0-D-VALUED FIELDS

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Abstract. In [Sca99], T. Scanlon proved a quantifier elimination result for valued D-fields in a three-sorted language by using angular component functions. Here we prove an analogous theorem in a different language $\mathcal{L}_2$ which was introduced by F. Delon in her thesis. This language allows us to lift the quantifier elimination result to a one-sorted language by a process described in the Appendix. As a byproduct, we state and prove a "positivstellensatz" theorem for the differential analogue of the theory of real-series closed fields in the valued D-field setting.
Keywords: valued D-fields, quantifier elimination, real-series closed fields, positivstellensatz.
Mathematics Subject Classification: 03C10, 12H05, 12J10.

1. Introduction

Let us recall that a valued field is a field together with a valuation $v : K \to \Gamma \cup \{\infty\}$, where $\Gamma := v(K^{\times})$ is a totally ordered abelian group which is called the value group. The subring $\mathcal{O}_K := \{x \in K : v(x) \geqslant 0\}$ of K is called the valuation ring of $\langle K, v\rangle$ and the residue field of K is $k_K := \mathcal{O}_K/\mathcal{M}_K$, where $\mathcal{M}_K := \{x \in K : v(x) > 0\}$. The residue map is denoted by $\pi : \mathcal{O}_K \longmapsto k_K$. In this paper, we will think of a valued field as the three-sorted structure $\langle K, k_K, \Gamma\rangle$.

We are interested in valued fields equipped with a derivation which induces a derivation on the residue field in a natural way, so there is a strong interaction between the derivation and the valuation topology which implies the continuity of the derivation D with respect to this topology. We will place ourselves in a particular case of the framework introduced by T. Scanlon (see [Sca00]). In [Sca00], T. Scanlon considered a valued field K equipped with an additive operator D which satisfies the identity $D(x.y) = D(x).y + x.D(y) + e.D(x).D(y)$ where e is an element of K such that $v(e) > 0$; this "twisted" derivation D is assumed to induce a derivation on the residue field (it is equivalent to the condition: $D(\mathcal{M}_K) \subseteq \mathcal{M}_K$). In fact, he assumes a stronger "continuity condition": for all x in K, $v(D(x)) \geqslant v(x)$, which implies that the twisted derivation D on K induces a derivation on the residue field k_K. These valued fields will be called valued D-fields.

In this paper, we are dealing with the pure differential residue field case, i.e. $e = 0$ in the previous terminology. T. Scanlon has dealt with the following problem. Fixing a differential residue field theory $\mathrm{Th}(\mathbf{k})$ and a totally ordered value group theory $\mathrm{Th}(\mathbf{G})$ which both admit quantifier elimination respectively in some suitable

Date: November 16, 2005.
[1] Research Fellow at the "Fonds National de la Recherche Scientifique"; partially supported by grants of the "National Bank of Belgium", FRFC contract 2.4.559.06 F.

expansions of languages of differential fields and of totally ordered abelian groups: under which additional conditions on the differential field $\mathbf{k}$ and the totally ordered abelian group $\mathbf{G}$ does the corresponding three-sorted theory of valued D-fields admit quantifier elimination (in a reasonable language)?

In [Sca00], T. Scanlon considered the following *Assumptions* on $\langle K, k, \Gamma\rangle$:

(1) the differential residue field k is linearly differentially closed,
(2) K has sufficiently many nth roots, i.e. for each natural number n and each element x of K if n divides $v(x)$ then there exists an element y of K such that $y^n = x$,
(3) a differential lifting principle called D-Hensel's Lemma,
(4) $v(C_K^{\times}) = \Gamma$ where C_K is the subfield of constants of K and,
(5) $k \models Th(\mathbf{k})$ and $\Gamma \models Th(\mathbf{G})$.

He called his models $(\mathbf{k}, \mathbf{G})$-D-henselian fields and showed a transfer principle analogous to the Ax-Kochen-Ersov theorem. In [Sca99], he generalized this framework in showing a similar transfer principle, removing *Assumption* 2 but by using angular component functions.

Under the same hypotheses on K as in [Sca99], we are able to show a similar transfer principle for an expansion by definition, denoted by $\mathcal{L}_2$, of a three-sorted language of valued D-fields. This language was introduced by F. Delon in [Del82]. It consists of adding new predicates $F_{\phi,\vec{n}}$ which combine the nth powers in the valued D-field and the residue field formulas. We will show an analogous quantifier elimination theorem in $\mathcal{L}_2$ which allows us to lift quantifier elimination results for the value group and the residue field to the whole three-sorted valued D-field structure. Moreover, it enables us to get a quantifier elimination result in a one-sorted language, often more suited for algebraic applications. The passage from a three-sorted language to a one-sorted language was done in Delon's thesis (see [Del82, p. 40]) and for the convenience of the reader, it can be found in the Appendix.

In Section 2, we recall the general setting of the problem in keeping the notations of [Sca00]. The important modification concerns the language since we do not want to use the angular component functions.

In Section 3, we prove our main result of quantifier elimination. The proof uses a standard back-and-forth construction. But since we do not suppose *Assumption* 2 (contrary to [Sca00]), the back-and-forth construction requires some subtlety in order to handle radical field extensions. In the construction of partial isomorphism extensions, we use the concept of an *efficient substructure* of a $(\mathbf{k}, \mathbf{G})$-D-henselian field (as introduced by F. Delon).

In Section 4, we consider the theory of closed ordered differential fields, denoted by $CODF$ (introduced by M.F. Singer in [Sin78]) and use this theory as differential residue field theory for valued D-fields. In [Sca00], under the previous assumptions on $\mathbf{k}$ and $\mathbf{G}$, T. Scanlon proves that $(\mathbf{k},\mathbf{G})$-D-henselian fields $\langle K, k, G\rangle$ admit quantifier elimination and so, a model completeness result holds. But in our context, $\mathbf{k}$ is a model of $CODF$ and we have that *Assumptions* 2 and 4 imply that k_K is root-closed (i.e. $(k_K^{\times})^n = {k_K}^{\times}\ \forall n \in \mathbb{N} \setminus \{0, 1\}$).

So we first prove a model completeness result for this theory. This model-theoretic tool allows us to transpose, in this valued D-field setting, a "positivstellensatz" theorem for real-closed series fields obtained in [Far91].

2. Preliminaries

Given any unitary ring R, we denote its units by $R^{\times} = \{x \in R : \exists y\,(xy = 1)\}$. If $\langle R, D\rangle$ is a differential ring then we denote the subring of constants $\{x \in R : D(x) = 0\}$ by C_R. The differential polynomial ring over the differential ring R equipped with the natural derivation extending that of R is denoted by $R\{X\}$. For any differential polynomial $P \in R\{X\}$, the partial derivative of the corresponding polynomial $P(X, X^{(1)}, \cdots, X^{(n)})$ at the variable $X^{(i)}$ is denoted by $\frac{\partial}{\partial X^{(i)}}P$ where $X^{(i)}$ is the ith derivative of X. We keep the same notations if K is a differential field; in particular, its subring of constants C_K is a field.

From now on, we always deal with fields of characteristic zero. First we recall the canonical example of valued D-fields (see also Section 6 in [Sca00]). We consider a differential field $\langle \mathbf{k}, \delta\rangle$ and a totally ordered abelian group $\mathbf{G}$. The set $\mathbf{k}((t^{\mathbf{G}}))$ is defined by $\{f : \mathbf{G} \to \mathbf{k} : \mathrm{supp}(f) := \{g \in \mathbf{G} : f(g) \neq 0\}$ is well-ordered in the ordering induced by $\mathbf{G}\}$. Each element of $\mathbf{k}((t^{\mathbf{G}}))$ can be viewed as a formal power series $\sum_{g\in\mathbf{G}} f(g)\cdot t^g$ with the addition and the multiplication defined as follows:

$$\begin{aligned}&(f+h)(g) := f(g) + h(g) \text{ and}\\ &(f.h)(g) := \sum_{g'+g''=g} f(g')h(g'') \text{ for any } g \in \mathbf{G}.\end{aligned}$$

It is easy to check that $\mathbf{k}((t^{\mathbf{G}}))$ is a field. There is a natural valuation v on $\mathbf{k}((t^{\mathbf{G}}))$ defined as $\min \mathrm{supp}(f)$ for any $f \in \mathbf{k}((t^{\mathbf{G}}))$. Moreover we can equip this field with the derivation D defined term by term:

$$(Df)(g) := \delta(f(g)).$$

From the general valuation theory, we know that $\mathbf{k}((t^{\mathbf{G}}))$ is a maximally complete valued field. It is clear that it satisfies the "continuity condition" for the derivation D. It is natural to consider valued fields in a three-sorted language where the three sorts are $\langle K, k, \Gamma\rangle$ in the context of this paper:

- K is equipped with the signature of a differential field: $\langle +, \cdot, -, ^{-1}, D, 0, 1\rangle$ where D is an unary function to be interpreted as a derivation.
- k is equipped with the signature of an expansion of the language $\langle +, -, ., ^{-1}, D, \{p_n\}_{n\in\mathbb{N}\setminus\{0,1\}}, 0, 1\rangle$ where p_n is an unary predicate to be interpreted as nth powers (as introduced by A. Macintyre in [Mac76]). The associated language will be denoted $\mathcal{L}_R$ in the sequel, independently of the added extra predicates. We will not use distinct symbols for the symbols of derivation in the sorts K and k since there is few risk of confusion.
- Γ is equipped with a signature $\langle +, -, 0, \leqslant\rangle$ of a totally ordered abelian group possibly with some additional predicates (for example, divisibility predicates $\{n|\cdot\}_{n\in\mathbb{N}\setminus\{0,1\}}$. The associated language will be denoted by $\mathcal{L}_V$, independently of the added extra predicates.

Moreover we add symbols to describe some relations between the three sorts. First we add an extra symbol ∞ both in the sort k and in the sort Γ; we will denote both by the same notation since it will be clear from the context in which sort we will be. The sorts are connected by the functions $\pi : K \longmapsto k \cup \{\infty\}$ and $v : K \longmapsto \Gamma \cup \{\infty\}$. Such a three-sorted first-order language will be always denoted by $\mathcal{L}_1$.

In this paper, we will consider classes $\mathcal{C}$ of $\mathcal{L}_1$-structures where the sort K will be a differential field, the sort k will be a differential field $\mathbf{k}$ which admits quantifier elimination in the language $\mathcal{L}_R$ and the sort Γ will be a totally ordered abelian group G which admits quantifier elimination in the language $\mathcal{L}_V$. Moreover, for convenience, we will require that all our structures satisfy: $0^{-1} = \infty$ (in the sort k) and $\forall \gamma \in \Gamma$ $\gamma < \infty$. The symbol v will be interpreted as a valuation map, i.e. v satisfies the following axioms:

- $\forall x, y \left[v(xy) = v(x) + v(y)\right]$,
- $\forall x \quad \left[v(x) = \infty \iff x = 0\right]$,
- $\forall x, y \left[v(x + y) \geqslant \min\{v(x), v(y)\}\right]$.

The sort Γ is the whole value group, i.e. $v(K) = \Gamma \cup \{\infty\}$. The symbol π is interpreted as the residue map, formally it means that $\pi : \mathcal{O}_K \longmapsto k_K$ is the canonical residue map and $\pi(K \setminus \mathcal{O}_K) = \infty$. We also require that our structures satisfy the continuity condition:

$$\forall x \in K \quad [v(D(x)) \geqslant v(x)]$$

and that the symbol of derivation D over the sort k is interpreted by the derivation induced by the derivation of the sort K:

$$D(\pi(x)) = \pi(D(x)).$$

Structures in such a class $\mathcal{C}$ are called *valued D-fields*. Let us note that a substructure of a valued D-field is itself a valued D-field if and only if π and v are surjective.

Now we describe F. Delon's language (which will be denoted $\mathcal{L}_2$ in the sequel). Let $P_n(x)$ be the formula with variables in K: $\exists y\, (y^n = x)$, and let $p_n(\eta)$ be the formula with variables in k_K: $\exists \epsilon\, (\epsilon^n = \eta)$.

With a language $\mathcal{L}_1$, we associate a language $\mathcal{L}_2$ obtained by adding to $\mathcal{L}_1$ the following set of new predicates (see [Del82, p. 38])

$$\left\{F_{\phi,n_1,\cdots,n_r}; \phi \text{ formula of } \mathcal{L}_R \text{ with } r+s \text{ variables, } n_1, \cdots, n_r \in \mathbb{N} \setminus \{0,1\}\right\}.$$

These predicates will be interpreted in the following way:

$$\begin{aligned} \forall x_1, \ldots, x_r \in K\, \forall \eta_1, \ldots, \eta_s \in k \big\{ F_{\phi,n_1,\cdots,n_r}(x_1, \cdots, x_r, \eta_1, \cdots, \eta_s) \iff \\ \exists z_1, \cdots, z_r \in K\ [\bigwedge_{i=1}^{r} v(z_i) = 0 \wedge \phi(\pi(z_1), \cdots, \pi(z_r), \eta_1, \cdots, \eta_s) \\ \wedge \bigwedge_{i=1}^{r} P_{n_i}(x_i z_i)]\big\}. \end{aligned}$$

Now we can naturally consider the classes $\mathcal{C}$ described above as classes of $\mathcal{L}_2$-structures, since the interpretation of "F. Delon predicates" are $\mathcal{L}_1$-definable.

From now on, we will always consider a class $\mathcal{C}$ as a class of $\mathcal{L}_2$-structures with the natural interpretation given for Delon's predicates.

Let us fix a differential field $\mathbf{k}$ of characteristic zero and a totally ordered abelian group $\mathbf{G}$ which satisfy all the previous requirements. The class $\mathcal{C}$ of valued D-fields such that the sort k is a model of $Th(\mathbf{k})$ and the sort Γ is a model of $Th(\mathbf{G})$ will be called the class of valued $(\mathbf{k}, \mathbf{G})$-D-fields.

Let us recall the interpretation of the divisibility predicates for the sort Γ:

$$n|x \iff \exists y \quad (\underbrace{y + \cdots + y}_{n \text{ times}} = x) \text{ for each } n \in \mathbb{N} \setminus \{0, 1\}.$$

Indeed these predicates will often be needed in order to be sure that $Th(\mathbf{G})$ admits quantifier elimination. Moreover, since the theory of value group eliminates quantifiers in $\mathcal{L}_V$, the divisibility predicate is equivalent to a quantifier-free formula in $\mathcal{L}_V$.

Definition 2.1. The valued D-field $\langle K, k_K, v(K^\times)\rangle$ is said to have enough constants if it satisfies $v(K^\times) = v(C_K^\times)$ (see Definition 7.3 in [Sca00]).

Now we define a subclass of valued $(\mathbf{k}, \mathbf{G})$-D-fields which will be called $(\mathbf{k}$,$\mathbf{G})$-D-henselian fields:

Axiom 1. Any non-zero linear differential equation operator $L \in \mathbf{k}[D]$ is surjective as a map $L : \mathbf{k} \longmapsto \mathbf{k}$. We call a differential field satisfying this condition *linearly differentially closed*.

Axiom 2. K has enough constants.

Axiom 3 (D-Hensel's Lemma). If $P \in \mathcal{O}_K\{X\}$ is a differential polynomial over $\mathcal{O}_K$, $a \in \mathcal{O}_K$ and $v(P(a)) > 0 = v(\frac{\partial}{\partial X_i} P(a))$ for some i, then there is some $b \in K$ with $P(b) = 0$ and $v(a - b) \geqslant v(P(a))$.

Axiom 1 is quite natural since it was shown in [Mic86] that every differential field which admits quantifier elimination in the language of pure differential rings is linearly differentially closed.

In fact, in Section 6 of [Sca00], it is shown that the generalized power series field $\mathbf{k}((t^{\mathbf{G}}))$ provides a canonical model for the theory of $(\mathbf{k}, \mathbf{G})$-D-henselian fields; the only non-trivial point is to prove that the D-Hensel's Lemma holds. It is done in Proposition 6.1 of [Sca00] by using Taylor expansion and the fact that $\mathbf{k}$ is linearly differentially closed.

3. Quantifier elimination result

In this section, we prove a quantifier elimination result, analogous the one of T. Scanlon in [Sca00], for $(\mathbf{k}, \mathbf{G})$-D-henselian fields in the previous language $\mathcal{L}_2$ by using a back-and-forth test. In Chapter 2 of [Del82], F. Delon proved a quantifier elimination result for maximally algebraic valued field (i.e. valued fields which do not have proper algebraic immediate extensions) and we are going to follow the general scheme of her proof. In particular, we use the concept of an efficient substructure of a $(\mathbf{k}, \mathbf{G})$-D-henselian field. This notion will allow us to construct radical field extensions in a back-and-forth process (see Proposition 3.11).

Now we recall the "*efficient*" concept introduced in [Del82, p. 48].

Definition 3.1. Let A be an $\mathcal{L}_1$-substructure of a $(\mathbf{k},\mathbf{G})$-D-henselian field M. We define the relation R^M over A: $R^M(a)$ iff $\forall n \in \mathbb{N}\setminus\{0,1\}$, if n divides $v(a)$, then there exists a_0 in A such that M satisfies $v(a_0)=0 \wedge P_n(aa_0)$. The ring A is said to be *efficient* in M if for every element a of A, A satisfies $R^M(a)$.

Remark 3.2.
- Let $a \in A$ satisfying $R^M(a)$ and $a' \in A$ with the same value, then we get $P_n(aa_0)$ iff $P_n(a'(a'^{-1}aa_0))$ and so, A is efficient iff $\forall w \in v(A^\times)\,\exists a \in A$, M satisfies $v(a)=w \wedge R^M(a)$.
- Let N be an elementary $\mathcal{L}_1$-substructure of a $(\mathbf{k},\mathbf{G})$-D-henselian field M and let A be an $\mathcal{L}_1$-substructure of N. Then A is efficient in N iff A is efficient in M.

Now we recall a lemma needed in Proposition 3.4.

Lemma 3.3. *(See Proposition 1 in* [Mic86]*) Let $\langle L,D\rangle$ be an $\aleph_1$-saturated differential field and let $\langle K,D\rangle$ be a countable differential subfield with a non-trivial derivation. Then we can find an element d of L which is differentially transcendental over K.*

In the sequel, if f is an isomorphism between two valued D-fields in the sort K then we denote the isomorphism corresponding to the residue field sort k (respectively the value group sort Γ) by f_R (respectively f_V). The domain of f will be denoted by $\mathrm{dom}(f)$.

Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k},\mathbf{G})$-D-henselian fields. Let $A_1 \subseteq_{\mathcal{L}_2} M_1$ and $A_2 \subseteq_{\mathcal{L}_2} M_2$ be two countable $\mathcal{L}_2$-substructures with a countable elementary submodel N of M_1 containing A_1 and a partial isomorphism $f : A_1 \to A_2$. By using quantifier elimination of $\mathrm{Th}(\mathbf{G})$ in $\mathcal{L}_V$, we can extend f_V to $v(N^\times)=\Gamma_N$. In the case of f_R, the situation is more complicated since the residue variables occur in the predicates $F_{\phi,\vec{n}}$. Since $k_N=\pi(\mathcal{O}_N)$ is countable and M_2 is $\aleph_1$-saturated, it suffices to satisfy any finite conjunction of formulas in n-types of elements of k_N over k_{A_1}.

That is, if $\vec{\lambda} \subseteq k_N$, $\vec{\alpha} \subseteq k_{A_1}$ and $\vec{x} \subseteq A_1$ satisfy

$$\psi(\vec{\lambda},\vec{\alpha}) \wedge F_{\phi,\vec{n}}(\vec{x},\vec{\lambda}^\frown\vec{\alpha}) \tag{*}$$

for residue formulas ψ and ϕ, then we get that

$$M_2 \models \exists\vec{\lambda}[\psi(\vec{\lambda},f_R(\vec{\alpha})) \wedge F_{\phi,\vec{n}}(f(\vec{x}),\vec{\lambda}^\frown f_R(\vec{\alpha}))].$$

This formula is exactly

$$F_{\exists\vec{\lambda}\psi(\vec{\lambda},f_R(\vec{\alpha}))\wedge\phi(-,\vec{\lambda}^\frown f_R(\vec{\alpha})),\vec{n}}(f(\vec{x}),f_R(\vec{\alpha}));$$

it can be easily deduced via f from (*). So the isomorphism f has been extended to the three-sorted structure $\langle A,k_N,v(N^\times)\rangle$. The previous isomorphisms f_R and f_V are implicitly extended whenever we handle extensions of the isomorphism f.

Now we proceed as in Proposition 2.17 of [Del82] in order to build differentially transcendental residue field extensions of a valued D-field. To this effect, we use Lemma 3.3 and the proof of Lemma 7.12 in [Sca00]. For the convenience of the reader, we give the details in the following proof.

Proposition 3.4. *Under the previous assumptions, we can extend f such that $dom(f)$ is a countable unramified valued D-field extension which has the same residue field as a countable elementary submodel $\widehat{N}$ of M_1 containing N and $dom(f)$, and is efficient in $\widehat{N}$.*

Proof. First we can assume that the derivation is non-trivial on A_1; otherwise we can extend the $\mathcal{L}_1$-isomorphism f such that its new domain, also denoted by A_1, has a non-trivial derivation. We proceed as follows. Since k_{M_1} is linearly differentially closed and M_1 is $\aleph_1$-saturated, we find an element a (in k_{M_1}) which is transcendental over k_{A_1} and $D(a) = 1$. By using Lemma 7.12 of [Sca00], we extend the $\mathcal{L}_1$-isomorphism f such that $\mathrm{dom}(f)$ has a non-trivial derivation.

Let us choose an enumeration $v(A_1^\times) = \{g_i; i < \omega\}$ and elements $(a_i)_{i<\omega}$ in A_1 with $v(a_i) = g_i$. We are now building a valued D-subfield $B := \bigcup_{i<\omega} B_i$ of M_1 (by induction on i) such that the following properties hold:

$$B_0 = A_1$$

$$B_{i+1} = B_i\langle e_i\rangle \qquad \text{with} \begin{cases} v(e_i) = 0 \\ \pi(e_i) \text{ is differentially transcendental over } \pi(\mathcal{O}_{B_i}) \\ P_n(a_ie_i) \text{ holds if and only if } n|v(a_i). \end{cases}$$

A priori the e_i's depend on the natural number n but we can use the $\aleph_1$-saturation of M_1 in order to avoid it. By using the proof of Lemma 7.12 of [Sca00] in the case of differentially transcendental residue field extension, each extension B_{i+1} of B_i is uniquely determined (up to $\mathcal{L}_1$-isomorphism) by the $\mathcal{L}_R$-type of $\pi(e_i)$ over $\pi(\mathcal{O}_{B_i})$, denoted by $tp(\pi(e_i)/\pi(\mathcal{O}_{B_i}))$. To prove the existence of such a e_i in M_1, it suffices to prove the consistency of:

$$\Sigma(e_i) := \begin{cases} v(e_i) = 0 \\ p(\pi(e_i)) \neq 0 \text{ for a differential polynomial } p \in \pi(\mathcal{O}_{B_i})\{X\} \\ P_n(a_ie_i) \text{ holds if and only if } n \text{ divides } v(a_i). \end{cases}$$

For the satisfiability of Σ, it suffices to check the satisfiability of each finite part of Σ in which case we use the following argument. Since n divides $v(a_i)$, there exist b_i, $c_i \in M_1$ satisfying $v(a_i) = v(b_i^n)$, $v(c_i) = 0$ and $\pi(c_i) = \pi(a_i^{-1}b_i^n)$. If $\pi(c_i)$ is differentially transcendental over $\pi(\mathcal{O}_{B_i})$, we have $\Sigma(c_i)$. Otherwise, by using Lemma 3.3, we can take $d_i \in M_1$ of value zero such that $\pi(d_i)$ is differentially transcendental over $\pi(\mathcal{O}_{B_i})$; so $\Sigma(d_i^nc_i)$ holds. Now we take a countable elementary submodel $\widehat{N}$ of M_1 containing N and B.

First, we extend the isomorphism f_R to $k_{\widehat{N}} = \pi(\mathcal{O}_{\widehat{N}})$ and then, we extend the isomorphism f such that its domain has residue field $k_{\widehat{N}}$.

We first extend the $\mathcal{L}_1$-isomorphism f from $B = \bigcup_{i<\omega} B_i$ onto a valued D-subfield $B' = \bigcup_{i<\omega} B'_i$ such that $B'_{i+1} = B'_i\langle e'_i\rangle$ with $\pi(e'_i) = f_R(\pi(e_i))$ and $P_n(f(a_i)e'_i)$ iff $n|v(f(a_i))$. We proceed by induction on i. Suppose that f is already extended to B_i. We want to find e'_i in M_2 such that $\pi(e'_i) \models f_R(tp(\pi(e_i)/\pi(\mathcal{O}_{B_i})))$ and $P_n(f(a_i)e'_i)$ iff $n|v(f(a_i))$. By the $\aleph_1$-saturation of M_2, it follows from the following facts:

- $M_2 \models F_{\phi(\eta),n}(f(a_i)) \iff \exists z\,[v(z) = 0 \wedge \phi(\pi(z)) \wedge P_n(f(a_i)z)]$,

- since f preserves the predicates $F_{\phi,n}$, $F_{\phi(\eta),n}(f(a_i))$ holds in A_2 for any $\phi \in f_R(tp(\pi(e_i)/\pi(\mathcal{O}_{B_i})))$.

As in Proposition 7.21 of [Sca00], we use Lemma 7.12 of [Sca00] in order to extend the $\mathcal{L}_1$-isomorphism such that its valued D-field domain, denoted by N', is efficient in $\widehat{N}$ with $k_{N'} = k_{\widehat{N}}$. Now we check that it is also an $\mathcal{L}_2$-isomorphism.

Claim (See Lemma 2.18 of [Del82]):

Let N' be the previous $\mathcal{L}_1$-structure which is efficient in $\widehat{N}$. The data of the relations $P_n(a_ie_i)$ for the previous a_i's and e_i's and of the $\mathcal{L}_1$-diagram of N' determine the $\mathcal{L}_2$-diagram.

Let $x_1, \cdots, x_n$ be elements in N'. If n_i divides $v(a_i)$ for some positive integer n_i, there exists $e_i \in N'$ verifying $v(e_i) = 0$, $v(a_i) = v(x_i)$ and $P_{n_i}(a_ie_i)$. We then have

$$P_{n_i}(zx_i) \iff P_{n_i}(zx_ia_i^{-1}e_i^{-1}) \iff p_{n_i}(\pi(z)\pi(e_i^{-1})\pi(x_ia_i^{-1})).$$

Hence $F_{\phi,n}(\bar{x}, \bar{\alpha})$ is equivalent to an $\mathcal{L}_R$-formula (with new parameters in N'), which proves the claim.

Since the valued D-field N' is efficient in $\widehat{N}$ and the data of the relations $P_n(a_ie_i)$ are preserved by the $\mathcal{L}_1$-isomorphism f, we obtain the required $\mathcal{L}_2$-isomorphism f. $\square$

The following lemma is an adaptation of Lemma 2.9 of [Del82] in the setting of our back-and-forth process.

Lemma 3.5. *Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k},\mathbf{G})$-D-henselian fields. Let $A_1 \subseteq_{\mathcal{L}_2} M_1$ and $A_2 \subseteq_{\mathcal{L}_2} M_2$ be countable $\mathcal{L}_2$-substructures with a countable elementary submodel N of M_1 containing A_1 such that $k_{A_1} = k_N$ and the value group $v(A_1^\times)$ of A_1 is pure in $v(N^\times)$. Let f be a partial $\mathcal{L}_1$-isomorphism from A_1 to A_2. Then f is also an $\mathcal{L}_2$-isomorphism.*

Proof. Let $x_1, \ldots, x_r$ be in A_1, let $\eta_1, \ldots, \eta_s$ be in k_{A_1} and let ϕ be an $\mathcal{L}_R$-formula. We have the following equivalences:

$$A_1 \models F_{\phi,n_1,\ldots,n_r}(x_1, \ldots, x_r, \eta_1, \ldots, \eta_s) \text{ iff } M_1 \models F_{\phi,n_1,\ldots,n_r}(x_1, \ldots, x_r, \eta_1, \ldots, \eta_s) \text{ iff}$$

$$M_1 \models \exists z_1, \ldots, z_r \{\bigwedge_{i=1}^r v(z_i) = 0 \wedge \phi(\pi(z_1), \ldots, \pi(z_r), \eta_1, \ldots \eta_s) \wedge \bigwedge_{i=1}^r P_{n_i}(x_iz_i)\}.$$

Since $N \prec_{\mathcal{L}_2} M_1$, this is equivalent to

$$N \models \exists z_1, \ldots, z_r \{\bigwedge_{i=1}^r v(z_i) = 0 \wedge \phi(\pi(z_1), \ldots, \pi(z_r), \eta_1, \ldots \eta_s) \wedge \bigwedge_{i=1}^r P_{n_i}(x_iz_i)\}.$$

Since $v(A_1^\times)$ is pure in $v(N^\times)$ and $k_{A_1} = k_N$, we can replace, in the previous equivalence, $N \models \bigwedge_{i=1}^r P_{n_i}(x_iz_i)$ by $N \models \bigwedge_{i=1}^r p_{n_i}(\pi(x_ia_ib_i^{-n_i}))$ for some $a_i, b_i \in A_1$ such that $v(x_i) = n_iv(b_i)$ and $\pi(a_i) = \pi(z_i)$. We apply the isomorphism f to this residue formula and by the previous process, we get the result. $\square$

The following lemma follows the same idea as in Corollary 2.6 of [Del82].

Lemma 3.6. *Let $A \subseteq_{\mathcal{L}_2} B$ be two $\mathcal{L}_2$-substructures of a $(\mathbf{k}, \mathbf{G})$-D-henselian field M such that B is an immediate valued D-field extension of A. Then the $\mathcal{L}_2$-diagram of B is a consequence of its $\mathcal{L}_1$-diagram and of the $\mathcal{L}_2$-diagram of A.*

Proof. Let $b_1, \dots, b_r$ be in B and let $\eta_1, \dots, \eta_s$ be in $k_B = k_A$. Since B is an immediate extension, there exist $a_1, \dots, a_r \in k_A$ such that $v(a_i) = v(b_i) < v(a_i - b_i)$ for all $i \in \{1, \cdots, r\}$.

So we have that

$$B \models F_{\phi,n_1,\dots,n_r}(b_1, \dots, b_r, \eta_1, \dots, \eta_s) \text{ iff } M \models F_{\phi,n_1,\dots,n_r}(b_1, \dots, b_r, \eta_1, \dots, \eta_s).$$

By Lemma 2.5 in [Del82], this is equivalent to

$$(*) \qquad M \models F_{\phi,n_1,\dots,n_r}(a_1, \dots, a_r, \eta_1, \dots, \eta_s).$$

Hence we get that (*) iff $A \models F_{\phi,n_1,\dots,n_r}(a_1, \dots, a_r, \eta_1, \dots, \eta_s)$ since A is an $\mathcal{L}_2$-substructure of M. So the result follows. ☐

Remark 3.7. Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k},\mathbf{G})$-D-henselian fields. Let $A_1 \subseteq_{\mathcal{L}_2} M_1$ and $A_2 \subseteq_{\mathcal{L}_2} M_2$ be two countable $\mathcal{L}_2$-substructures with a countable elementary submodel N of M_1 containing A_1 and a partial isomorphism $f : A_1 \to A_2$. Thanks to Lemma 7.11 in [Sca00] and the preceding lemma, we can extend the isomorphism f to the Henselization of A_1.

The two following lemmas will help us to extend (in the main quantifier elimination Theorem 3.13) the $\mathcal{L}_2$-isomorphism f such that $\mathrm{dom}(f)$ has the same value group as a countable $(\mathbf{k}, \mathbf{G})$-D-henselian field.

The first lemma is a differential adaptation of the proof of Lemma 2.11 in [Del82] where we deal with a cross-section of the valuation in the constant field and the second lemma is the differential adaptation of Lemma 2.12 in [Del82].

Let us first recall that a cross-section β of a valued field $\langle K, v\rangle$ on its value group is a group homomorphism $v(K^\times) \to K^\times$ such that $v(\beta(\gamma)) = \gamma$ for all $\gamma \in v(K^\times)$ and the notion of a cross-section on a subgroup H of $v(K^\times)$ is defined similarly. In [Koc74], S. Kochen expands the three-sorted language of valued fields by a cross-section in order to approach the problem of quantifier elimination for henselian valued fields of finite ramification index. We will use the following facts about cross-sections (see Section 8 in [Koc74]):

- every $\aleph_1$-saturated valued field admits a cross-section on its value group,
- a cross-section on a pure subgroup of the value group can be extended to a cross-section on the whole value group.

In these two following results, the valued constant field plays an important role.

Lemma 3.8. *Let $K \subseteq M$ be two valued D-fields and let β be a cross-section of v in C_M such that $\beta(v(M^\times)) \subseteq C_M^\times$ and $\beta(v(K^\times)) \subseteq C_K^\times$. Let $G := \beta(v(M^\times))$ and let H be a subgroup of G containing $\beta(v(K^\times))$. Then we have $k_{K\langle H\rangle} = k_K$, $v(K\langle H\rangle^\times) = v(H)$ and the $\mathcal{L}_1$-diagram of $K\langle H\rangle$ is a consequence of the $\mathcal{L}_1$-diagram of K, the $\mathcal{L}_V$-diagram of H and the fact that $H \subseteq C_M^\times$.*

Moreover, if M is a (non differential) valued field with the same hypotheses on the valued D-field K then $K\langle H\rangle$ admits a valued D-field structure such that the derivation D is zero on H.

Proof. For the convenience of the reader, we give the details of the proof below.

Take an element x in $K[H]$. Write x as $\sum_i k_ih_i$ with each $k_i \in K$ and each $h_i \in H$. I claim that we may assume that the values $v(k_ih_i)$ are strictly increasing.

Suppose that $v(k_ih_i) = v(k_jh_j)$, for some $i \neq j$. So we have $v(h_jh_i^{-1}) = v(k_ik_j^{-1})$ and $h_jh_i^{-1}$ is in $\beta(v(K^\times)) \subseteq C_K^\times$. So we can write $k_ih_i + k_jh_j = h_i(k_i + k_jh_jh_i^{-1})$ where $k_i + k_jh_jh_i^{-1} \in K$. This proves the claim and the equality $v(K\langle H\rangle^\times) = v(H)$.

Now we take an element x in $K(H)$ such that $x = (\sum_{i=1}^n k_ih_i)(\sum_{j=1}^{n'} k'_jh'_j)^{-1}$, $v(x) = 0$, $k_i, k'_j \in K$, $h_i, h'_j \in H$, with the $v(k_ih_i)$'s and the $v(k'_jh'_j)$'s strictly increasing. So we get $v(k_1h_1) = v(k'_1h'_1)$ and $h'_1h_1^{-1} \in K$; and conclude $\pi(x) = \pi((k_1{k'_1}^{-1})(h_1{h'_1}^{-1})) \in k_K$. Therefore, if the valuation on K and the subgroup H are given then the valuation on $K(H)$ is completely determined.

Now we show that the field structure is completely determined (and so the valued D-field structure is determined). We consider the ring of generalized power series $K[t^h; h \in H]$ and the substitution f which sends t^h to h. If we show that the kernel is the ideal I generated by the $(t^g - g)_{g\in\beta(v(K^\times))}$, we obtain an isomorphism and it is finished. Clearly $\ker(f)$ contains I. Let x be an element in $\ker(f)$ with the form $\sum_{i=1}^n k_it^{h_i}$ and the $v(k_ih_i)$'s increasing. We have $v(k_1h_1) = v(k_2h_2)$ since $\sum_{i=1}^n k_ih_i = 0$, so $h_2h_1^{-1} \in \beta(v(K^\times))$ and x is equal modulo I to $y = t^{h_1}(k_1 + k_2h_2h_1^{-1}) + \sum_{i=3}^n k_it^{h_i}$, and y is an element of $\ker(f)$ with a monomial less. We iterate this and obtain the congruence $x \equiv 0$.

It remains to show that $K(H)$ admits a structure of valued D-field determined only by $D(h) = 0$ for all $h \in H$, i.e. satisfies the property $v(D(x)) \geqslant v(x)$ ($\Diamond$) for all element x of $K(H)$. Let $x = \sum_i k_ih_i$ be an element of $K[H]$ with the $v(k_ih_i)$'s strictly increasing. Then $D(x) = \sum_i D(k_i)h_i$ since $H \subseteq C_K^\times$. Hence we get that

$$v(D(x)) \geqslant \min_i\{v(D(k_i)h_i)\} \geqslant \min_i\{v(k_ih_i)\} = v(k_1h_1) = v(x)$$

since K is a valued D-field. So the property ($\Diamond$) is satisfied for all element x of $K[H]$ and by Lemma 7.7 of [Sca00], $K(H)$ is a valued D-field. □

Lemma 3.9. *Let $K \subseteq M$ be two valued D-fields and let $G \subseteq C_M^\times$ be a multiplicative subgroup of constants on which the valuation is injective and such that $v(G) \cap v(K^\times) = \{0\}$. Then*

- *$v(K\langle G\rangle^\times) = v(K^\times) \oplus v(G)$,*
- *$k_{K\langle G\rangle} = k_K$,*
- *the $\mathcal{L}_1$-diagram of K is determined by the $\mathcal{L}_V$-diagram of $v(K^\times) \oplus v(G)$ and the fact that $G \subseteq C_M^\times$.*

Proof. The proof is similar to the one of Lemma 3.8. □

Now we recall a special case of Proposition 9.8 in [Sca99].

Proposition 3.10. *Let K be a valued D-field. Let L be a field extension of K such that $z_1^n = c$ where $z_1 \in L$, $c \in K$ and $v(c) \notin m \cdot v(K^\times)$ for each positive integer m dividing n. Then there exists a unique (up to $\mathcal{L}_1(K)$-isomorphism) extension of valued D-fields of the form $K\langle \sqrt[n]{c}\rangle$.*

Proof. Let $z_1 = \sqrt[n]{c}$. Since the extension $K(z_1)$ is totally ramified, the valuation structure is completely determined. We claim that the valued D-field structure is completely determined by $v(Dc) \geqslant v(c)$ and $z_1^n = c$. Indeed, since $z_1^n = c$, we get by differentiating:

$$Dz_1 = \frac{Dc}{c \cdot n} \cdot z_1.$$

Hence we have $v(Dz_1) \geqslant v(z_1)$ since the residue field is of characteristic zero and $v(Dc) \geqslant v(c)$. We check that this prescription correctly defines a valued D-field. Let $x = \sum_{i=0}^{n-1} x_i z_1^i \in K\langle z_1\rangle$ for some $x_i \in K$. Then $v(x) = \min_i\{v(x_i) + \frac{i}{n} \cdot v(c)\}$. Since $Dz_1 = \frac{Dc}{c \cdot n} \cdot z_1$, we get

$$Dx = \sum_{i=0}^{n-1} \underbrace{(Dx_i + \frac{Dc \cdot i}{c \cdot n} \cdot x_i)}_{\in K} z_1^i.$$

So we conclude that $v(Dx) \geqslant v(x)$. □

Proposition 3.11. *Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k},\mathbf{G})$-D-henselian valued fields. Let $A_1 \subseteq_{\mathcal{L}_2} M_1$ and $A_2 \subseteq_{\mathcal{L}_2} M_2$ be countable substructures with a countable elementary submodel N of M_1 containing A_1 and a partial isomorphism $f : A_1 \to A_2$. Suppose in addition that $k_{A_1} = k_N$ and A_1 is efficient in N. Then we can extend f such that its countable domain N' has a value group which is pure in $v(N^\times) = \Gamma_N$, $N' \subseteq N$ and N' is a totally ramified valued D-field extension of A_1.*

Proof. First we prove the following claim:

Claim: Let w be in $v(N^\times)$ such that $n \cdot w \in v(A_1^\times)$ for some minimal positive integer n. Then there exists $x \in N$ such that $v(x) = w$ and $x^n = c \in A_1$.

Let $w = v(y)$ for some $y \in N$ and $v(y^n) = n \cdot w = v(a)$ for some $a \in A_1$. Then we get that $\pi(y^n a^{-1}) \in k_N = k_{A_1}$, so $\pi(y^n a^{-1}) = \pi(b)$ for some $b \in A_1$; we take $c = ab$. Since N is henselian, the relation $v(c) = v(y^n) < v(c - y^n)$ proves the existence of a nth root x of c in N, and the claim is proved.

Let $\widehat{v(A_1^\times)}$ be the pure hull of $v(A_1^\times)$ in $v(N^\times)$. We choose, in order to generate $\widehat{v(A_1^\times)}$, elements $u_i \in N$ such that u_i is radical over $A_1\langle u_j ; j < i\rangle$.

So we build inductively a substructure $N' = \bigcup_{i<\omega} N'_i$ of N such that $N'_{i+1} = N'_i\langle u_i\rangle$ where $n \cdot v(u_i) \in v(N_i'^\times)$ for some positive integer n, minimal with this property and $N'_0 = A_1$. By using the *Claim* and Proposition (3.10), $N'_i\langle u_i\rangle$ is a valued D-field extension of N'_i uniquely determined up to $\mathcal{L}_1(N'_i)$-isomorphism. Now we extend, inductively on i, the isomorphism f to N'. Assume that the isomorphism is already extended to N'_i ($i \geqslant 0$ and $N'_0 = A_1$) and extend the isomorphism to N'_{i+1}. Assume that $m \cdot v(u_i) \in \widehat{v(A_1^\times)}$ for some minimal natural number m. By the previous claim, there exists y_i in $\widetilde{A_1(u_i)}$ such that $y_i^m = d_i \in A_1$ where $\widetilde{A_1(u_i)}$ is the Henselization of

$A_1(u_i)$. So $u_i = \epsilon \cdot y_i$ where $v(\epsilon) = 0$ and $\epsilon \in \widetilde{A_1(u_i)}$. Since f respects the predicates $F_{\phi,\vec{n}}$, the predicates P_n are preserved by f (because we have $P_n(x) \iff F_{\phi,n}(x)$ for $\phi(\eta) := (\eta = 1)$). So $f(d_i)$ is a mth power in M_2. Now we can uniquely extend f over $A_1\langle y_i\rangle$ in taking for $f(y_i)$ a root y'_i of order m of $f(d_i)$. Moreover the isomorphism can be extended to $\widetilde{A_1(y_i)}$. Now we let, for $f(u_i)$, the element $f(\epsilon)\cdot y'_i$ and the isomorphim f is extended to $N'_{i+1} = N'_i\langle u_i\rangle$. By induction we obtain that f is an $\mathcal{L}_1$-isomorphism such that $\mathrm{dom}(f) = N'$. Moreover, by Lemma 3.5, f is an $\mathcal{L}_2$-isomorphism since $v(N'^{\times}) = \widehat{v(A_1^{\times})}$ is pure in $v(N^{\times})$. $\square$

The following proposition allows us to extend the isomorphism f such that $\mathrm{dom}(f)$ has enough constants in the sense of Definition 2.1.

Proposition 3.12. *Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k},\mathbf{G})$-D-henselian valued fields. Let $A_1 \subseteq_{\mathcal{L}_2} M_1$ and $A_2 \subseteq_{\mathcal{L}_2} M_2$ be countable substructures with a countable elementary submodel N of M_1 containing A_1 and a partial $\mathcal{L}_2$-isomorphism $f : A_1 \to A_2$. Suppose in addition that $k_{A_1} = k_N$, A_1 is efficient in N and the value group $v(A_1^{\times})$ is pure in $v(N^{\times}) = \Gamma_N$. Let γ be in $v(A_1^{\times}) \setminus v(C_{A_1}^{\times})$.*

Then the $\mathcal{L}_2$-isomorphism f can be extended to a countable valued D-field extension A'_1 of A_1 which has the same residue field as a countable elementary submodel N' of M_1 containing N, has a value group which is pure in $v(N'^{\times}) = \Gamma_{N'}$ and $C_{A'_1}$ has an element of value γ.

Proof. We use the same argument as in the proof of Proposition 7.18 of [Sca00]. Let $\gamma \in v(A_1^{\times}) \setminus v(C_{A_1}^{\times})$ and $a \in A_1$ such that $v(a) = \gamma$. Proposition 7.16 in [Sca00] allows us to see that finding an element x such that $D(x) = 0$ and $v(x) = v(a)$ is equivalent to finding an element $y(= a/x)$ such that $v(y) = 0$ and $D(y) = (D(a)/a) \cdot y$.

Let us consider the following partial type Σ in the sort k

$$\{D(x) = \pi(D(a)/a) \cdot x\} \cup \{x \neq b |\, b \in k_{A_1}\}.$$

This partial type p is consistent since we are in characteristic zero and k_N is linearly differentially closed. Now we choose an element $b_1 \in k_{M_1}$ such that $b_1 \models \Sigma$ and we let the type $p := tp(b_1/k_{A_1})$. By the saturation hypothesis, $f_R(p)$ is realized in k_{M_2} by some b_2. By the D-Hensel's Lemma, there is some c_1 in M_1 and some c_2 in M_2 such that $\pi(c_i) = b_i$, $D(c_1) = (D(a)/a) \cdot c_1$ and $D(c_2) = f(D(a)/a) \cdot c_2$. By Proposition 7.16 of [Sca00], the extension of f given by $c_1 \to c_2$ is an $\mathcal{L}_1$-isomorphism and the element a/c_1 is a constant with value $v(a)$. Now we take a countable elementary submodel N' of M_1 containing N and $A_1\langle c_1\rangle$. We note that the new domain $A_1\langle c_1\rangle$ has the same value group as A_1 and so, it is efficient in N'. As usual, we extend the $\mathcal{L}_1$-isomorphism f such that its domain has the same residue field as N'. By the Claim of Proposition 3.4, we have that f is also an $\mathcal{L}_2$-isomorphism. Now we use Proposition 3.11 to finish the proof. $\square$

Now we apply the previous results in order to prove our main result of quantifier elimination. We will follow the lines of the proof of Theorem 2.2 in [Del82].

Theorem 3.13. *The $\mathcal{L}_2$-theory of $(\mathbf{k},\mathbf{G})$-D-henselian fields eliminates quantifiers.*

Proof. We prove this by a standard back-and-forth test. Let M_1 and M_2 be two $\aleph_1$-saturated $(\mathbf{k},\mathbf{G})$-D-henselian fields. Let $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ be countable $\mathcal{L}_2$-substructures. Let $f : A_1 \to A_2$ be an isomorphism of $\mathcal{L}_2$-structures. Let $b \in M_1$. We have to prove that f extends to a partial isomorphism from M_1 to M_2 having b in its domain. Before extending f, we fix a countable elementary submodel N of M_1 containing A_1 and b. So we have that k_N is a linearly differentially closed field of characteristic zero equal to $\pi(\mathcal{O}_N)$, $k_{A_1} \subseteq k_N$ and $\Gamma_{A_1} \subseteq \Gamma_N$ where $\Gamma_N = v(C_N^\times)$.

First, we extend the partial residue field isomorphism f_R (respectively the partial value group isomorphism f_V) to k_N (respectively to $v(N^\times)$).

By using Proposition 3.4 and Proposition 3.11, we can extend f such that its domain, also denoted by A_1, has the same residue field as a countable elementary submodel N_1 of M_1 containing N, is efficient in N_1 and its value group is pure in $v(N_1^\times)$. We can also extend f_R and f_V to k_{N_1} and respectively to $v(N_1^\times)$ as in the paragraph after Lemma 3.3.

Now we have to extend f such that its domain A_1' has the same residue field as a countable elementary submodel N_1' of M_1 containing N_1, its value group is pure in $v({N_1'}^\times)$ and A_1' has enough constants. For this purpose, we proceed by induction on ω to take $N^{(i-1)} \preceq N^{(i)}$ a countable elementary submodel of M_1 and find a countable extension $A^{(i)}$ of $A^{(i-1)}$ on which an extension of f is defined such that $v(C^\times_{A^{(i)}})$ contains $v({A^{(i-1)}}^\times)$ and $v({A^{(i)}}^\times)$ is pure in $\Gamma_{N^{(i)}} = v({N^{(i)}}^\times)$, $\pi(\mathcal{O}_{A^{(i)}}) = k_{A^{(i)}} = k_{N^{(i)}}$, $A^{(i)} \subseteq N^{(i)}$ and $A^{(i)}$ is efficient in $N^{(i)}$.

We begin the induction with $A^{(0)} := A_1$, $N^{(0)} := N_1$, $N^{(-1)} := A_1$ (the required properties are ignored at this initial stage) and by the first step, we have that $k_{A_1} = k_{N_1}$, $v(A_1^\times)$ is pure in $v(N_1^\times)$ and A_1 is efficient in N_1. Eventually we let N_1' (respectively A_1') be the direct limit of the $N^{(i)}$'s (respectively of the $A^{(i)}$'s) and we get that N_1' is a countable elementary submodel of M_1 and A_1' is an efficient $\mathcal{L}_2$-substructure of N_1' such that $k_{A_1'}$ is equal to $k_{N_1'}$, A_1' has enough constants and its value group is pure in $v({N_1'}^\times)$.

Assume $A^{(m)}$ and $N^{(m)}$ have been built for each $m \leqslant i$ (for some natural number i). We enumerate the following set $v({A^{(i)}}^\times) \setminus v(C^\times_{A^{(i)}})$ by a sequence of elements $(\gamma_n)_{n\in\omega}$.

We proceed by induction on ω to take $N^{(i,n-1)} \preceq N^{(i,n)} \prec M_1$ a countable model and find a countable extension $A^{(i,n)}$ of $A^{(i,n-1)}$ on which an extension of f is defined such that, for each element γ_n in $v({A^{(i)}}^\times) \setminus v(C^\times_{A^{(i)}})$, $A^{(i,n)}$ has a value group which is pure in $v({N^{(i,n)}}^\times)$, contains a constant with value γ_n and is efficient in $N^{(i,n)}$ such that $\pi(\mathcal{O}_{A^{(i,n)}}) = k_{A^{(i,n)}} = k_{N^{(i,n)}}$ and $A^{(i,n)} \subseteq N^{(i,n)}$. We begin the induction with $A^{(i,0)} := A^{(i)}$, $N^{(i,0)} := N^{(i)}$, $N^{(i,-1)} := A^{(i)}$ (and the required properties are ignored at this initial stage). Eventually we let $N^{(i+1)}$ (respectively $A^{(i+1)}$) be the direct limit of the $N^{(i,n)}$'s (respectively of the $A^{(i,n)}$'s) and the couple $(N^{(i+1)}, A^{(i+1)})$ is as required.

Assuming $A^{(i,m)}$ and $N^{(i,m)}$ have been built for each $m \leqslant n$ (for some natural number $n \geqslant 0$), we are now building the required $A^{(i,n+1)}$ and $N^{(i,n+1)}$. By Proposition 3.12, we can extend the $\mathcal{L}_2$-isomorphism f such that its domain $A^{(i,n+1)}$ has the same residue field as a countable elementary submodel $N^{(i,n+1)}$ of M_1 (containing $N^{(i,n)}$), is efficient in $N^{(i,n+1)}$ and $v(A^{(i,n+1)^\times})$ is a pure subgroup of $v(N^{(i,n+1)^\times})$ containing γ_n.

Now we may assume that the domain of the $\mathcal{L}_2$-isomorphism f, denoted by A_1', has enough constants, the same residue field as a countable elementary submodel N_1' of M_1 and has a value group which is pure in $v({N_1'}^\times)$.

Since $C_{A_1'} \hookrightarrow C_{N_1'}$ as valued fields, we take an $\aleph_1$-saturated ultrapower $\mathcal{U}$ of this valued fields embedding: $C_{A_1'}^{\mathcal{U}} \hookrightarrow C_{N_1'}^{\mathcal{U}}$, and we have $C_{A_1'}^{\mathcal{U}}$ (which is $\aleph_1$-saturated) admits a cross-section β_1 which can be extended to a cross-section $\widetilde{\beta_1}$ of $C_{N_1'}^{\mathcal{U}}$ by the $\aleph_1$-saturation of $C_{N_1'}^{\mathcal{U}}$ and the pureness of $v(C_{A_1'}^\times)$ in $v(C_{N_1'}^\times)$. Since $v(C_{A_1'}^\times) = v({A_1'}^\times)$ and $v(C_{N_1'}^\times) = v({N_1'}^\times)$, we can consider β_1 (respectively $\widetilde{\beta_1}$) as a cross-section of ${A_1'}^{\mathcal{U}}$ (respectively ${N_1'}^{\mathcal{U}}$). In the language $\mathcal{L}_2$ enriched by a symbol for β ($\widetilde{\beta}$) and a predicate for ${A_1'}^{\mathcal{U}}$, we consider a countable elementary substructure L_1' of ${N'}_1^{\mathcal{U}}$ containing N_1'. We obtain:

$$\begin{tikzcd}
D_1' = {A'}_1^{\mathcal{U}} \cap L_1' \arrow[r, hook] \arrow[d, "f^{\mathcal{U}}\lceil_{D_1'}"', "\sim"] & L_1' \arrow[r, hook, "\prec"] & {N'}_1^{\mathcal{U}} \arrow[r, hook, "\prec"] & M_1^{\mathcal{U}} \\
D_2' \arrow[rrr, hook] & & & M_2^{\mathcal{U}}
\end{tikzcd}$$

Since $L_1' \prec {N'}_1^{\mathcal{U}}$ and ${N'}_1^{\mathcal{U}} \models \forall x\, \exists y \in {A'}_1^{\mathcal{U}}\, [\pi(x) = \pi(u)]$, we get that $k_{D_1'} = k_{L_1'}$ and L_1' admits a cross-section β on its value group satisfying $\beta(v({D_1'}^\times)) \subseteq C_{D_1'}$ and $\beta(v({L_1'}^\times)) \subseteq C_{L_1'}$. Now we are extending $f^{\mathcal{U}}\lceil_{D_1'}$ to an $\mathcal{L}_2$-isomorphism f_1' such that its domain is $E_1' = D_1'(\beta(v({L_1'}^\times)))$. By applying Lemma 3.8, we get that $k_{E_1'} = k_{L_1'}$ and $v({E_1'}^\times) = v({L_1'}^\times)$ and the valued D-field structure is characterized by these data. Since $v({A'}_1^{\mathcal{U}\times})$ is a direct summand of $v({N'}_1^{\mathcal{U}\times})$, there exists a subgroup H of ${L_1'}^\times$ such that $v({L'}_1^\times) = v({D'}_1^\times) \oplus v(H)$ (because $L_1' \prec {N'}_1^{\mathcal{U}} \models \forall x\, \exists y \in {A'}_1^{\mathcal{U}}\, \forall z \in {A'}_1^{\mathcal{U}} [v(x) = v(y) \vee v(x) - v(y) \neq v(z)]$). By using the isomorphism $(f_1')_V$ to $v(H)$ and the fact that $C_{M_2}^{\mathcal{U}}$ admits a cross-section β_2 on its value group (because $C_{M_2}^{\mathcal{U}}$ is $\aleph_1$-saturated and $v(M_2^\times) = v(C_{M_2}^\times)$), we apply Lemma 3.9 to $\beta_2((f_1')_V(v(H)))$. Hence we get an $\mathcal{L}_1$-isomorphism f_1' which will respect the language $\mathcal{L}_2$ by Lemma 3.5. Let us denote by D_2' (respectively E_2') the image of D_1' (respectively E_1') by the $\mathcal{L}_2$-isomorphism f_1'.

Since L_1' is not necessarily included in M_1 and E_2' is not necessarily included in M_2, we have to use the $\aleph_1$-saturation of M_1 and M_2 and the countability of L_1' (respectively E_2') in order to find an image L_1 (respectively E_2) isomorphic to L_1' (respectively E_2') by an isomorphism f_1 (respectively f_2) which pointwise fixes A_1' (respectively A_2'). We can represent the situation by the following diagram:

$$
\begin{tikzcd}
D_1 \arrow[r, hook] & E_1 \arrow[r, hook] & L_1 \arrow[r, hook] & M_1 \arrow[d, "\prec"] \\
D'_1 \arrow[u, "f_1\restriction_{D'_1}", "\sim"'] \arrow[r, hook] \arrow[d, dash, "f^{\mathcal{U}}\restriction_{D'_1}"', "\sim"] & E'_1 \arrow[u, "f_1\restriction_{E'_1}"] \arrow[r, hook] \arrow[d, dash, "f'_1"', "\sim"] & L'_1 \arrow[u, "f_1"] \arrow[r, hook, "\prec"] & M_1^{\mathcal{U}} \\
D'_2 \arrow[r, hook] \arrow[d, "f_2^{\mathcal{U}}\restriction_{D'_2}"', "\sim"] & E'_2 \arrow[rr, hook] \arrow[d, "f_2"', "\sim"] & & M_2^{\mathcal{U}} \\
D_2 \arrow[r, hook] & E_2 \arrow[rr, hook] & & M_2 \arrow[u, "\prec"']
\end{tikzcd}
$$

Now we denote by E_1 the image of E'_1 by the isomorphism f_1. We have extended the isomorphism f such that $\mathrm{dom}(f)$ is equal to E_1 and E_1 is an $\mathcal{L}_2$-substructure of a countable elementary substructure of M_1, namely L_1.

Now we may assume that the domain of the isomorphism f, denoted by A^*, has enough constants and that N^* is an immediate extension of A^* as in the proof of Theorem 6.4 in [Sca99]. Moreover, N^* is a countable elementary substructure of M_1 and so, we use the same proof as in [Sca00]. As in Lemma 7.49 of [Sca00], we take an immediate extension of A^* in M_1 which is in fact an ∞-full extension of A^* (following the terminology in Definition 7.26 of [Sca00]). So this immediate extension contains N^* and so b. In [Sca00], Lemma 7.50 allows us to express the form of this immediate extension of valued D-fields and Proposition 7.52 of [Sca00] shows that there is an extension of f to an $\mathcal{L}_1$-embedding of N^* into M_2. Now by using the fact that $\mathrm{dom}(f)$ is an immediate extension of A^* and Lemma (3.6), we have that this $\mathcal{L}_1$-isomorphism is also an $\mathcal{L}_2$-isomorphism. So we have finished the proof. $\square$

The following result is a reformulation of [Del82, Corollary 2.21] in the valued D-field setting.

Corollary 3.14. *Let M be a $(\mathbf{k}, \mathbf{G})$-D-henselian field. Assume that for all $n \in \mathbb{N} \setminus \{0,1\}$, $k_M^\times/(k_M^\times)^n$ is finite with j_n its number of elements.*

Then M eliminates quantifiers in $\mathcal{L}_1 \cup \{C_{n,j}; n \in \mathbb{N} \setminus \{0,1\}, 1 \leqslant j \leqslant j_n\} \cup \{P_n; n \in \mathbb{N} \setminus \{0,1\}\}$, the $C_{n,j}$'s are constants of the field and for each n, the $\pi(C_{n,j})$'s are a set of representatives of $k_M^\times/(k_M^\times)^n$.

Proof. There is quantifier elimination in $\mathcal{L}_2$, so in $\mathcal{L}_2 \cup \{C_{n,j}; P_n\}$. But the predicates are quantifier-free definable in $\mathcal{L}_1 \cup \{C_{n,j}; P_n\}$:

$$M \models F_{\phi,n_1,\cdots,n_r}(x_1,\cdots,x_r,\eta_1,\cdots,\eta_s) \iff$$

$$M \models \exists z_1,\ldots,z_r\big[\bigwedge_{i=1}^{r} v(z_i)=0 \wedge \phi(\pi(z_1),\ldots,\pi(z_r),\eta_1,\ldots\eta_s) \wedge \bigwedge_{i=1}^{r} P_{n_i}(x_iz_i)\big] \iff$$

$$k_M \models \exists \zeta_1,\ldots,\zeta_r\, \phi(\zeta_1,\ldots,\zeta_r,\eta_1,\ldots\eta_s) \wedge \bigwedge_{i=1}^{r}\big\{\bigvee_{j=0}^{j_{n_i}}[p_{n_i}(\zeta_i\pi(C_{n_i,j})) \wedge P_{n_i}(C_{n_i,j}^{-1}x_i)]\big\}.$$

This formula is equivalent to a disjunction of terms where the variables with quantifiers concern only residue sort formulas. □

4. Application

We keep the notations of the previous sections. If K is a field equipped with two valuations v and w then we add a subscript v in order to distinguish the valuation rings, maximal ideals and residue fields of the valuation v with those of w (i.e. $\mathcal{O}_{K,v}$, $\mathcal{M}_{K,v}$ and $k_{K,v}$). To a valuation defined on K, we can associate a binary relation $\mathcal{D}$ which is interpreted as the set of 2-tuples (a,b) of K^2 such that $v(a) \leqslant v(b)$. So this relation $\mathcal{D}$ satisfies the following:

- $\mathcal{D}$ is transitive, $\neg\mathcal{D}(0,1)$,
- $\mathcal{D}$ is compatible with $+$ and $\cdot$ and,
- either $\mathcal{D}(a,b)$ or $\mathcal{D}(b,a)$ for all $a,b \in K$ (*).

Such a relation is called a *linear divisibility relation* (l.d. relation).

If A is a subring of K with fraction field K and a relation $\mathcal{D}$ which satisfies the properties (*) then, by extending $\mathcal{D}$ to K as follows:

$$\mathcal{D}(\frac{a}{b},\frac{c}{d}) \iff \mathcal{D}(ad,bc),$$

we get that the l.d. relation on K induces a valuation v on K by defining $v(a) \leqslant v(b)$ if $\mathcal{D}(a,b)$.

This section concerns a differential analogue of a positivstellensatz result for real-series closed fields (see [Far91] or Chapter 1 in [Far93]). So we introduce the differential counterpart of the theory of real-closed fields. More precisely, we are interested in D-henselian valued fields with differential residue fields which are models of $CODF$ (see [Sin78]) and $\mathbb{Z}$-groups as value groups. By using our quantifier elimination result, we prove the model completeness of this theory which allows us to solve the problem (see Theorem 4.13).

First, we are interested in ordered valued fields with a compatibility condition between the order and the valuation topology.

Definition 4.1. Given a valuation v on the ordered field $\langle K, \leqslant\rangle$, we will say that v is *convex* or *compatible with the order* $\leqslant$, if v satisfies the following equivalent conditions:

(1) $\mathcal{O}_K$ is a convex subset of $\langle K, \leqslant\rangle$,

(2) $v(x) > v(y)$ implies $|x| < |y|$ for all $x, y \in K^\times$,
(3) $1 + \mathcal{M}_K \subseteq P$ where P is the positive cone of $\langle K, \leqslant \rangle$.

In the sequel, a real-closed field k is a field such that the set of squares form a positive maximal cone P, i.e. P is equal to the set of positive elements for a total ordering $\leqslant$, for which the intermediate value property holds.

Following the terminology of [Jac80], we can now define a real-series closed field.

Definition 4.2. A field K with a distinguished non-zero element is said to be *real-series closed* if K carries a henselian valuation v with real-closed residue field and as value group a $\mathbb{Z}$-group with $v(a) = 1$ is the least positive element of $v(K^\times)$.

Since the valuation v on K is henselian and the residue field k_K is real-closed, we can show that there exists an order $\leqslant$ on K such that v is convex and so, this order induces the initial order on k_K. Moreover, if v and w are two such valuations, their valuation rings $\mathcal{O}_{K,v}$ and $\mathcal{O}_{K,w}$ are convex for every order of K and therefore, $\mathcal{O}_{K,v} \subseteq \mathcal{O}_{K,w}$ or $\mathcal{O}_{K,v} \supseteq \mathcal{O}_{K,w}$. Suppose $\mathcal{O}_{K,v} \subseteq \mathcal{O}_{K,w}$. By the conditions $v(a) = 1$ and $w(a) = 1$, we have that $\mathcal{M}_{K,v} = a \cdot \mathcal{O}_{K,v} \subseteq a \cdot \mathcal{O}_{K,w} = \mathcal{M}_{K,w}$ and $\mathcal{O}_{K,v} = \mathcal{O}_{K,w}$. Such a field has exactly two orders determined by the conditions $a > 0$ or $a < 0$ and with positive cones $K^2 \cup aK^2$ and $K^2 \cup -aK^2$ respectively. For a proof of these previous general facts, the reader can refer to Chapter 0 in [Far93].

From now on, we deal with $\langle K, a \rangle$ a field with a distinguished element a; we will add a constant $\underline{a}$ to the language of valued fields. When we will deal with discrete valuation v on a field K, we will denote the least positive element of $v(K^\times)$ by 1.

In the theory of real-closed series fields, an operator $\gamma(X)$ (defined in the following lemma) plays an important role as the Kochen's operator in the p-adic field case (see [Koc67]). The next lemma determines whenever an element of the maximal ideal of a valued field $\langle K, v \rangle$ has the least positive value.

Lemma 4.3. *(See Lemma 2.3 in* [Far91]*) Let $\langle K, v \rangle$ be a valued field and let a be a non-zero element of K. Let γ be the operator defined by $\gamma(X) = \frac{X}{X^2 - a}$. Then the following are equivalent:*

(1) *$v(a) = 1$,*
(2) *$\gamma(K) \subseteq \mathcal{O}_K$ and $a \in \mathcal{M}_K$.*

So, it follows from (2) that the valuation is discrete.

The statement of the following lemma can be found in [Far91].

Lemma 4.4. *Let $\langle K, a \rangle$ be a real-series closed field. Then the valuation ring $\mathcal{O}_K$ is equal to $\gamma(K)$.*

Proof. By definition of real-series closed field, $v(a)$ is equal to 1 in $v(K^\times)$, and by Lemma 4.3, we have that $\gamma(K) \subseteq \mathcal{O}_K$. Let k be in $\mathcal{O}_K$. We consider the polynomial $f(X) = (X^2 - a)k - X$. Clearly 0 is a simple residue root of f. So by Hensel's Lemma, we find $b \in \mathcal{O}_K$ such that $k = \gamma(b)$. $\square$

Notation 4.5. If A is a subset of K then we denote by $SG(A)$ the semigroup generated by A and by $S(A)$ the semiring (closed under addition and multiplication) generated by A.

With these notations we can state Theorem 1.1 in [Far91].

Proposition 4.6. *Let K be a field of characteristic zero, A, M and B subsets of K such that $\mathbb{Z} \cdot M \subseteq M$ and $A \cdot M \subseteq M$. Then $S(K^2 \cup B \cup 1 + M) = \bigcap \mathcal{P}$ where $\mathcal{P} = \{P/P$ is the positive cone of an order in K containing B and such that there exists a compatible valuation v with $A \subseteq \mathcal{O}_{K,v}$ and $M \subseteq \mathcal{M}_{K,v}\}$.*

In [Sin78], M.F. Singer introduced the theory of closed ordered differential fields, denoted by $CODF$. This theory admits quantifier elimination in the language of differential ordered fields i.e. $\mathcal{L}_D \cup \{\leqslant\}$ where $\mathcal{L}_D$ is the language of differential fields $\mathcal{L}_{\text{fields}} \cup \{D\}$, $\mathcal{L}_{\text{fields}}$ is the language of fields, $\leqslant$ is a binary relation to be interpreted as a total ordering and D is an unary function to be interpreted as a derivation.

Now we recall an axiomatization of $CODF$:

(1) the theory RCF of real-closed fields,
(2) D is a derivation,
(3) for any positive integer n, for any differential polynomial $f(X, \cdots, X^{(n)})$ of order n,

$$\forall \epsilon \, \forall b_0, \cdots, b_n \left[f^*(b_0, \cdots, b_n) = 0 \wedge \frac{\partial}{\partial X^{(n)}} f^*(b_0, \cdots, b_n) \neq 0 \right.$$
$$\left. \Rightarrow \exists z \, f(y) = 0 \wedge \bigwedge_{i=0}^{n} |y^{(i)} - b_i| < \epsilon \right]$$

where f^* is the differential polynomial seen as an ordinary polynomial in the differential indeterminates and $|x|$ is the absolute value operator.

By using $CODF$ as differential residue theory, we can introduce the valued D-field analogue of the theory of real-series closed fields.

Definition 4.7. A differential field $\langle K, D, \underline{a} \rangle$ is called *a closed real-series differential field* if $\langle K, D \rangle$ admits a D-henselian valuation v such that k_K is a model of $CODF$ and K has a $\mathbb{Z}$-group of value with $v(a) = 1$ and $D(a) = 0$.

Remark 4.8. A model of this theory was given in Section 2 where $\mathbf{k}$ is a model of $CODF$ and $\mathbf{G}$ is a $\mathbb{Z}$-group. In this example, t plays the role of a in the previous definition (so $D(a) = 0$). Generally, when such a triple is given, we consider K as an ordered field where the positive cone is given by $K^2 \cup aK^2$ (i.e. $a > 0$ in K).

Now we apply Corollary 3.14 in order to prove a model completeness result for the theory of closed real-series differential fields in a suitable language $\mathcal{L}_a$. In the sequel, we will use this result about this three-sorted theory of D-henselian fields. In the algebraic application, we want to use a one-sorted language. Hence we will use a l.d. relation $\mathcal{D}$ in order to express the axioms of $(\mathbf{k}, \mathbb{Z})$-D-henselian fields such that $\mathbf{k} \models CODF$.

We consider the language $\mathcal{L}_a := \mathcal{L}_D \cup \{\mathcal{D}, \underline{a}\}$ where $\mathcal{D}$ will be interpreted as a l.d. relation with respect to a valuation v on a field K and $\underline{a}$ is a constant symbol. The axioms for the $\mathcal{L}_a$-theory of closed real-series differential fields, denoted by $CSDF$, are the following (the reader can be easily verified that these following properties can be written in the first-order language $\mathcal{L}_a$):

K is a model of $CSDF$ iff

- K is a valued D-field of equicharacteristic zero;
- $K \models \forall x\, \exists y\, [Dy = 0 \wedge \mathcal{D}(x,y) \wedge \mathcal{D}(y,x)]$ which means that $v(K^{\times})$ is equal to $v(C_K^{\times})$;
- K satisfies the D-Hensel's Lemma which can be formulated as follows: for each positive integer n, for each differential polynomial $f(X, \cdots, X^{(n)})$ of order n with coefficients in the valuation ring $\mathcal{O}$ (i.e. $\mathcal{O} = \{x |\, \mathcal{D}(1,x)\}$),
$$\forall b \quad \big[\mathcal{D}(1,b) \wedge \bigvee_{i=0}^{n} (\mathcal{D}(1, \frac{\partial}{\partial X^{(i)}} f(b)) \wedge \mathcal{D}(\frac{\partial}{\partial X^{(i)}} f(b), 1)) \wedge \neg \mathcal{D}(f(b), 1)\big]$$
$$\Rightarrow \exists z \big\{ f(z) = 0 \wedge \mathcal{D}(f(b), b - z) \big\};$$
- k_K is a model of $CODF$ and $v(K^{\times})$ is a $\mathbb{Z}$-group;
- K satisfies the following axiom which says that the constant element a has value 1 in $v(K^{\times})$:
$$\mathcal{D}(1,a) \wedge \forall x\, [\mathcal{D}(1,x) \wedge \neg\mathcal{D}(x,1) \Rightarrow \mathcal{D}(a,x)] \wedge D(a) = 0.$$

Now we want to establish a model-theoretic result needed in the proof of Theorem 4.13 which is a differential positivstellensatz for closed real-series differential fields.

Lemma 4.9. *In the $\mathcal{L}_a$-theory of real-series closed fields, the nth power predicates and their negations are existentially definable in the language of rings with the distinguished element $\underline{a}$.*

Proof. Let us consider an element $x \neq 0$ of a model K of $CSDF$ such that $v(x) \geqslant 0$ (otherwise we use that $P_n(x) \iff P_n(x^{-n+1})$). Then for each natural number n, we get that
$$K \models \exists y\, [\bigvee_{i=0}^{n-1} v(x) = v(a^i y^n)].$$
Since $\mathcal{O}_K$ satisfies the Hensel's Lemma and k_K is real-closed, this is equivalent to
$$k_K \models \exists z\, \{ \bigvee_{i=0}^{n-1} z^n = x \cdot a^{-i} \cdot y^{-n} \vee \bigvee_{i=0}^{n-1} z^n = -x \cdot a^{-i} \cdot y^{-n} \}.$$
So we get that $K = \bigcup_{i=0}^{n-1} \bigcup_{k \in \{0,1\}} (-1)^k a^i K^n$ if n is even and $K = \bigcup_{i=0}^{n-1} a^i K^n$ if n is odd (and the unions are disjoint). $\square$

Proposition 4.10. *The $\mathcal{L}_a$-theory $CSDF$ is model complete.*

Proof. Let $\mathbf{k}$ be a model of $CODF$. It is well-known that the theory of $\mathbb{Z}$-groups admits quantifier elimination in the language $\mathcal{L}_V$ of totally ordered abelian groups with divisibility predicates (see Section 2) and that the theory of closed ordered differential fields admits quantifier elimination in the language $\mathcal{L}_R$ of differential ordered fields. We have to show that any $\mathcal{L}_a$-formula is equivalent to an existential formula.

Let $\phi(\bar{x})$ be an $\mathcal{L}_a$-formula with $\bar{x}$ the free variables. By using the Appendix, we can translate this $\mathcal{L}_a$-formula to an $(\mathcal{L}_D, \mathcal{L}_V, \mathcal{L}_R)$-formula $\phi_*(\bar{x})$. Now we apply Corollary 3.14 to obtain an $(\mathcal{L}_D, \mathcal{L}_V, \mathcal{L}_R)$-quantifier-free formula equivalent to $\phi_*(\bar{x})$.

Since the divisibility predicates $n|.$ of the language of $\mathbb{Z}$-groups and their negations are existentially definable in the language $\{+,-,.,0,1\}$; and the order, the predicates of nth powers and their negations are existentially definable in the language of fields in $CODF$, we get by using the previous lemma and the Appendix, an existential $\mathcal{L}_a$-formula $\psi(\bar{x})$ equivalent to $\phi(\bar{x})$ (we also used $v(a)=1$). $\square$

Proposition 4.11. *Let $\langle K, D, \leqslant, v, a\rangle$ be an ordered valued D-field. Assume that the valuation v is convex and $v(a)=1$ in $v(K^\times)$. Then we can extend the ordered valued D-field $\langle K, D, \leqslant, v, a\rangle$ to a model $\langle L, D, \leqslant, w, a\rangle$ of $CSDF$.*

Proof. We know that if H is a discrete totally ordered abelian group and $\alpha = 1_H$ is the least positive element of H then there exists G an extension of H contained in $\widetilde{H}$, the divisible hull of H such that G is a $\mathbb{Z}$-group with first positive element α (see Lemma 4 in [Koc67]). First we build an henselian unramified valued D-field extension K' of K such that its differential residue field is a model of $CODF$. Since $CODF$ is the model completion of the theory of ordered differential fields, we can consider an extension k' of k_K which is a model of $CODF$. Using the existence part of Lemma 7.12 in [Sca00], we obtain our extension K'. Moreover, by Lemma 1.2 of [Far91], we can equip K' with an order which extends the one of K, is compatible with the valuation on K' and induces the order on k' (moreover, as in Proposition 1.3 of [Far91], we can assume K' henselian). Then we build an ordered totally ramified valued D-field extension K'' of K' such that its value group $v(K''^\times)$ is equal to G. Indeed let g in $G \setminus v(K''^\times) = v(K^\times)$ be such that $n \cdot g \in v(K^\times)$ for some minimal positive integer n. We choose an element x and define the extension $K'(x)$ by $x^n = k \in K'$ such that $v(k) = n \cdot g \in v(K)$. By Lemma 3.10, we obtain a totally ramified valued D-field extension of K' which contains an element of value h. By transfinite induction, we obtain K'' with the required properties. To obtain an ordered extension of K', it suffices to take the element x in the real closure of K' as in Proposition 1.3 of [Far91]. Now by using the construction in Proposition 3.12 and the first step of the proof we obtain an unramified valued D-field extension K''' of K'' which has enough constants and has a model of $CODF$ as differential residue field. To finish the proof, we proceed as in [Sca00], more precisely we use Lemma 7.25 of [Sca00] to produce the necessary pseudo-convergent sequence in K''' and then use Proposition 7.32 of [Sca00] to actually find a solution in an immediate valued D-field extension. So we obtain the required valued D-field extension L. Since the extension is immediate, the valuation v is henselian on L and $k_L \models CODF$ with $v(L^\times)$ a $\mathbb{Z}$-group. If we put $L^2 \cup aL^2$ as positive cone for the order $\leqslant_L$ on L then v is convex for the order $\leqslant_L$; so L is also an ordered extension of $K\langle \underline{X}\rangle$. $\square$

Notation 4.12. In the sequel we use the operator $\dagger$ defined as $x^\dagger = D(x)/x$ which is the logarithmic derivative. We will denote the differential field of the differential rational functions in n differential indeterminates by $K\langle \underline{X}\rangle$.

Now we prove a differential positivstellensatz for closed real-series differential fields which is the analogue of Theorem 2.6 in [Far91].

Theorem 4.13. *Let $\langle K, D, a\rangle$ be a closed real-series differential field and $\leqslant$ the order for $a>0$. Let $f, g_1, \cdots, g_r \in K\langle \underline{X}\rangle$.*

Assume that $f(\bar{x}) \geqslant 0$ *for every* $\bar{x} \in K^n$ *such that* $f(\bar{x}), g_1(\bar{x}), \cdots, g_r(\bar{x})$ *are defined and* $g_i(\bar{x}) > 0$ *for all* $i \in \{1, \cdots, r\}$ *(*).*

Then there exist $k \in \mathbb{N}$, δ_i, $\delta_{i,j} \in \{0,1\}$ *for* $i = 1, \cdots, k, j = 1, \cdots, r$, $h_1, \cdots, h_k \in K\langle \underline{X} \rangle$, $u_1, \cdots, u_k \in SG(\gamma(K\langle \underline{X} \rangle) \cup (K\langle \underline{X} \rangle)^\dagger)$ *such that* $f = \sum_{i=1}^k h_i^2 a^{\delta_i} g_1^{\delta_{i,1}} \cdots g_r^{\delta_{i,r}} (1 + u_i)$.

Proof. We follow the lines of the proof of Theorem 2.6 in [Far91]. We suppose that f does not admit such an expression i.e. $f \notin S(K\langle \underline{X} \rangle^2 \cup \{a, g_1, \cdots, g_r\} \cup 1 + a \cdot SG(\gamma(K\langle \underline{X} \rangle) \cup (K\langle \underline{X} \rangle)^\dagger))$. We set $A = \gamma(K\langle \underline{X} \rangle)$ and $M = a \cdot SG(\gamma(K\langle \underline{X} \rangle), (K\langle \underline{X} \rangle)^\dagger))$. From the fact $a \in \gamma(K)$ and $\mathbb{Z} \subseteq \gamma(K)$, one can easily deduce $M \cdot M \subseteq M$, $A \cdot M \subseteq M$ and $\mathbb{Z} \cdot M \subseteq M$. Thus, by Theorem 4.6, there exists in $K\langle \underline{X} \rangle$ an order and a valuation w compatible such that $f < 0$, $g_1, \cdots, g_r > 0$, $a \in \mathcal{M}_{K,v}$ and $\gamma(K\langle \underline{X} \rangle), (K\langle \underline{X} \rangle)^\dagger \subseteq \mathcal{O}_{K\langle \underline{X} \rangle, w}$. In particular, w makes $K\langle \underline{X} \rangle$ into a valued D-field and we have that $w(a) = 1$ by Lemma 4.3. By Proposition 4.11, we take $\langle L, D, \leqslant, \widetilde{w} \rangle$ an extension of $\langle K\langle \underline{X} \rangle, D, \leqslant, w \rangle$ with $\langle L, D, \widetilde{w} \rangle$ D-henselian, $k_{L,\widetilde{w}}$ is a model of $CODF$, $\langle L, \leqslant \rangle$ an ordered extension of $\langle K\langle \underline{X} \rangle, \leqslant \rangle$ and $w(L^\times)$ a $\mathbb{Z}$-group such that $\widetilde{w}(a) = 1_{\widetilde{w}(L^\times)}$. By the henselian property of v and $\widetilde{w}$, we deduce that $\mathcal{O}_{K,v} = \gamma(K)$ and $\mathcal{O}_{L,\widetilde{w}} = \gamma(L)$; so $\mathcal{O}_{K,v} \subseteq \mathcal{O}_{L,\widetilde{w}}$. But v and $\widetilde{w}$ are such that $v(a) = 1$ and $\widetilde{w}(a) = 1$ then we have that $\mathcal{M}_{K,v} = a \cdot \mathcal{O}_{K,v} \subseteq a \cdot \mathcal{O}_{L,\widetilde{w}} = \mathcal{M}_{L,\widetilde{w}}$. Applying Proposition 4.10, we deduce that $K \prec_{\mathcal{L}_a} L$. Keeping in mind that as well K as in L the order for which $a > 0$ is defined by the formula $x > 0$ if and only if $\exists y\,(x = y^2 \vee x = ay^2)$, we have that $K \prec_{\mathcal{L}_a \cup \{\leqslant\}} L$. But $f < 0$, $g_1, \cdots, g_r > 0$ in L implies $f(\underline{X}) < 0$, $g_1(\underline{X}) > 0, \cdots, g_r(\underline{X}) > 0$ and hence the formula ϕ expressing $\exists \bar{x}\, f(\bar{x}), g_1(\bar{x}), \cdots, g_r(\bar{x})$ is defined and $f(\bar{x}) < 0$, $g_i(\bar{x}) > 0$, $i = 1, \cdots, r$ holds in L. By the elementary inclusion, ϕ holds in $\langle K, D, \leqslant \rangle$ showing that (*) is false. $\square$

Appendix

In this section we recall the process used to transfer the quantifier elimination theorem from a three-sorted language to a one-sorted language.

Let L_0 be the language of fields with a binary symbol $\mathcal{D}$ which stands for the l.d. relation related to a valuation v. Let $\mathcal{L}_{oR}$ be the symbols of language of differential fields in the residue language $\mathcal{L}_R$, let $\mathcal{L}_{oV}$ be the symbols of language of ordered groups in the value group language $\mathcal{L}_V$ and $\mathcal{L}_0$ is the three-sorted language of valued D-fields with respect to $\mathcal{L}_{oR}$ and $\mathcal{L}_{oV}$. Suppose that $\mathcal{L}_R \setminus \mathcal{L}_{oR}$ and $\mathcal{L}_V \setminus \mathcal{L}_{oV}$ contains only relation symbols, then $\mathcal{L}_2$ can be translated to an one-sorted language L_2 which we are going to define.

To every relation symbol $r \in \mathcal{L}_2 \setminus \mathcal{L}_0$ we associate a symbol r^* and L_2 is the union $L_0 \cup \{r^*; r \in \mathcal{L}_2 \setminus \mathcal{L}_0\}$. In order to be clear, we distinguish the L_2-structure $\mathcal{M}$ and the $\mathcal{L}_2$-structure $\langle M, k_M, v(M^\times) \rangle$. For $r \in \mathcal{L}_2 \setminus \mathcal{L}_0$, we define r^* in the following way:

- if $r \in \mathcal{L}_R$, $\mathcal{M} \models r^*(\vec{x})$ iff $\langle M, k_M, v(M^\times) \rangle \models \bigwedge_i v(x_i) = 0 \wedge r(\vec{x})$,
- if $r \in \mathcal{L}_V$, $\mathcal{M} \models r^*(\vec{x})$ iff $\langle M, k_M, v(M^\times) \rangle \models r(\overrightarrow{v(x_i)})$,
- if $r = F_{\phi,\vec{n}}$, $\mathcal{M} \models r^*(\vec{x}, \vec{y})$ iff $\langle M, k_M, v(M^\times) \rangle \models \bigwedge_i v(y_i) \geqslant 0 \wedge r(\vec{x}, \pi(\vec{y}))$,

Let T be the translation in L_2 of the $\mathcal{L}_2$-theory $\mathcal{T}$ of the $(\mathbf{k}, \mathbf{G})$-D-henselian fields, then Theorem 3.13 becomes

Theorem 1. *T admits quantifier elimination in L_2.*

Proof. • The application * can be extended to a lifting in L_2 of any formula which contains only field variables (quantifier variables are included):

1) lifting for atomic formula's

– we have defined the lifting of symbols in $(\mathcal{L}_V \setminus \mathcal{L}_{oV}) \cup (\mathcal{L}_R \setminus \mathcal{L}_{oR})$

– if r is a symbol of the language of differential fields then $r^* = r$,

– for the symbols of $\mathcal{L}_{oR}$: $(\pi(x) = \pi(y))^* = \mathcal{D}(1,x) \wedge \mathcal{D}(1,y) \wedge \mathcal{D}(1, x - y) \wedge \neg\mathcal{D}(x-y, 1)$, $0^* = 0$, $1^* = 1$, $+^* = +$, $.^* = .$, $(^{-1})^* = {}^{-1}$, $D^* = D$.

– for the symbols of $\mathcal{L}_{oV}$: $[v(x) = v(y)]^* = \mathcal{D}(x,y) \wedge \mathcal{D}(y,x)$, $0^* = 1$, $+^* = .$, $[v(x) \leqslant v(y)]^* = \mathcal{D}(x,y)$.

2) Extension by natural induction on the complexity of the formula's.

• Conversely we can translate a formula ϕ in L_2 to a formula ϕ_* in $\mathcal{L}_2$:

– for any symbol of the language of differential fields r, we define $r_* = r$,

– $[\mathcal{D}(x,y)]_* = v(x) \leqslant v(y)$,

– $[r^*(\vec{x})]_* = r(\pi(\vec{x}))$ if $r \in \mathcal{L}_R \setminus \mathcal{L}_{oR}$,

– $[r^*(\vec{x})]_* = r(\overrightarrow{v(x_i)})$ if $r \in \mathcal{L}_V \setminus \mathcal{L}_{oV}$.

We continue by induction on all formulas in L_2. The only formulas which are concerned with are those whose all variables are in the differential fields and we get that

$$\mathcal{M} \models \phi(\vec{x}) \iff \langle M, k_M, v(M^\times)\rangle \models \phi_*(\vec{x}).$$

Now we use the result of quantifier elimination of Theorem 3.13. Let ϕ be a formula in L_2. We apply Theorem 3.13 to the formula ϕ_*, i.e.

$$\mathcal{M} \models \phi(\vec{x}) \iff \langle M, k_M, v(M^\times)\rangle \models \phi_*(\vec{x}).$$

It is equivalent to the fact that $\langle M, k_M, v(M^\times)\rangle$ satisfies a boolean combination of formulas (or the negation)

$$P(\vec{x}) = 0, \phi_R(\pi(P_1(\vec{x}))), \phi_V(\overrightarrow{v(P_2(\vec{x}))}), F_{\phi,\vec{n}}(\overrightarrow{P_3(\vec{x})}, \overrightarrow{P_4(\vec{x})})$$

where $P, \vec{P_1}, \vec{P_2}, \vec{P_4}$ and $\vec{P_4}$ are differential polynomials with constant coefficients, ϕ_R is a formula in $\mathcal{L}_R$, ϕ_V is a formula in $\mathcal{L}_V$. This is equivalent to $\langle M, k_M, v(M^\times)\rangle$ satisfies a boolean combination of formulas

$$P(\vec{x}) = 0,\ \phi_R^*(\pi(P_1(\vec{x}))),\ \phi_V^*(\overrightarrow{P_2(\vec{x})}),\ F^*_{\phi,\vec{n}}(\overrightarrow{P_3(\vec{x})}, \overrightarrow{P_4(\vec{x})}).$$

□

Acknowlegments

I would like to thank Professor Thomas Scanlon for the reading of a first version of this work. I am also greatful to Professor Françoise Delon for improving my understanding of her thesis. This work was done under the supervision of my advisors Françoise Point and Christian Michaux.

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