]^2, f_\beta)$ in the case that $\vartheta = (\beta, \beta, \tau, \tau) \in P^4_{all}$ and β a reciprocal of a PV number. We concentrate on the variational principle for Hausdorff dimension.

Theorem 10.1.1.

If β is the reciprocal for a PV number we have:
(1) The variational principle for Hausdorff dimension does not hold for the Fat Baker's transformation $([-1,1]^2, f_\beta)$.
(2) The variational principle for Hausdorff dimension does not hold for the attractors $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ where $\vartheta = (\beta, \beta, \tau, \tau)$ and τ is sufficient small.
(3) For the repellers $(\Lambda_\vartheta, T_\vartheta)$ with $\vartheta = (\beta, \beta, \tau, \tau)$ and τ sufficient small Bernoulli measures do not have full Hausdorff dimension.

Remark

This theorem compared with our results in chapter seven shows that the dimensional theoretical properties of a dynamical systems can considerably change because of number theoretical peculiarities. Particular looking at the attractors $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ for τ small and on the systems $([-1,1]^2, f_\beta)$ we see that in situations, where the variational principle for Hausdorff dimension generically holds, it does not have to hold generally because of such peculiarities. Looking at the repellers $(\Lambda_\vartheta, T_\vartheta)$ for τ small, we see that generically a Bernoulli measure of full dimension is available but if the parameters have special number theoretical properties then such a measure does not exist. This provides substantial difficulties. We can not decide with our technique whether there exists a measure full dimension for $(\Lambda_\vartheta, T_\vartheta)$ if β is the reciprocal of a PV number .

Now we want to proof theorem 10.1.1.

Proof of 10.1.1.

(1) Let $\mu \in M(\Sigma, \sigma)$. By 5.3.5. and 9.3.2. we have:

$$\dim_H \bar{\mu}_\beta \leq 1 + \dim_H (pr^+\mu)_\beta \leq 1 + G_\beta(pr^+\mu)/\log \beta^{-1}.$$

By 9.2.2. and 9.2.3. $G_\beta(pr^+\mu)/\log\beta^{-1} \leq c_1 < 1$ holds for all μ in hole weak* neighborhood U of b in $M(\Sigma, \sigma)$. Hence $\dim_H \bar{\mu}_\beta \leq c_1 + 1 < 2$ holds for all μ in U.

On the other hand we have by the properties of the metric entropy $h_\mu(\sigma)/\log 2 \leq c_2 < 1$ on the complement of U. With 5.3.4. it follows that $\dim_H \bar{\mu}_\beta \leq c_2 + 1 < 2$ holds for all $\mu \in M(\Sigma^+, \sigma)\backslash U$. Putting these facts together we obtain:

$$\dim_H \bar{\mu}_\beta \leq \max\{c_1, c_2\} + 1 < 2 = \dim[-1,1]^2 \qquad \forall \mu \in M(\Sigma, \sigma).$$

But by 3.2.4. all ergodic measures for the system $([-1,1]^2, f_\beta)$ are of the form $\bar{\mu}_\beta$ for some $\mu \in M(\Sigma, \sigma)$. So the proof is complete.

(2) Let $\mu \in M(\Sigma, \sigma)$. By 5.3.2. we have:

$$\dim \hat{\mu}_\vartheta \leq \frac{h_\mu(\sigma)}{\log 2} + \frac{h_\mu(\sigma)}{\log \tau^{-1}} + (1 - \frac{\log \beta}{\log \tau}) \dim_H (pr^+\mu)_\beta.$$

This implies

$$\dim \hat{\mu}_\vartheta \leq 1 + \frac{h_\mu(\sigma)}{\log 2} + \frac{\log 2\beta}{\log \tau^{-1}}$$

and combined with 9.3.2.

$$\dim \hat{\mu}_\vartheta \leq 1 + \frac{G_\beta(pr^+\mu)}{\log 2} + \frac{\log 2\beta}{\log \tau^{-1}}.$$

By the same arguments we used in (1) we now see

$$\dim_H \hat{\mu}_\beta \leq \max\{c_1, c_2\} + 1 + \frac{\log 2\beta}{\log \tau^{-1}} \qquad \forall \mu \in M(\Sigma, \sigma)$$

where the constants c_1, c_2 are the same as in (1). If τ is sufficient small we get

$$\dim_H \hat{\mu}_\beta \leq c < 2 \leq \dim_H \hat{\Lambda}_\vartheta$$

for all $\mu \in M(\Sigma, \sigma)$. But by 3.2.5. all ergodic measures for the system $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ are of the form $\bar{\mu}_\beta$ for some $\mu \in M(\Sigma, \sigma)$. This completes the proof.

(3) From 5.3.3. we know

$$\dim_H b^p_\vartheta = \frac{-p\log p - (1-p)\log(1-p)}{\log \tau^{-1}} + (1 - \frac{\log \beta}{\log \tau}) \dim_H b^p_\beta \qquad \forall p \in (0,1).$$

From 9.1.1. we have $\dim_H b^p_\beta \leq c < 1$ for all $p \in (0,1)$. This implies

$$\dim_H b^p_\vartheta \leq c + \frac{\log 2 + c\log \beta}{\log \tau^{-1}} \qquad \forall p \in (0,1).$$

If τ is sufficient small, we get our result:

$$\dim_H b^p_\vartheta \leq \bar{c} < 1 \leq \dim_H \Lambda_\vartheta \qquad \forall p \in (0,1).$$

□

10.2. Invariant Sets

We prove here upper bounds on the Hausdorff dimension of the repellers Λ_ϑ and attractors $\hat{\Lambda}_\vartheta$ in the symmetric situation $\vartheta = (\beta, \beta, \tau, \tau) \in P^4_{all}$ under the assumption, that β^{-1} is a PV number. An important consequence of this upper bound is the following theorem:

Theorem 10.2.1.

If β is the reciprocal for a PV number, $\tau \in (0, 0.5)$ and $\vartheta = (\beta, \beta, \tau, \tau)$ we have:

$$\dim_H \Lambda_\vartheta < \dim_B \Lambda_\vartheta \quad \text{and} \quad \dim_H \hat{\Lambda}_\vartheta < \dim_B \hat{\Lambda}_\vartheta.$$

Remark

If we compare this result with 7.1. we learn that dimensional theoretical properties of invariant sets of a dynamical system can considerably change because of number theoretical peculiarities of parameter values. For our classes of attractors and repellers we generically have the identity for Hausdorff dimension and box-counting dimension, but for parameter values with special number theoretical properties this identity does not hold.

To get 10.2.1. we prove now explicit upper bounds on the Hausdorff dimension of Λ_ϑ.

Proposition 10.2.2.

If β is the reciprocal for a PV number, $\tau \in (0, 0.5)$ and $\vartheta = (\beta, \beta, \tau, \tau)$ we have:

$$\dim_H \Lambda_\vartheta \leq \frac{\log(\sum_{P \in \Pi_{n,\beta}} (\sharp P)^{\frac{\log \beta}{\log \tau}})}{n \log \beta^{-1}} \qquad \forall n \geq 1$$

where $\Pi_{n,\beta}$ is the partition of Σ^+ defined in 9.2. and $\sharp P$ denotes the number of cylinder sets of length n contained in an element of this partition.

Proof

Fix a reciprocal of a PV number β, $\tau \in (0, 0.5)$ and $\vartheta = (\beta, \beta, \tau, \tau)$. Let $n \geq 1$ and set

$$u_n = \frac{\log(\sum_{P \in \Pi_{n,\beta}} (\sharp P)^{\frac{\log \beta}{\log \tau}})}{n \log \beta^{-1}}.$$

Consider the set of cylinders in Σ^+ given by $C_n = \{[\tilde{s}_1\tilde{s}_2\dots\tilde{s}_m]_0 \mid \tilde{s}_i \in \{-1,1\}^n \ i = 1\dots m\}$. Define a set function η on C_n by

$$\eta([\tilde{s}]_0) = \frac{\sharp P(\tilde{s})^{\log\beta/\log\tau}}{\sharp P(\tilde{s})}\beta^{nu_n} \quad \text{and} \quad \eta([\tilde{s}_1\tilde{s}_2\dots\tilde{s}_m]_0) = \eta([\tilde{s}_1]_0)\cdot\eta([\tilde{s}_2]_0)\cdot\dots\cdot\eta([\tilde{s}_m]_0)$$

where $\tilde{s}, \tilde{s}_1, \dots \tilde{s}_m$ are elements of $\{-1,1\}^n$ and $P(\tilde{s})$ denotes the element of the partition $\Pi_{n,\beta}$ containing the cylinder $[\tilde{s}]_0$.

Note the facts that C_n is a basis of the metric topology of Σ^+ and that $\sum_{\tilde{s}\in\{-1,1\}^n}\eta([\tilde{s}]_0) = 1$ by the definition of u_n. Thus we can extend η to a Borel probability measure on Σ^+.

Now recall that the map $\pi_\vartheta = \pi_{\beta,\beta,\tau,\tau}$ given by

$$\pi_\vartheta(\underline{s}) = (\sum_{i=0}^{\infty} s_i(1-\beta)\beta^i, \sum_{i=0}^{\infty} s_i(1-\tau)\tau^i)$$

is a homeomorphism from Σ^+ onto Λ_ϑ. Thus $\eta_{\beta,\tau} := \eta \circ \pi_\vartheta^{-1}$ defines a Borel probability measure on Λ_ϑ.

Given $m \geq 1$ we set $q(m) = \lceil m(\log\beta/\log\tau)\rceil$. Given a sequence $\tilde{s}_i \in \{-1,1\}^n$ for $i = 1\dots m$ we define a subset of Λ_ϑ by

$$R_{\tilde{s}_1\dots\tilde{s}_n} = \{(\sum_{i=0}^{\infty} s_i(1-\beta)\beta^i, \sum_{i=0}^{\infty} t_i(1-\tau)\tau^i) \mid s_i, t_i \in \{-1,1\}$$

$$(s_{(i-1)n}, \dots, s_{in-1}) = \tilde{s}_i \quad i = 1\dots m \quad \text{and} \quad (t_{(i-1)n}, \dots, t_{in-1}) = \tilde{s}_i \quad i = 1\dots q(m)\}.$$

We see that $R_{\tilde{s}_1\dots\tilde{s}_m}$ is "almost" a square in Λ_ϑ of side length β^{mn}. We have:

$$c_1\beta^{mn} \leq \operatorname{diam} R_{\tilde{s}_1\dots\tilde{s}_m} \leq c_2\beta^{mn} \qquad (1)$$

where the constants c_1, c_2 are independent of the choice of $\tilde{s}_i$.

Now let as examine the $\eta_{\beta,\tau}$ measure of the sets $R_{\tilde{s}_1\dots\tilde{s}_m}$.
Assume that $\tilde{t}_i \sim_{n,\beta} \tilde{s}_i$ for $i = q(m)+1\dots m$ where $\sim_{n,\beta}$ is the equivalence relation introduced in 9.2. . The rectangles $\pi_\vartheta([\tilde{s}_1\dots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\dots\tilde{t}_m]_0)$ are all disjoint and lie above each other in the set $R_{\tilde{s}_1\dots\tilde{s}_m}$. Hence we have

$$\eta_{\beta,\tau}(R_{\tilde{s}_1\dots\tilde{s}_m}) \geq \eta(\bigcup_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \quad i=q(m)+1\dots m} \pi_\vartheta([\tilde{s}_1\dots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\dots\tilde{t}_m]_0) =$$

$$= \sum_{\tilde{t}_i\sim_{n,\beta}\tilde{s}_i \quad i=q(m)+1\dots m} \eta([\tilde{s}_1\dots\tilde{s}_{q(m)}\tilde{t}_{q(m)+1}\dots\tilde{t}_m]_0).$$

Using the fact $\tilde{s} \sim_{n,\beta} \tilde{t} \Rightarrow \sharp P(\tilde{s}) = \sharp P(\tilde{t}) \Rightarrow \eta([\tilde{s}]_0) = \eta([\tilde{t}]_0)$ this equals

$$\prod_{i=1}^{m} \eta([\tilde{s}_i]_0) \sum_{\tilde{t}_i \sim_{n,\beta} \tilde{s}_i \quad i=q(m)+1\ldots m} 1 = \prod_{i=1}^{m} \frac{\sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\sharp P(\tilde{s}_i)} \beta^{mnu_n} \sum_{\tilde{t}_i \sim_{n,\beta} \tilde{s}_i \quad i=q(m)+1\ldots m} 1 =$$

$$= \frac{\prod_{i=1}^{m} \sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\prod_{i=1}^{q(m)} \sharp P(\tilde{s}_i)} \beta^{mnu_n} = (\phi_{\tilde{s}_1\ldots\tilde{s}_m} \beta^{nu_n})^m$$

where

$$\phi_{\tilde{s}_1\ldots\tilde{s}_m} = \left(\frac{\prod_{i=1}^{m} \sharp P(\tilde{s}_i)^{\log\beta/\log\tau}}{\prod_{i=1}^{q(m)} \sharp P(\tilde{s}_i)}\right)^{1/m}.$$

Now fix an $\epsilon > 0$ We use the sets $R_{\tilde{s}_1\ldots\tilde{s}_m}$ to construct a good cover of Λ_ϑ in the sense for Hausdorff dimension. To this end set

$$R_m := \{R_{\tilde{s}_1\ldots\tilde{s}_m} | \phi_{\tilde{s}_1\ldots\tilde{s}_m} \geq \beta^{n\epsilon}\}.$$

We have an upper bound on the cardinality of R_m. If $R \in R_m$ then $\eta_{\beta,\tau}(R) \geq \beta^{mn(u_n+\epsilon)}$ and since $\eta_{\beta,\tau}$ is a probability measure we see:

$$\text{card}(R_m) \leq \beta^{-mn(u_n+\epsilon)} \qquad (2).$$

Now let $R(M) = \bigcup_{m\geq M} R_m$. We want to prove that $R(M)$ is a cover of Λ_ϑ for all $M \geq 1$.

For $\underline{s} = (s_k) \in \Sigma^+$ we define the function ϕ_m by $\phi_m(\underline{s}) = \phi_{s_0\ldots s_{mn-1}}$. In addition we need two auxiliary functions on Σ^+:

$$f_m(\underline{s}) = \frac{\prod_{i=0}^{m} \sharp P((s_{(i-1)n}, \ldots, s_{in-1}))^{1/m}}{\prod_{i=0}^{q(m)} \sharp P((s_{(i-1)n}, \ldots, s_{in-1}))^{1/q(m)}},$$

$$g_m(\underline{s}) = (\prod_{i=1}^{q(m)} \sharp P((s_{(i-1)n}, \ldots, s_{in-1})))^{1/q(m)(\log\beta\log\tau - q(m)/m)}.$$

Since $1 \leq \sharp P(\tilde{s}) \leq 2^n$ we have $1 \leq g_m(\underline{s}) \leq 2^{n(\log\beta/\log\tau - q(m)/m)}$. Thus by the definition of $q(m)$ we have $g_m(\underline{s}) \longrightarrow 1$. Moreover we have $\overline{\lim}_{m\longrightarrow\infty} f_m(\underline{s}) \geq 1$ because $\prod_{i=0}^{t} \sharp P((s_{i-1}n, \ldots, s_{in-1}))^{1/t} \geq 1 \;\; \forall t \geq 1$.
A simple calculation shows $\phi_m(\underline{s}) = (f_m(\underline{s}))^{\log\beta/\log\tau} g_m(\underline{s})$. The properties of f and g thus imply:

$$\overline{\lim}_{m\longrightarrow\infty} \phi_m(\underline{s}) \geq 1 \qquad \forall\ \underline{s} \in \Sigma^+.$$

This will help us to show that $R(M)$ is a cover of Λ_ϑ. For all $\underline{s} = (s_k) \in \Sigma^+$ there is an $m \geq M$ such that $\phi_m(\underline{s}) \geq \beta^{n\epsilon}$ and thus $\pi_\vartheta(\underline{s}) \in R_{s_0,\ldots,s_{mn-1}} \in R(M)$. Since π_ϑ

is onto Λ_ϑ we see that $R(M)$ is indeed a cover of Λ_ϑ.

We are now able to complete the proof. For every $\epsilon > 0$ and every $M \in \mathbb{N}$ we have:

$$\sum_{R\in R(M)} (\mathrm{diam} R)^{u_n+2\epsilon} = \sum_{m\geq M} \sum_{R\in R_m} (\mathrm{diam} R)^{u_n+2\epsilon}$$
$$\leq^{(1)} \sum_{m\geq M} \sum_{R\in R_m} (c_2\beta^{mn})^{u_n+2\epsilon} = \sum_{m\geq M} \mathrm{card}(R_m)(c_2\beta^{mn})^{u_n+2\epsilon}$$
$$\leq^{(2)} c_2^{u_n+2\epsilon} \sum_{m\geq M} \beta^{mn\epsilon}.$$

The last expression goes to zero with $M \longrightarrow 0$. By the definition for Hausdorff dimension we thus get $\dim_H \Lambda_\vartheta \leq u_n + 2\epsilon$ and since ϵ is arbitrary, we have $\dim_H \Lambda_\vartheta \leq u_n$.

□

Some ideas we have used here are to due the prove of McMullen's theorem (2.1.2.) by Pesin in [PE2].

Now we use strategies developed in the proof of 9.2.3. to get:

Proposition 10.2.3.

If β is the reciprocal for a PV number, $\tau \in (0, 0.5)$ and $\vartheta = (\beta, \beta, \tau, \tau)$ we have:

$$\exists\ N\ \ \forall\ n > N \qquad \frac{\log(\sum_{P\in\Pi_{n,\beta}} (\sharp P)^{\frac{\log\beta}{\log\tau}})}{n\log\beta^{-1}} < \frac{\log(2\beta/\tau)}{\log(1/\tau)}.$$

Proof

Fix a reciprocal of a PV number β. Consider the proof of 9.2.3. for the equal weighted Bernoulli measure b. Recall that we denote by x_i^n $i = 1 \ldots \sharp(n)$ the distinct points of the form $\sum_{k=0}^{n-1} \pm(1-\beta)\beta^k$ and by m_i^n the b measure of corresponding element P_n^i from the partition $\Pi_{n,\beta}$.

By the singularity of b_β we have more than we used in 9.2.3. :
$\forall C \in (0,1)\ \forall\epsilon > 0\ \exists$ disjoint intervals $(a_1, b_1), \ldots, (a_u, b_u)$ with

$$\sum_{l=1}^{u} (b_l - a_l) < \epsilon \quad \text{and} \quad b_\beta(O) > C \ \text{ where } \ O := \bigcup_{l=0}^{u} (a_l, b_l).$$

By the same arguments we used in the proof of 9.2.3., we conclude:
$\exists c > 0\ \forall C \in (0,1)\ \forall \epsilon > 0\ \exists N = N(\epsilon, C)\ \forall n \geq N$:

$$\sum_{x_i^n \in \bar{O}} m_i^n > C \text{ and } \hat{\sharp}(n) := \text{card}\{x_i^n \in \bar{O}\} \leq \epsilon c \beta^{-n}.$$

Since $m_i^n = b(P_n^i) = \sharp P_n^i / 2^n$, where $\sharp P$ denotes the number of cylinder sets of length n contained in P, it follows that there is a subset $\hat{\Pi}_{n,\beta}$ of $\Pi_{n,\beta}$ with $\hat{\sharp}(n)$ elements such that

$$\sum_{P \in \hat{\Pi}_{n,\beta}} \sharp P \geq C 2^n$$

Now we estimate:

$$\sum_{P \in \Pi_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} = \sum_{P \in \hat{\Pi}_{n,\beta}} (\sharp P)^{\log\beta/\log\tau} + \sum_{P \in \Pi_{n,\beta} \backslash P i_{n,\beta}} (\sharp P)^{\log\beta/\log\tau}$$

$$\leq \hat{\sharp}(n)^{1-\log\beta/\log\tau} \Big(\sum_{P \in \hat{\Pi}_{n,\beta}} \sharp P \Big)^{\log\beta/\log\tau} (\sharp(n) - \hat{\sharp}(n))^{1-\log\beta/\log\tau} + \Big(\sum_{P \in \Pi_{n,\beta} \backslash P i_{n,\beta}} \sharp P \Big)^{\log\beta/\log\tau}$$

$$\leq (\epsilon c \beta^{-n})^{1-\log\beta/\log\tau} 2^{n\log\beta/\log\tau} + (c\beta^{-n})^{1-\log\beta/\log\tau} ((1-C)2)^{n\log\beta/\log\tau}$$
$$= \beta^{n(\log\beta/\log\tau - 1)} 2^{n\log\beta/\log\tau} ((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau}).$$

Now choose ϵ and C such that $((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau}) < 1$.
For all $n \geq N(\epsilon, C)$ we have:

$$\frac{\log(\sum_{P \in \Pi_{n,\beta}} (\sharp P)^{\frac{\log\beta}{\log\tau}})}{n \log\beta^{-1}} < \frac{\log(2\beta/\tau)}{\log(1/\tau)} + \frac{\log((\epsilon c)^{1-\log\beta/\log\tau} + c^{1-\log\beta/\log\tau}(1-C)^{\log\beta/\log\tau})}{n \log\beta^{-1}}.$$

The last term in this sum is negative and hence our proof is complete.

□

Now the proof of our theorem is obvious:

Proof of 10.2.1.

From 4.1. we know that the box-counting dimension of Λ_ϑ is given by $\log(2\beta/\tau)/\log(1/\tau)$ in the situation we study here. Thus 10.2.2. and 10.2.3. immediately imply $\dim_H \Lambda_\vartheta < \dim_B \Lambda_\vartheta$. The inequality $\dim_H \hat{\Lambda}_\vartheta < \dim_B \hat{\Lambda}_\vartheta$ follows from this with the help of proposition A5.

□

We end this work with three problems concerning number theoretical peculiarities that we were not able to solve.

Open problems

(1) What is the Hausdorff dimension of Λ_ϑ if β is the reciprocal of a PV number, $\tau \in (0, 0.5)$ and $\vartheta = (\beta, \beta, \tau, \tau)$?

(2) Does the variational principle for Hausdorff dimension hold for the systems $(\Lambda_\vartheta, T_\vartheta)$ in this situation?

(3) Are there number theoretical peculiarities for the systems $(\Lambda_\vartheta, T_\vartheta)$, $(\hat{\Lambda}_\vartheta, \hat{f}_\vartheta)$ and $([-1, 1], f_{\beta_1,\beta_2})$ in the asymmetric situation, $\vartheta = (\beta_1, \beta_2, \tau_1, \tau_2) \in P^4_{all}$ with $\beta_1 \neq \beta_2$?

Appendix A: General facts in dimension theory

We will here first define the most important quantities in dimension theory and then collect some basic facts. We refer to the book of Falconer [FA1] and the book of Pesin [PE2] for a more detailed discussion of dimension theory.
Let $Z \subseteq I\!R^q$. We define the **s-dimensional Hausdorff measure** $H^s(Z)$ of Z by

$$H^s(Z) = \lim_{\lambda \longrightarrow 0} \inf\{\sum_{i\in I} (\mathrm{diam} U_i)^s | Z \subseteq \bigcup_{i\in I} U_i \text{ and } \mathrm{diam}(U_i) \leq \lambda\}.$$

The **Hausdorff dimension** $\dim_H Z$ of Z is given by

$$\dim_H Z = \sup\{s|H^s(Z) = \infty\} = \inf\{s|H^s(Z) = 0\}.$$

Let $N_\epsilon(Z)$ be the minimal number of balls of radius ϵ that are needed to cover Z. We define
the **upper box-counting dimension** $\overline{\dim}_B$ resp. **lower box-counting dimension** $\overline{\dim}_B$ of Z by

$$\overline{\dim}_B Z = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon} \qquad \underline{\dim}_B Z = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{\log N_\epsilon(Z)}{-\log \epsilon}.$$

We remark that these quantities are not changed if we replace $N_\epsilon(Z)$ by the minimal number of squares parallel to the axis with side length ϵ that are needed to cover Z. Furthermore we note that limit in the definition exists, if it exists for some exponential decreasing sequence.

Now let μ be a Borel probability measure on $I\!R^q$. We define the dimensional theoretical quantities for μ by

$$\dim_H \mu = \inf\{\dim_H Z|\mu(Z) = 1\}$$

and

$$\underline{\overline{\dim}}_B \mu = \lim_{\rho \longrightarrow 0} \inf\{\underline{\overline{\dim}}_B Z|\mu(Z) \geq 1 - \rho\}.$$

We introduce one more notion of dimension for a measure μ. Let $h_\mu(\epsilon) = \inf\{H_\mu(\Pi)|\Pi \text{ a partition with } \mathrm{diam}\Pi \leq \epsilon\}$ where $H_\mu(\Pi)$ is the usual entropy of Π. We define the upper **Rényi dimension** $\overline{\dim}_R$ resp. **lower Rényi dimension** $\overline{\dim}_R$ of Z by

$$\overline{\dim}_R Z = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{h_\mu(\epsilon)}{-\log \epsilon} \qquad \underline{\dim}_R Z = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{h_\mu(\epsilon)}{-\log \epsilon}.$$

The **upper local dimension** $\overline{d}(x, \mu)$ resp. **lower local dimension** $\underline{d}(x, \mu)$ of the measure μ in a point x is defined by

$$\overline{d}(x, \mu) = \overline{\lim}_{\epsilon \longrightarrow 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon} \qquad \underline{d}(x, \mu) = \underline{\lim}_{\epsilon \longrightarrow 0} \frac{\mu(B_\epsilon(x))}{\log \epsilon}.$$

Basic relations of the dimensions defined here are stated in the following proposition

Proposition A1

(1) $\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z$ holds for all $Z \subseteq I\!R^q$.
(2) $\dim_H \mu \leq \underline{\dim}_B \mu \leq \overline{\dim}_B \mu$ holds for all Borel probability measures μ on $I\!R^q$.
(3) $\overline{\dim}_R \mu \leq \overline{\dim}_B \mu$ holds for all Borel probability measures μ on $I\!R^q$.

The first two inequalities are obvious and third one is proved in [YO]. The relations between the local dimension and the other notion of dimension of measures are described in the following theorem:

Theorem A2

(1) $\underline{d}(x,\mu) \leq c$ -a.e. $\Rightarrow$ $\dim_H \mu \leq c$.
(2) $\underline{d}(x,\mu) \geq c$ -a.e. $\Rightarrow$ $\dim_H \mu \geq c$ and $\underline{\dim}_R \mu \geq c$.
(3) $\overline{d}(x,\mu) \leq c$ -a.e. $\Rightarrow$ $\overline{\dim}_B \mu \leq c$.
(4) $\overline{d}(x,\mu) = \underline{d}(x,\mu) = c$ a.e. $\Rightarrow$ $dim_H \mu = \dim_B \mu = \dim_R \mu = c$.

A proof of this theorem is contained in the work of Young [YO]. If the condition in part (4) of the last theorem holds, the measure μ is called **exact dimensional** and the common value of the dimensions is denoted by $\dim \mu$. In particular absolute continuous measures are exact dimensional:

Proposition A3

If μ is an absolutely continuous Borel probability measure on $I\!R^q$ then $\overline{d}(x,\mu) = \underline{d}(x,\mu) = q$ μ-a.e. .

One basic fact we have to mention is that dimensional theoretical quantities are not increased by Lipschitz maps and are hence bi-Lipschitz invariants.

Proposition A4

Let f be a Lipschitz map from $I\!R^q$ into itself then we have:
(1) $\dim_{B/H} f(Z) \leq \dim_{B/H} Z$ for all $Z \subseteq I\!R^q$.
(2) $\dim_{B/H} \mu \circ f^{-1} \leq \dim_{B/H} \mu$ for all Borel probability measures μ on $I\!R^q$.
Here $\dim_B$ can be both upper and lower box-counting dimension.

The proof of this proposition is obvious from the definitions. Especially we see that a projection on a linear subspace of $I\!R^q$ does not increase Hausdorff and box-counting dimension of a set or a measure.

There is one other elemental fact we use in our work:

Proposition A5

If $Z \subseteq \mathbb{R}^q$ and I is an interval then $\dim_{H/B}(Z \times I) = \dim_{H/B} + 1$, where $\dim_B$ can be both upper and lower box-counting dimension.

The statement for Hausdorff dimension follows from proposition 7.4. of [FA1] and the statement for box-counting dimension is easy to see using 3.1. of [FA1].

At the end of this appendix we like to remark that the terminology in dimension theory is not unique. What we called box-counting dimension is also known as Minkowsky dimension or as capacity. The Rényi dimension is often called information dimension.

Appendix B: Pisot-Vijayarghavan numbers

A **Pisot-Vijayarghavan number** (short: PV number) is by definition the root of an algebraic equitation whose conjugates lie all inside the unit circle in the complex plane. Salem [SA] showed that the set of PV numbers is a closed subset of the reals and that 1 is an isolated element.
In our context we are interested in numbers $\beta \in (0.5, 1)$ such that β^{-1} is a PV number. We list some examples including all reciprocals of PV numbers with minimal polynomial of degree two and three and a sequence of such numbers decreasing to 0.5.

$x^2 + x - 1$	$(\sqrt{5} - 1)/2$
$x^3 + x^2 + x - 1$	0.5436898...
$x^3 + x^2 - 1$	0.754877 ...
$x^3 + x - 1$	0.6823278...
$x^3 - x^2 + 2x - 1$	0.5698403...
$x^4 - x^3 - 1$	0.7244918...
$x^n + x^{n-1} \ldots + x - 1$	$r_n \longrightarrow 0.5$

Table 1: Reciprocals of PV numbers

An important property of PV numbers is that their powers are near integers. More precise:

Proposition B1

If α is a PV number then there is a constant $0 < \theta < 1$ such that $||\alpha^n||_{\mathbb{Z}} \leq \theta^n$ $\forall n \geq 0$ where $||.||$ denotes the distance to the nearest integer.

This statement can be found in [ER1]. There is an another property of PV numbers that is of great importance for us. For $\beta \in (0, 1)$ we denote by $\sharp_\beta(n)$ the number of distinct points of the for $\sum_{k=0}^{n-1} \pm\beta^k$ and by $\omega_\beta(n)$ the minimal distance between two of those points.

Proposition B2

If $\beta \in (0.5, 1)$ is the reciprocal of a PV number then there are positive constants $\bar{c} > 0$ and $\bar{C} > 0$ such that $\omega_\beta(n) \geq \bar{c}\beta^n$ and $\sharp_\beta(n) \geq \bar{C}\beta^{-1}$ holds for all $n \geq 0$.

For the first inequality we refer to [GA2] lemma 1.6. and for the second inequality see (15) of [PU]. Finally we like to mention that there is a whole book about Pisot and Salem numbers [BDGPS]. Certainly the reader will find much more information about the role of these numbers in algebraic number theory in this book than we provided here for our purposes.

General notations

$I\!N$	denotes the set of natural numbers $\{1, 2, 3, 4, 5, \ldots\}$
$I\!N_0$	$:= I\!N \cup \{0\}$
$\mathbb{Z}$	denotes the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^-$	denotes the set of negative integers $\{-1, -2, -3, -4, \ldots\}$
$\mathbb{Z}_0^-$	$:= \mathbb{Z}^- \cup \{0\}$
$I\!R$	denotes the set of real numbers
$\sup(A)$	denotes the supremum of a set $A \subseteq I\!R$
$\inf(A)$	denotes the infimum of a set $A \subseteq I\!R$
$\overline{\lim}$	denotes the limes superior
$\underline{\lim}$	denotes the limes inferior
$\lceil x \rceil$	denotes the smallest integer bigger then $x \in I\!R$
$\lfloor x \rfloor$	denotes the biggest integer smaller then $x \in I\!R$
$\lvert x \rvert$	denotes the absolute value of $x \in I\!R$
$d(x, y)$	denotes the distance between two points x and y in a metric space
$B_\epsilon(x)$	denotes the open ball of radius ϵ around x in a metric space
$\text{diam}(A)$	denotes the diameter of a subset A of a metric space $:= \sup\{d(x, y) \mid x \in A \quad y \in A\}$
$\text{card}(A)$	denotes the cardinality of a set A
closure(A)	denotes the closure of the set A with respect to a given topology
$pr_X(A)$	denotes the projection of $A \subseteq I\!R^q$ onto the first component
$pr_Y(A)$	denotes the projection of $A \subseteq I\!R^q$ onto the second component

$pr_Z(A)$	denotes the projection of $A \subseteq \mathbb{R}^q$ onto the third component
$pr_{XY}(A)$	denotes the projection of $A \subseteq \mathbb{R}^q$ onto the first two components
$pr_{XZ}(A)$	denotes the projection of $A \subseteq \mathbb{R}^q$ onto the first and the third component
$pr_{YZ}(A)$	denotes the projection of $A \subseteq \mathbb{R}^q$ onto the second and the third component
Σ	$:= \{-1,1\}^{\mathbb{Z}}$
$\bar{\Sigma}$	$:= \Sigma \backslash \{(s_k) \mid \exists k_0 \forall k \leq k_0 : s_k = 1\}) \cup \{(1)\}$
Σ^+	$:= \{-1,1\}^{\mathbb{N}_0}$
Σ_A	denotes a Markov chain in Σ ; see chapter eight
Σ_A^+	denotes a Markov chain in Σ^+ ; see chapter eight
pr^+	denotes the projection from Σ onto Σ^+
σ	denotes the shift map; $\sigma((s_k)) = (s_{k+1})$
b^p	denotes the Bernoulli measure on Σ resp. Σ^+ which is the product of the discrete measure giving 1 the probability p and -1 the probability $(1-p)$
b	$:= b^{0.5}$
ℓ	denotes the normalized Lebesgue measure on the interval $[-1,1]$
m^p	denotes a Markov measure; see chapter eight
$h_{top}(T)$	denotes the topological entropy of a continuous transformation on a topological space; see [KH] for definition
$h_\mu(T)$	denotes the metric entropy of a transformation T with respect to an invariant measure μ; see [KH] for definition
$\dim_B$	denotes the box-counting dimension; see appendix A
$\dim_H$	denotes the Hausdorff dimension; see appendix A

$\dim_R$ denotes the Renyi dimension; see appendix A

$d(x, \mu)$ denotes the local dimension of a measure μ in a point x; see appendix A

Some notations and basic relations in our work

system	$(\hat{\Lambda}_\vartheta, f_\vartheta)$	$(\Lambda_\vartheta, T_\vartheta)$	$([-1,1]^2, f_{\beta_1,\beta_2})$
type	attractor	repeller	endomorphism
parameters	$\vartheta \in P^4_{all}$	$\vartheta \in P^4_{all}$	$(\beta_1, \beta_2) \in P^2_{olapp}$
projections	$pr_{XY} \circ f_\vartheta = f_{\beta_1,\beta_2}$ $pr_{XZ}\hat{\Lambda}_\vartheta = \Lambda_\vartheta$		
see	page 14	page 9	page 12
coding system	(Σ, σ^{-1})	(Σ^+, σ)	(Σ, σ^{-1})
coding map	$\hat{\pi}_\vartheta$	π_ϑ	$\bar{\pi}_{\beta_1,\beta_2}$
projections	$pr_{XZ} \circ \hat{\pi}_\vartheta = \pi_\vartheta \circ pr^+$ $pr_{XY} \circ \hat{\pi}_\vartheta = \bar{\pi}_{\beta_1,\beta_2}$	$pr_X \circ \pi_\vartheta = \pi_{\beta_1,\beta_2}$	$pr_X \circ \bar{\pi}_{\beta_1,\beta_2} =$ $\pi_{\beta_1,\beta_2} \circ pr^+$
see	page 17	page 16	page 18
ergodic measures	$\hat{\mu}_\vartheta$	μ_ϑ	$\bar{\mu}_{\beta_1,\beta_2}$
projections	$pr_{XY}\hat{\mu}_\vartheta = \bar{\mu}_\vartheta$	$pr_X\mu_\vartheta = \mu_{\beta_1,\beta_2}$	$pr_X\bar{\mu}_{\beta_1,\beta_2} = (pr^+\mu)_{\beta_1,\beta_2}$
see	page 23	page 21	page 21

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Zusammenfassung der Ergebnisse

In dieser Dissertation werden dimensionstheoretische Eigenschaften einiger Klassen affiner dynamischer System behandelt. Wir untersuchen eine Klasse von selbstaffinen Repellern und eine Klasse von Attraktoren stückweise affiner Abbildungen, die jeweils von vier Parametern abhängen. Darüber hinaus betrachten wir verallgemeinerte Baker's Transformationen, ein Klasse von Endomorphismen abhängig von zwei Parametern. Bei unserer dimensionstheoretischen Analyse leiten uns im wesentlichen zwei Fragestellungen. Erstens fragen wir, ob die Hausdorff Dimension der betrachteten invarianten Mengen mit deren Box-Counting Dimension übereinstimmt. Zweitens fragen wir, ob auf den betrachteten invarianten Mengen ein ergodisches Maß voller Hausdorff Dimension existiert bzw. ob das Variationsprinzip der Hausdorff Dimension gilt, was bedeutete, daß sich die Dimension der betrachteten Menge durch die Dimension ergodischer Maße auf der Menge approximieren lässt. Im Rahmen dieser Arbeit konnten wir ein ganze Reihe neuer Ergebnisse erzielen, die interessante Phänomene im Bereich der Dimensionstheorie dynamischer Systeme, anhand der von uns gewählten Beispiele, aufzeigen. Wir denken, daß unsere Ergebnisse und Methoden auch bei der Entwicklung einer allgemeinen Theorie relevant sein könnten. Wir werden nun unsere Hauptergebnisse zusammenfassend darstellen. Die Berechnung der Box-Counting Dimension der Attraktoren und Repellern, die wir betrachten, ist mit elementaren Überdeckungs Argumenten möglich und wir erhalten eine allgemein gültige Formel. Weiterhin zeigen wir, daß die Box-Counting Dimension der Repeller und Attraktoren generisch (im Sinne des Lebesgue Maßes auf Teilen des Parameterraums) mit deren Hausdorff Dimension übereinstimmt. Für die Repeller finden wir generisch ergodische Maße voller Hausdorff Dimension. Auf der anderen Seite zeigen wir, daß das Variationsprinzip für die Attraktoren nicht generisch gilt. Für die verallgemeinerte Baker's Transformation gibt es Parameterbereich in denen generisch ein ergodisches Maß voller Hausdorff Dimension existiert und Bereiche in denen das Variationsprinzip nicht gilt. Die Beweise dieser generischen Resultate basieren zum einen auf einer geeigneten Anwendung der allgemeinen Dimensionstheorie ergodischer Masse und zum anderen auf einem Studium bestimmter selbst- ähnlicher Maße. Weitere Hauptergebnisse unserer Arbeit beziehen sich auf zahlentheoretische Ausnahmen zu unseren generischen Resultaten in einer symmetrischen Situation. Wir zeigen, daß die Identität zwischen Hausdorff und Box-Counting Dimension der Attraktoren und der Repeller nicht gilt, wenn die Parameter bestimmte zahlentheoretische Eigenschaften besitzen. Weiterhin zeigen wir, daß für die symmetrische Attraktoren sowie für die Fat Baker's Transformationen das Variationsprinzip der Hausdorff Dimension unter bestimmten zahlentheoretischen Bedingungen nicht gilt, obwohl es in diesem symmetrischen Fall generisch gilt. Für die Reppeller konnten wir unter diesen Bedingungen nur zeigen, daß kein Bernoulli Maß voller Dimension existieren kann.

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