# On 2nd Order Calculi of Individuals

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ABSTRACT: From early work of N. Goodman to recent approaches by H. Field and D. Lewis, there have been attempts to combine 2nd order languages with calculi of individuals. This paper is a contribution, containing basic definitions and distinctions and some metatheorems, to the development of a general metatheory of such theories.

Keywords: nominalism, mereology, calculus of individuals, 2nd order logic, Goodman, Lewis, Field.

## 1. *Calculi of individuals and nominalistic theories*

### 1.1. *Motivation and historical background*

Nelson Goodman is well known for being a nominalist; see, in particular Goodman (1951). One of the lasting contributions to this philosophical position has been his work on the development and promotion of nominalistic theories, in particular of *calculi of individuals*. "Calculus of individuals" is the term Goodman preferred for a theory which, platonistically speaking, deals with the part-whole relation and is therefore also sometimes called a "mereology". It should be noted, however, that Goodman's early study of calculi of individuals, as it is presented in Leonard and Goodman (1940), was probably not primarily motivated by nominalistic concerns.[1] So called "individuals" play an important role in that paper; but their connection to nominalism is almost nowhere addressed. Yet, the one exception, i.e., "The dispute between nominalist and realist as to what actual entities are individuals and what are classes [...]" Leonard and Goodman (1940, p. 55) is interesting in so far as it anticipates Goodman's conception of nominalism as that position which admits only *individuals*, which would become so important and characteristic for his later writings —see Goodman (1951, 1986) and Niebergall (2005).[2]

Already in its syntactical features, however, the "calculus of individuals" presented by Leonard and Goodman is different from most of the theories which later on appeared under this heading. It is granted that the latter's language—let's call it "$L^1[\circ]$"—and the language L[G] used by Leonard and Goodman may both be taken to contain the same non-logical vocabulary: the 2-place predicate "$\circ$" (read "overlaps").[3] But whereas

[1] It is also well known that it was not Goodman (or H. Leonard) who first formulated and investigated such theories, but S. Leśniewski: see Leśniewski (1927) (this is acknowledged in Leonard and Goodman (1940)). Moreover, already in the 1920s A. Tarski had presented a reformulation of Leśniewski's approach which was free from the pecularities of the latter's background "logic"; see Tarski (1929).

[2] In his dissertation *A Study of Qualities* from 1941, Goodman does care for nominalism; but there, he understands it in more "classical" terms—as the rejection of universals.

[3] To be precise, Leonard and Goodman had a sign for "is disjoint from" as their primitive predicate; and one could also choose "$\sqsubseteq$" (read "is a part of"), as was done in Tarski (1929), for example. But

$L^1[\circ]$ is a 1st order language, L[G] was a higher order language, i.e., a type theoretic language;[4] see especially Leonard and Goodman (1940, p. 130). Actually, in this paper I have neither found a passage where variables of higher than 2nd order are employed nor one where 2nd order variables of more than one places are used. Thus, it seems that the formal framework of Leonard and Goodman (1940) did, in effect, not go beyond a *monadic 2nd order language* with "$\circ$" as its only predicate. In what follows, I will call that language "$L^2[\circ]$".

Apart from Goodman's own early work, an additional motivation for the study of *2nd order versions of calculi of individuals* is provided by two more recent contributions to the nominalistic program: H. Field's *Science without numbers* (1980) and D. Lewis' *Parts of Classes* (1991). In both of these books, 2nd order languages in general and 2nd order mereologies in particular figure prominently. Moreover, the mereological theories favoured by these philosophers are again formulated in *monadic* 2nd order languages. Presumably, this is so since of all the 2nd order variables the monadic ones seem to be particularly amenable to a nominalistic interpretation: be it by equipping them with a plural reading (as done in Lewis (1991)), be it by interpreting them as ranging over regions (as suggested in Field (1980)).

In sum, we have three influential philosophical approaches where monadic 2nd order versions of calculi of individuals are promoted. Moreover, in each case these are regarded as nominalistic theories.

In principle, I share the nominalistic inclinations of Goodman, Lewis and Field. Actually, it seems to be often agreed upon that the avoidance of any commitment to universals or abstract objects is intuitively plausible and should be systematically attractive; but many philosophers are persuaded by indispensability arguments that such a nominalistic program cannot be carried out. In particular, there are doubts that a reduction of such mathematical theories as Peano Arithmetic (PA) or Zermelo-Fraenkel Set Theory (ZF) to nominalistic theories is possible.

Now in Lewis (1991), Lewis develops a theory formulated in a dyadic 2nd order language (with "$\circ$") to which ZF *is* reducible—in some sense; and in an appendix, he presents methods for defining ordered pairs in $L^2[\circ]$, thereby reducing ZF to a theory (which I call TL) stated in $L^2[\circ]$.[5] If TL were a nominalistic theory, Lewis' result should be viewed as playing a considerable role for the above mentioned nominalistic reduction program. And as I understand Lewis —see Lewis (1991, 1993), he has no problems in regarding TL as a nominalistic theory. But certainly not all will agree with him. In fact, there used to be times when, in particular under the influence of W. V. O. Quine, the claim

all of these predicates are interdefinable, given their intended readings: in particular, $x \sqsubseteq y :\longleftrightarrow \forall z\,(x \circ z \rightarrow y \circ z)$.

4 At any rate, that is how I understand Leonard and Goodman. Since they did not lay down the vocabulary and formation rules for L[G] explicitly, it cannot be ruled out that they conceived of it as a 1st order language containing set theoretic vocabulary.

5 All of this is done informally and sometimes only sketchily; in particular, the reduction of ZF to TL is not explicitly presented. But I assume that Lewis' reasoning is correct and can be carried out in all formal detail.

($\alpha$) No theory formulated in a 2nd order language is a nominalistic theory

seemed to be received wisdom. As far as I know, this was shaken only with the discovery of the plural reading of the monadic 2nd order variables —see Boolos (1984, 1985). Yet, above doubts as to an independent understandability of the plural idiom —see Resnik (1988), the fact that the 2nd order quantifiers can be read "plurally" does not imply that *each* theory stated in, say, $L^2[\circ]$, can be correctly classified as a nominalistic theory.[6] Thus, even if ($\alpha$) is rejected, what is still needed is a specific argument for the nominalistic acceptability of TL. Now it has to be granted that Lewis does present one: "Structuralist set theory [i.e., TL] is nominalistic set theory, in the special sense of Goodman" —see Lewis (1993, p. 17). It is, however, highly questionable that in order to be a nominalistic theory, it suffices to be a nominalistic theory in the sense of Goodman; for more on this see Niebergall (2005).

Let me also address the theme of calculi of individuals and their relation to nominalistic theories. I think that in distinction to "$T$ is a nominalistic theory", which especially for 1st order $T$ is difficult to explain, a reasonable explication for "$T$ is a (1st order) calculus of individuals" is possible. The one I have put forward in Niebergall (2007a) is repeated in section 1.2 of this paper. Although an *explicans* for "$T$ is a nominalistic theory" is missing, it moreover seems that at least for 1st order theories,

($\beta$) Each calculus of individuals is a nominalistic theory

is widly, if not universally accepted. I, for one, agree. If, however, ($\beta$) is also accepted for 2nd order theories, we have the inconvenient consequence that under the additional assumption of ($\alpha$) the investigation of 2nd order calculi of individuals is futile right from the start (because of their nonexistence). Moreover, I think that ($\beta$) is not as evident for 2nd order theories as for 1st order theories. One reason for this may be that intuitively it is just not so clear what exactly a 2nd order calculus of individuals could be.[7] Thus, giving up ($\beta$) for 2nd order theories may be worth a thought.

Actually, I think that also in $L^2[\circ]$ there are clear cases of theories which are[8] and for theories which fail to be calculi of individuals. And I grant that the investigation of 2nd order calculi of individuals would carry a particular weight if ($\beta$) were the case. But whatever the final word on ($\alpha$) and ($\beta$) may be, the monadic 2nd order theories developed and investigated by Goodman, Lewis and Field seem to be of independent interest.

This paper is inspired by the above mentioned texts. In distinction to these, however, it is laid out to be a *first beginning* in the development of a *general framework* for the investigation of theories $T$ formulated in $L^2[\circ]$ which are extensions of the 1st order calculi of individuals.[9] *Being a general framework* is supposed to convey that I am interested in formulating and proving metatheorems about arbitrary extensions of 1st order calculi of

[6] The beginning of section 1.2 contains an argument which is relevant also here.

[7] See also section 2.3.

[8] The 2nd order theory from Leonard and Goodman (1940) is an example.

[9] In Leonard and Goodman (1940), Field (1980) and Lewis (1991), no attempt to obtain *general metalogical results* about such theories can be found. On the one hand, Leonard and Goodman state axioms for one such theory: they comprise the system of *Principia mathematica* (whatever that *exactly* is; Leonard and Goodman (1940) does not tell us) plus sentences which amount to what is called $\mathrm{CI} \cup \{\text{FUS-Ax}\}$

individuals which are formulated in $\mathrm{L}^2[\circ]$. *Being a first beginning* means, however, that many natural questions concerning metalogical features of such theories are not addressed in it, let alone answered (cf. the end of section 4.4 for more concrete suggestions). At several points, I am therefore content in obtaining results for 2nd order theories which are particularly close to the 1st order calculi of individuals.

In sections 1 and 2, the background for the main theme of this paper—the investigation of 2nd order analogues of calculi of individuals—is laid. The remainder of section 1 is a survey of 1st order calculi of individuals; and sections 2.1 and 2.2 contain an introduction into the syntax and semantics of $\mathrm{L}^2[\circ]$, with a special emphasis on the distinction between standard 2nd order and generalized 2nd order models. In the rest of section 2 and in sections 3 and 4, the main metalogical results of this paper are stated and (in most of the cases) proved: they concern the consistency, satisfiability, categoricity, maximal-consistency and decidability of some of the 2nd order theories formulated in $\mathrm{L}^2[\circ]$ which extend the above mentioned 1st order calculi of individuals. In the concluding section 5, I present a discussion of Field's use of the expressions "the complete logic of the part/whole relation" and "the complete logic of Goodmanian sums" in Field (1980). Indeed, my aim to find out what Field may have meant by these expressions provided the trigger for writing this paper.

### *1.2. 1st order calculi of individuals*

Let $\mathrm{L}^1[\circ]$ be the 1st order language mentioned above, with the 2-place predicate "$\circ$" as its sole non-logical primitive expression.[10] As 1st order calculi of individuals I will regard

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below (see section 1.2). Whether the theory induced by these axioms is determined syntactically or semantically, and if the latter, by using $g2$-structures or $s2$-structures (cf. section 2.1), is not addressed in Leonard and Goodman (1940), however. On the other hand, Lewis deliberately avoids the use of formal languages altogether in his work (1991). Certainly, that book contains several ingenious constructions; but it is therefore not easy to evaluate whether they work properly. Finally, unlike Leonard and Goodman and Lewis, Field (1980) does address the distinction between theories as determined by $g2$-structures vs. $s2$-structures. Yet, although he does that at several places, his remarks are somewhat sketchy and remain, in my opinion, unclear (cf. section 5).

[10] The treatment of identity in this context needs a comment. If "=" is defined through

$$x = y :\longleftrightarrow \forall z\,(z \circ x \leftrightarrow z \circ y),$$

its usual principles—reflexivity and substitutivity (in $\mathrm{L}^1[\circ]$)—are consequences of the axiom-set $\mathrm{Ax(CI)}$ introduced below. But if $\mathrm{L}^1[\circ]$ is extended to some language L, substitutivity in L is not guaranteed by this definition. In this situation, further instances of substitutivity have to be assumed (they may, for example, be regarded as logical truths in L and thus be taken as belonging to its "background logic").

Alternatively, one may, as it is usual, view "=" as a logical sign whose use is governed by reflexivity and substitutivity in L, whatever (extensional) language L may be. This, however, has the disadvantage that "$x = y \longleftrightarrow \forall z\,(z \circ x \leftrightarrow z \circ y)$" is missing now: it has to be postulated additionally.

Unfortunately, I have been rather sloppy in distinguishing these two options in Niebergall (2007a). Here, I choose the first one. Let me also point out that in this case, too, "=" is assumed to be evaluated by the identity relation (over a model $\mathcal{M}$).

only theories which are formulated in $\mathrm{L}^1[\circ]$.[11] Being expressible in $\mathrm{L}^1[\circ]$ is certainly no sufficient reason for a theory $T$ to be called a "calculus of individuals", however. Thus, consider ZF (i.e., Zermelo-Fraenkel set theory), which—given my intuitions and probably Goodman's, too—is a theory which is not a calculus of individuals.[12] Let's rewrite ZF by replacing "$\in$" by "$\circ$". The resulting theory (let's call it "$\mathrm{ZF}^\circ$") *is* stated in $\mathrm{L}^1[\circ]$. But $\mathrm{ZF}^\circ$ is hardly a calculus of individuals, being a mere notational variant of ZF. In my view, in order for a theory $T$ to be rightfully called a "calculus of individuals", enough sentences involving "$\circ$" must belong to $T$ which are supposed to be true if "$a \circ b$" is read as "$a$ overlaps $b$". Moreover, we should be disposed to accept these sentences already because of our usual understanding of "$a$ overlaps $b$".[13] And it seems clear to me that such a $T$ can not be consistent with $\mathrm{ZF}^\circ$.

I think that at this point, however, we should try to "cultivate our intuitions" (to use R. Eberle's words; see Eberle (1970)) on what a calculus of individuals is supposed to be by having a look at paradigmatic examples. The following two are taken from the relevant literature.

The first example is the theory ACI from Hodges and Lewis (1968); it is axiomatized by the following list $\mathrm{Ax}(\mathrm{ACI})$ of sentences:[14]

$$\begin{array}{ll}
\mathrm{O} & \forall xy\,(x \circ y \longleftrightarrow \exists z\,(z \sqsubseteq x \wedge z \sqsubseteq y)),\\
\mathrm{SUM} & \forall xy \exists z \forall u\,(u \circ z \longleftrightarrow u \circ x \vee u \circ y),\\
\mathrm{NEG} & \forall x\,(\neg \forall v\; v \circ x \longrightarrow \exists y \forall v\,(v \sqsubseteq y \leftrightarrow \neg v \circ x)),\\
\mathrm{AT} & \forall x \exists y\,(y \sqsubseteq x \wedge At(y)).
\end{array}$$

In stating AT, I have used this definition of "$x$ is an atom":

$$At(x) :\longleftrightarrow \forall y\,(y \sqsubseteq x \rightarrow x \sqsubseteq y).$$

AT is an axiom of atomicity: in each of its models, each object has a part which is an *atom*. Goodman and his followers have often dealt with theories in $\mathrm{L}^1[\circ]$ containing AT; but, according to Goodman (1951), there is no need to regard only such theories as calculi of individuals. Actually, the theory suggested in Goodman (1951)—my second

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[11] Whether theories stated in 1st order languages with additional predicates or function signs should be called "calculi of individuals" is debatable. In Goodman (1951), Goodman is reluctant to do so; in fact, he seems to apply that term only to the theory named "$\mathrm{CI} + \mathrm{FUS}^1$" below. The problem is, of course, that it is so unclear which extensions of the vocabulary of $\mathrm{L}^1[\circ]$ should still be considered nominalistically admissible. I am not aware of a general solution of this difficulty.

[12] For Goodman, classes were the archetypical examples of non-individuals. It is also clear that at least in Goodman (1951), the *calculus of classes* is taken to be no calculus of individuals. But I have not found any explicit pronouncement in that text on whether ZF is regarded as a calculus of classes.

[13] Of course, these claims are rather vague. I doubt that they can be made really precise; but this may be nothing to worry about, anyway —see Niebergall (2005) for more on this topic.

[14] For (axiomatizable) 1st order theories $T$, $\mathrm{Ax}(T)$ is always supposed to be *a* set of axioms for $T$; i.e., $\overline{\mathrm{Ax}(T)} = T$ has to be the case. Here, $\overline{\Sigma} := \{\psi \mid \psi \text{ is a sentence in } \mathrm{L}^1[\circ] \wedge \Sigma \vdash \psi\}$, where "$\vdash$" is derivability in classical 1st order logic with identity. Moreover, as usual I use "$\Sigma + \Sigma'$" for "$\overline{\Sigma \cup \Sigma'}$" if $\Sigma$ and $\Sigma'$ are sets of 1st order formulas.

example of a calculus of individuals—does not contain AT: it can be axiomatized by O and a certain 1st order axiom-schema, the so-called *fusion-schema* $\mathrm{FUS}^1$.[15] Employing the common procedure of identifying a schema with the set of "its instances", $\mathrm{FUS}^1$ can be precisely formulated as follows:

Let $\psi$ be a formula in $\mathrm{L}^1[\circ]$; then set

$\mathrm{FUS}_\psi$ : (the universal closure of)[16] $\exists x\, \psi \longrightarrow \exists z \forall y (z \circ y \leftrightarrow \exists x (x \circ y \wedge \psi))$.

$\mathrm{FUS}^1 := \{\mathrm{FUS}_\psi \mid \psi \text{ is a } \mathrm{L}^1[\circ]\text{-formula}\}$.

The theory CI axiomatized by O, SUM and NEG (a set of axioms I call "Ax(CI)") is a common part both of ACI and Goodman's theory, which therefore amounts to $\mathrm{CI} + \mathrm{FUS}^1$. As a matter of fact, it can be shown that $\mathrm{CI} + \mathrm{FUS}^1$ is a subtheory of ACI —see Niebergall (2007a,b).

Thus, we have CI, ACI and $\mathrm{CI} + \mathrm{FUS}^1$ as calculi of individuals; yet, they are far from being all consistent or all interesting extensions of CI (in $\mathrm{L}^1[\circ]$). ACI and $\mathrm{CI} + \mathrm{FUS}^1$, for example, imply nothing about the number of the atoms; there may even be infinitely many of them.[17] Alternatively, a calculus of individuals may have models which are atom-free, i.e., models which satisfy

AF $\quad \forall x \exists y\, (y \sqsubset x)$.[18]

Finally, instead of AT or AF, their negations may be added to CI. These remarks motivate the consideration of further theories $T$ extending CI (in $\mathrm{L}^1[\circ]$).[19]

Here are their axiom-sets.

$$
\begin{aligned}
&\mathrm{Ax}(\mathrm{ACI}_{\geq n+1}) := \mathrm{Ax}(\mathrm{ACI}) \cup \{\exists_{\geq n+1} At\} \quad (n \in \mathbb{N}),\\
&\mathrm{Ax}(\mathrm{ACI}_{n+1}) := \mathrm{Ax}(\mathrm{ACI}) \cup \{\exists_{= n+1} At\} \quad (n \in \mathbb{N}),\\
&\mathrm{Ax}(\mathrm{ACI}_{\infty}) := \mathrm{Ax}(\mathrm{ACI}) \cup \{\exists_{\geq n+1} At \mid n \in \mathbb{N}\},\\
&\mathrm{Ax}(\mathrm{FCI}) := \mathrm{Ax}(\mathrm{ACI}) \cup \{\mathrm{AF}\},\\
&\mathrm{Ax}(\mathrm{MCI}) := \mathrm{Ax}(\mathrm{ACI}) \cup \{\neg\, \mathrm{AT}, \neg\, \mathrm{AF}\},\\
&\mathrm{Ax}(\mathrm{MCI}_{\geq n+1}) := \mathrm{Ax}(\mathrm{MCI}) \cup \{\exists_{\geq n+1} At\} \quad (n \in \mathbb{N}),\\
&\mathrm{Ax}(\mathrm{MCI}_{n+1}) := \mathrm{Ax}(\mathrm{MCI}) \cup \{\exists_{= n+1} At\} \quad (n \in \mathbb{N}),\\
&\mathrm{Ax}(\mathrm{MCI}_{\infty}) := \mathrm{Ax}(\mathrm{MCI}) \cup \{\exists_{\geq n+1} At \mid n \in \mathbb{N}\}.
\end{aligned}
$$

[15] The other sentences considered as axioms in Goodman (1951) can be proved from O and $\mathrm{FUS}^1$.

[16] I prefer axioms to be sentences; it usually simplifies the metalogical reasoning.

[17] Again, in Goodman (1951, p. 53), Goodman is clear in stating that nominalism, as he understands it, is not committed to finitism. And in Goodman and Quine (1947), finitism is also additional.

[18] Here I use the definition: $x \sqsubset y :\longleftrightarrow x \sqsubseteq y \wedge y \not\sqsubseteq x$.

[19] In stating their axioms, I employ two abbreviations:

$$
\begin{aligned}
\exists_{\geq n+1} At :\longleftrightarrow\ & \exists x_1 \ldots x_{n+1} (At(x_1) \wedge \cdots \wedge At(x_{n+1}) \wedge x_1 \neq x_2 \wedge \ldots \\
& x_1 \neq x_{n+1} \wedge \ldots\ x_n \neq x_{n+1}),\\
\exists_{= n+1} At :\longleftrightarrow\ & \exists_{\geq n+1} At \wedge \neg \exists_{\geq n+2} At.
\end{aligned}
$$

Moreover, arbitrary instances of $\mathrm{FUS}^1$ may be added to each of these sets as axioms.

I think it is quite natural to take the theories $\mathrm{ACI}_{n+1}$ and $\mathrm{MCI}_{n+1}$ (for $n \in \mathbb{N}$), $\mathrm{ACI}_\infty$, FCI and $\mathrm{MCI}_\infty$ induced by these sets of axioms into consideration and to regard them as *calculi of individuals*. But could there be other theories which deserve that classification? For one, think of those $T$ which are stated in a language L different from $\mathrm{L}^1[\circ]$. The construction and investigation of appropriate candidates which are formulated in $(\mathrm{L} =)\, \mathrm{L}^2[\circ]$ is the main topic of the next sections of this paper. But then, already in $\mathrm{L}^1[\circ]$ there are still uncountably many consistent extensions $T$ of CI which are different from the theories just mentioned. All of them may also be viewed as calculi of individuals.

Nonetheless, it is not at all arbitrary to put a particular emphasis on the above theories. For the following result can be shown:

**Theorem A**[20] *The maximally consistent extensions of* $\mathrm{CI} + \mathrm{FUS}^1$ *in* $\mathrm{L}^1[\circ]$ *are exactly the* $\mathrm{ACI}_{n+1}$ *and the* $\mathrm{MCI}_{n+1}$ *(*$n \in \mathbb{N}$*), plus* $\mathrm{ACI}_\infty$, FCI *and* $\mathrm{MCI}_\infty + \mathrm{FUS}^1$.

Let's call the consistent extensions of $\mathrm{CI} + \mathrm{FUS}^1$ in $\mathrm{L}^1[\circ]$ "(1st order) calculi of individuals". Then Theorem A comes close to a *classification result* (as it may be called) of the (1st order) calculi of individuals: although no full list of these theories is obtained, we have a good grasp of them, since each of them has a maximally consistent extension—which is one of the above theories.

Theorem A implies that no (1st order) calculus of individuals can relatively interpret[21] PA nor even its weak subtheory Q.[22] This means that the theories which most often are regarded as *the* nominalistic theories cannot interpret even a small amount of what is accepted mathematics. If the nominalist did not have any other theories at his disposal, this would almost be a provable refutation of his reduction program. But, gladly, there *are* theories which are commonly viewed as nominalistic which save the day. One sort of them is provided by token concatenation theories: here we have samples which interpret even ZF —see Niebergall (2005). And other examples may be found among the 2nd order calculi of individuals.

Now, whatever the answer to the question whether there are such $T$ in $\mathrm{L}^2[\circ]$ which, moreover, are nominalistically admissible will be, the proper framework in which to formulate those $T$ has to be developed anyway. Since one has to be somewhat careful with respect to the different types of semantics which can be used to interpret 2nd order languages, let me therefore first give a short, but in part quite explicit, introduction to 2nd order languages and their possible models.

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[20] For a proof, see Niebergall (2007b); it is sketched in Niebergall (2007a). See also Hodges and Lewis (1968) and Hendry (1982) for similar, but weaker results.

[21] For relative interpretations, see Tarski, et al. (1953) and Feferman (1960). For a defense of the claim that relative interpretability provides a good *explicans* for "reducibility", see Niebergall (2000).

[22] See Niebergall (2007a,b). For Q, i.e., Robinson Arithmetic, see Tarski, et al. (1953).

## 2. *Syntax and semantics of* $\mathrm{L}^2[\circ]$

### 2.1. *2nd order semantics: standard and generalized*

Apart from minor variations there is exactly one generally accepted way[23] to define "$\langle M, \mathcal{I}\rangle, \beta \models \psi$" (i.e., *satisfaction in a structure* $\mathcal{M}$) when $\psi$ is a formula of a (1-sorted) 1st order language. For 2nd order formulas, in contrast, at least two nonequivalent types of definitions of *satisfaction in a structure* are common.[24] Moreover, for one of them two different *presentations* are in use. Let me address the latter topic first.

To be specific, take the monadic 2nd order extension $\mathrm{L}^2[\circ]$ of $\mathrm{L}^1[\circ]$[25] (which, as already mentioned, results from the latter through the addition of *one-place* 2nd order variables ("$X$", $\ldots$)), and let $\psi$ be a formula from $\mathrm{L}^2[\circ]$. In order to define "variable assignment $\beta$ satisfies $\psi$ in structure $\langle M, \mathcal{I}\rangle$", "structure" and "variable assignment" have to be explained.

In one presentation, the structures in which formulas from $\mathrm{L}^2[\circ]$ are to be evaluated are just the same relational systems or sets plus interpretation functions that are employed to interpret 1st order formulas: i.e., they are of the form $\langle M, \mathcal{I}\rangle$, with $M \neq \emptyset$ and $\mathcal{I}(\circ) \subseteq M^2$. A variable assigment $\beta$, however, must also have 2nd order variables $X$ in its domain, with $\beta(X) \subseteq M$ in place of $\beta(x) \in M$ (where $x$ is a 1st order variable). Given that, satisfaction is defined as in the 1st order case, with the following clause added:

$$\langle M, \mathcal{I}\rangle, \beta \models \forall X \psi \iff \forall C\, (C \subseteq M \Longrightarrow \langle M, \mathcal{I}\rangle, \beta(X:C) \models \psi).^{26}$$

In the second, in some sense equivalent, presentation the structures taken to interpret $\mathrm{L}^2[\circ]$ are of the form $\langle M, \wp(M), \mathcal{I}\rangle$. Yet, as before, $M \neq \emptyset$ and $\mathcal{I}(\circ) \subseteq M^2$, and $\beta$ is again defined also on the 2nd order variables $X$ and has to satisfy $\beta(X) \subseteq M$. Since this may be rephrased as "$\beta(X) \in \wp(M)$", $\wp(M)$ may be taken as the domain of the 2nd order variables now. Here, the evaluation of 2nd order quantification is formulated as

$$\langle M, \wp(M), \mathcal{I}\rangle, \beta \models \forall X \psi \iff \forall C\, (C \in \wp(M) \Longrightarrow \langle M, \wp(M), \mathcal{I}\rangle, \beta(X:C) \models \psi).$$

In comparison, the second presentation seems to be more flexible: it more easily suggests generalizations. Thus, instead of taking the full powerset $\wp(M)$ of the domain $M$ of the 1st order variables as the domain of the 2nd order variables, one might consider

[23] For model theory, see for example Chang and Keisler (1973). Definitions of "logical truth" which do not deliver the set of logical truths of classical 1st order logic (for some language L) are not taken into account here.

[24] For more on this topic, see Leivant (1994) and Shapiro (1991).

[25] For explicitness, I will mostly deal only with the language $\mathrm{L}^2[\circ]$. Of course, all the definitions presented in this and the following subsection can be extended to arbitrary 2nd order languages; and usually, the metatheorems can be transferred to these, too.

[26] $\beta(X:C)$ is a *variant* of $\beta$: it is that function which maps the variable $X$ to $C$ and agrees with $\beta$ on all other arguments.

an arbitrary (but, of course, non-empty) subset $\Omega$ of $\wp(M)$ in that role. This leads to structures of the form $\langle M, \Omega, \mathcal{I}\rangle$ as possible models of 2nd order languages, where $\emptyset \neq \Omega \subseteq \wp(M)$. Such structures are called "generalized 2nd order structures" (in short: $g2$-*structures*), whereas those with $\Omega = \wp(M)$ are the so-called "standard 2nd order structures" (in short: $s2$-*structures*).[27] Evaluation in generalized 2nd order structures is explained as 1st order evaluation extended by the following condition:

$$\langle M, \Omega, \mathcal{I}\rangle, \beta \models \forall X\psi \iff \forall C\,(C \in \Omega \Longrightarrow \langle M, \Omega, \mathcal{I}\rangle, \beta(X : C) \models \psi).^{28}$$

With the distinction between $g2$- and $s2$-structures at hand, the metalogical vocabulary, such as "satisfiability", "logical truth" and "logical consequence", for example, can be explained in two versions. Let me illustrate this by explicitly stating the definitions of the important relation of *logical consequence*:

$$\Sigma \models^{s2} \psi :\iff \forall \mathcal{M}\,(\mathcal{M} \text{ is a } s2\text{-structure} \Longrightarrow (\mathcal{M} \models \Sigma \Longrightarrow \mathcal{M} \models \psi)),$$
$$\Sigma \models^{g2} \psi :\iff \forall \mathcal{M}\,(\mathcal{M} \text{ is a } g2\text{-structure} \Longrightarrow (\mathcal{M} \models \Sigma \Longrightarrow \mathcal{M} \models \psi))$$

for $L^2[\circ]$-sentences $\psi$ and sets $\Sigma$ of $L^2[\circ]$-sentences.

Since each $s2$-structure is a $g2$-structure, this implies

$$\Sigma \models^{g2} \psi \Longrightarrow \Sigma \models^{s2} \psi.$$

But for 2nd order languages in general, the converse is not at all true. In fact, for some of them, standard 2nd order semantics and generalized 2nd order semantics are *very* different.

On the one hand, the notorious incompleteness and categoricity results, resp., which are so often attributed to 2nd order logic and to certain 2nd order theories, are only true under the presupposition that the 2nd order languages for which they are stated are interpreted *via* $s2$-structures. If all $g2$-structures are admitted as interpretations instead, these "results" are not only false: in this case, 2nd order logic turns out to be no more than a notational variant of (2-sorted) 1st order logic. In particular, we have a completeness theorem for 2nd order logical consequence with respect to $g2$-structures. That is, finitely many axioms (or axiom-schemata) and rules can be stated (see Shapiro (1991) for a workable list) such that for each set $\Sigma$ of $L^2[\circ]$-sentences and each $L^2[\circ]$-sentence $\psi$, it is the case that

$$\Sigma \models^{g2} \psi \iff \Sigma \vdash \psi.$$

---

[27] That is, in the case of $L^2[\circ]$,

$x$ is a $s2$-structure $:\iff \exists M\mathcal{I}\,(x = \langle M, \wp(M), \mathcal{I}\rangle \wedge M \neq \emptyset \wedge \mathcal{I}$ is an interpretation of $L^2[\circ]$),

$x$ is a $g2$-structure $:\iff \exists M\Omega\mathcal{I}\,(x = \langle M, \Omega, \mathcal{I}\rangle \wedge M \neq \emptyset \wedge \emptyset \neq \Omega \subseteq \wp(M) \wedge \mathcal{I}$ is an interpretation of $L^2[\circ]$).

[28] For the sake of completeness, let me mention that the new atomic 2nd order formulas are evaluated as follows:

$$\langle M, \Omega, \mathcal{I}\rangle, \beta \models Xy \iff \beta(y) \in \beta(X).$$

It should be noted that in this case, $\beta$ is supposed to be a function which maps 2nd order variables only to those subsets of $M$ which are elements of $\Omega$.

It should be noted that "$\Sigma \vdash \psi$" is defined in the ususal style: as *there is a finite sequence of* $\mathrm{L}^2[\circ]$*-formulas ending with* $\psi$ *such that all its entries are logical axioms, elements of* $\Sigma$ *or are obtained from earlier lines by the rules of inference*. Therefore, we also have compactness and the various Löwenheim-Skolem theorems if all $g2$-structures are admissible; and the set of the $g2$-consequences of a recursively enumerable set is recursively enumerable.

On the other hand, it is not only the interplay between the languages and the models that depends on the different types of 2nd order semantics. The theories determined by these different classes of structures have to be clearly distinguished from each other, too. To be more explicit, consider the set $Cn^{g2}(\Sigma)$ of $g2$-consequences of $\Sigma$ and the set $Cn^{s2}(\Sigma)$ of $s2$-consequences of $\Sigma$.[29] Certainly, there are cases in which $Cn^{g2}(\Sigma) = Cn^{s2}(\Sigma)$ (see section 4.2 for examples). But let me stress that often $Cn^{g2}(\Sigma)$ is vastly different from $Cn^{s2}(\Sigma)$.

*2nd order number theory* provides a beautiful example of that phenomenon. Thus, let $\mathrm{L}^2[\mathrm{PA}]$ be the monadic 2nd order language with the non-logical vocabulary "$S$", "$+$", "$\cdot$" and "0" and let $\mathrm{Ax}(\mathrm{ACA}_0)$ be the usual set of axioms for the theory $\mathrm{ACA}_0$, stated in this language —see Simpson (1999). Now, consider $Cn^{g2}(\mathrm{Ax}(\mathrm{ACA}_0))$ and $Cn^{s2}(\mathrm{Ax}(\mathrm{ACA}_0))$.

On the one side, $Cn^{g2}(\mathrm{Ax}(\mathrm{ACA}_0))$ is a recursively enumerable theory which is highly incomplete. More explicitly, $Cn^{g2}(\mathrm{Ax}(\mathrm{ACA}_0))$ is a conservative extension of PA and therefore much weaker than, e.g., ZF. In particular, "$Con_{aca_0}$" (i.e., the (natural) consistency assertion for $Cn^{g2}(\mathrm{Ax}(\mathrm{ACA}_0))$) is not contained in $Cn^{g2}(\mathrm{Ax}(\mathrm{ACA}_0))$.

$Cn^{s2}(\mathrm{Ax}(\mathrm{ACA}_0))$, on the other side, is not only a proper extension of $Cn^{g2}$ $(\mathrm{Ax}(\mathrm{ACA}_0))$: actually, it is maximally consistent, even in $\mathrm{L}^2[\mathrm{PA}]$. It contains, in particular, both "$Con_{aca_0}$" and "$Con_{zf}$" (if ZF is consistent). As a matter of fact, $Cn^{s2}(\mathrm{Ax}(\mathrm{ACA}_0))$ contains each true arithmetic sentence from the 1st order language $\mathrm{L}^2[\mathrm{PA}]$; therefore, it is also far from being recursively enumerable.[30]

Section 4.3 contains examples of "$Cn^{g2}(\Sigma) \neq Cn^{s2}(\Sigma)$" in which $\Sigma$ is formulated in $\mathrm{L}^2[\circ]$.

So much for a general overview of the distinction between the $g2$- and the $s2$-semantics for 2nd order languages. Since I have no preference for either of them, I will consider both types of models. In fact, the comparison between them is at the centre of this investigation.

### *2.2. 2nd order theories, maximal-consistency and categoricity*

Let's continue the topic of the dependency of metalogical notions on the type of 2nd order semantics presupposed and deal with the problem of what proper *definientia* of "categoricity", but also of "maximal-consistency" and even of "theory" could be.

**Definition 1.** *Let* $\mathcal{M}$ *be a* $g2$*-structure for* $\mathrm{L}^2[\circ]$. *Then*

$$\mathrm{Th}^2(\mathcal{M}) := \{\varphi \mid \varphi \textit{ is a } \mathrm{L}^2[\circ]\textit{-sentence} \wedge \mathcal{M} \models \varphi\}.$$

[29] That is, for $\Sigma$ from $\mathrm{L}^2[\circ]$, $Cn^{g2}(\Sigma) = \{\psi \mid \psi \text{ is a } \mathrm{L}^2[\circ]\text{-sentence} \wedge \Sigma \models^{g2} \psi\}$ and $Cn^{s2}(\Sigma) = \{\psi \mid \psi \text{ is a } \mathrm{L}^2[\circ]\text{-sentence} \wedge \Sigma \models^{s2} \psi\}$.

[30] See, e.g., Shapiro (1991) and Simpson (1999).

**Definition 2.** *Let* $\Sigma$ *be a set of* $\mathrm{L}^2[\circ]$*-formulas. Then*

$$\Sigma \textit{ is maximal-consistent: } \iff \forall\psi\ (\psi \textit{ is a } \mathrm{L}^2[\circ]\textit{-sentence} \Longrightarrow \psi \in \Sigma \ \vee\ \ulcorner\neg\psi\urcorner \in \Sigma) \textit{ and } \Sigma \textit{ is consistent.}$$

The following lemma may be regarded as sort of a justification of these definitions.

**Lemma 3.** *Let* $\Sigma$ *be a set of* $\mathrm{L}^2[\circ]$*-sentences.*

*(i) If* $\Sigma$ *is maximal-consistent, then* $Cn^{g2}(\Sigma) \subseteq \Sigma$.
*(ii) If* $\mathcal{M}$ *be a* $g2$*-structure for* $\mathrm{L}^2[\circ]$*, then* $\mathrm{Th}^2(\mathcal{M})$ *is maximal-consistent.*
*(iii) If* $\Sigma$ *is maximal-consistent,* $\Sigma'$ *is a consistent set of* $\mathrm{L}^2[\circ]$*-sentences and if* $\Sigma \subseteq \Sigma'$*, then* $\Sigma = \Sigma'$.
*(iv) If* $\Sigma$ *is maximal-consistent, then there is a* $g2$*-structure* $\mathcal{M}$ *for* $\mathrm{L}^2[\circ]$ *such that* $\Sigma = \mathrm{Th}^2(\mathcal{M})$.

*Proof.* (i) – (iii) are as usual.

(iv) If $\Sigma$ is maximal-consistent, then there is a $g2$-structure $\mathcal{M}$ for $\mathrm{L}^2[\circ]$ such that $\mathcal{M} \models \Sigma$. Now, $\mathrm{Th}^2(\mathcal{M})$ is consistent, and we also have $\Sigma \subseteq \mathrm{Th}^2(\mathcal{M})$. By (iii), the claim follows.

There are several possible *definientia* for "$\Sigma$ is a 2nd order theory" (in $\mathrm{L}^2[\circ]$). Here are two extreme suggestions; others could be extracted from what is done in section 2.3.

(i) $\Sigma$ is a 2nd order theory: $\iff$ $Cn^{g2}(\Sigma) \subseteq \Sigma$,
(ii) $\Sigma$ is a 2nd order theory: $\iff$ $Cn^{s2}(\Sigma) \subseteq \Sigma$.

In this text, I accept (i). I prefer it intuitively over (ii). Furthermore, given (i), a maximal-consistent set of sentences (in $\mathrm{L}^2[\circ]$) is a 2nd order theory (by Lemma 3(i)). Finally, there are consistent sets of sentences $\Sigma$ such that $Cn^{s2}(\Sigma)$ is inconsistent (see the end of section 4.3 for an example); I take it to be implausible that such a $\Sigma$ could never be a theory.

**Definition 4.** *Let* $\Sigma$ *be a set of* $\mathrm{L}^2[\circ]$*-sentences. Then*

$$\Sigma \textit{ is } g2\textit{-categorical: } \iff \forall\mathcal{M}\mathcal{M}'(\mathcal{M},\mathcal{M}' \textit{ is a } g2\textit{-structure} \ \wedge\ \mathcal{M},\mathcal{M}' \models \Sigma \Longrightarrow \mathcal{M} \cong \mathcal{M}'),$$

$$\Sigma \textit{ is } s2\textit{-categorical: } \iff \forall\mathcal{M}\mathcal{M}'(\mathcal{M},\mathcal{M}' \textit{ is a } s2\textit{-structure} \ \wedge\ \mathcal{M},\mathcal{M}' \models \Sigma \Longrightarrow \mathcal{M} \cong \mathcal{M}').$$

Let $\kappa$ be a cardinal. Then

$$\Sigma \text{ is } \kappa\text{-}g2\text{-categorical: } \iff \forall\mathcal{M}\mathcal{M}'(|M| = |M'| = \kappa \wedge \mathcal{M},\mathcal{M}' \text{ is a } g2\text{-structure} \ \wedge\ \mathcal{M},\mathcal{M}' \models \Sigma \Longrightarrow \mathcal{M} \cong \mathcal{M}').$$

$$\Sigma \text{ is } \kappa\text{-}s2\text{-categorical: } \iff \forall\mathcal{M}\mathcal{M}'(|M| = |M'| = \kappa \wedge \mathcal{M},\mathcal{M}' \text{ is a } s2\text{-structure} \ \wedge\ \mathcal{M},\mathcal{M}' \models \Sigma \Longrightarrow \mathcal{M} \cong \mathcal{M}').$$

**Lemma 5.** *Let* $\Sigma$ *be a set of* $\mathrm{L}^2[\circ]$*-sentences.*

*(i) If* $\Sigma$ *is* $g2$*-categorical, then* $\Sigma$ *is* $s2$*-categorical.*
*(ii) If* $\Sigma$ *is* $\kappa$-$g2$*-categorical, then* $\Sigma$ *is* $\kappa$-$s2$*-categorical.*
*(iii) If* $\Sigma$ *is* $s2$*-categorical and* $Cn^{s2}(\Sigma)$ *is consistent, then* $Cn^{s2}(\Sigma)$ *is maximal-consistent.*
*(iv) If* $\Sigma$ *is* $g2$*-categorical and* $Cn^{g2}(\Sigma)$ *is consistent, then* $Cn^{g2}(\Sigma)$ *is maximal-consistent.*
*(v) Let* $\kappa$ *be an infinite cardinal. If* $\Sigma$ *is* $\kappa$-$g2$*-categorical and has only infinite* $g2$*-models, and if* $Cn^{g2}(\Sigma)$ *is consistent, then* $Cn^{g2}(\Sigma)$ *is maximal-consistent.*

*Proof.* (iii) Assume $\psi$ is a $\mathrm{L}^2[\circ]$-sentence such that $\psi \notin Cn^{s2}(\Sigma)$. Then there exists a $s2$-structure $\mathcal{A}$ such that (a) $\mathcal{A} \models \Sigma$ and (b) $\mathcal{A} \models \neg\psi$. Now let $\mathcal{B}$ be a $s2$-structure such that $\mathcal{B} \models \Sigma$. Then since $\Sigma$ is $s2$-categorical, $\mathcal{A} \cong \mathcal{B}$ by (a). But in this case, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same 2nd order formulas, whence by (b), $\mathcal{B} \models \neg\psi$.

This shows: $\forall\mathcal{B}(\mathcal{B}$ is a $s2$-structure$\wedge\mathcal{B} \models \Sigma \Longrightarrow \mathcal{B} \models \neg\psi)$, i.e., $\ulcorner\neg\psi\urcorner \in Cn^{s2}(\Sigma)$.

(iv) Like (iii).

(v) Assume that $\Sigma$ is $\kappa$-$g2$-categorical and $Cn^{g2}(\Sigma)$ is consistent, but not maximal-consistent: i.e., there are $\mathrm{L}^2[\circ]$-sentences $\psi, \ulcorner\neg\psi\urcorner$ such that $\psi \notin Cn^{g2}(\Sigma)$ and $\ulcorner\neg\psi\urcorner \notin Cn^{g2}(\Sigma)$.

Then $\Sigma \cup \{\neg\psi\}$ and $\Sigma \cup \{\psi\}$ have $g2$-models $\mathcal{A}$ and $\mathcal{A}'$ which, by assumption, must be infinite. Now, by an appropriate version of the Löwenheim-Skolem theorem, take $g2$-structures $\mathcal{M}$ and $\mathcal{M}'$, both of cardinality $\kappa$, which satisfy the same sentences as $\mathcal{A}$ and $\mathcal{A}'$, resp. Thus, we have $\mathcal{M}, \mathcal{M}'$ with

$$|M| = |M'| = \kappa \text{ and } \mathcal{M}, \mathcal{M}' \models \Sigma.$$

By the $\kappa$-$g2$-categoricity of $\Sigma$, $\mathcal{M} \cong \mathcal{M}'$—whence $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same $\mathrm{L}^2[\circ]$-sentences. Yet $\mathcal{M} \models \neg\psi$ and $\mathcal{M} \models \psi$. Contradiction.

It should be interesting to find out whether the claims from Lemma 5 can be simplified or strengthened. This question will be taken up at the end of section 4.3.

### *2.3. Calculi of individuals in* $\mathrm{L}^2[\circ]$

The investigation of subsections 2.1 and 2.2 is carried out for the *language* $\mathrm{L}^2[\circ]$ (its syntax and semantics) and for arbitrary theories in that language. In particular, it is not only concerned with those theories formulated in $\mathrm{L}^2[\circ]$ that could be regarded as reasonable choices for 2nd order analogues of the 1st order calculi of individuals. Let's come to this topic now.

As a start, let me note that it is in some sense trivial to obtain such 2nd order analogues. Thus, take Ax(ACI) as an example. Now, simply treat it as being stated in $\mathrm{L}^2[\circ]$: the set of $\mathrm{L}^2[\circ]$-sentences following from Ax(ACI) using some version of 2nd order consequence is a 2nd order theory which is *very close* to (the 1st order) ACI, but is different from it for the trivial reason that it contains $\mathrm{L}^2[\circ]$-sentences. Of course, this procedure leads to rather uninteresting 2nd order variants of 1st order theories. A much more important and illuminating connection between 1st order languages and their 2nd order extensions (and theories stated in them) is there where a 1st order *schema* can be replaced by a 2nd order *sentence*. Thus, in our case it is the substitution of a 2nd order *fusion-axiom* for the 1st order fusion-schema $\mathrm{FUS}^1$ which is of prime relevance.[31] The former is stated as a single sentence of $\mathrm{L}^2[\circ]$:

FUS-Ax $\forall X\,(\exists x\, Xx \longrightarrow \exists z \forall y\,(z \circ y \leftrightarrow \exists x(x \circ y \wedge Xx)))$.

Of course, stronger 2nd order versions of fusion-principles are easily formulated, too. In particular, there is the "full" *2nd order fusion-schema* $\mathrm{FUS}^2$:

Let $\psi$ be a formula in $\mathrm{L}^2[\circ]$; then set

$\mathrm{FUS}_\psi$ : (the universal closure of)[32] $\exists x\, \psi \longrightarrow \exists z \forall y (z \circ y \leftrightarrow \exists x(x \circ y \wedge \psi))$.
$\mathrm{FUS}^1 := \{\mathrm{FUS}_\psi \mid \psi$ is a $\mathrm{L}^1[\circ]$-formula$\}$.

FUS-Ax is an instance of $\mathrm{FUS}^2$. But what about the feeling that $\mathrm{FUS}^1$ should be a consequence of FUS-Ax? It may be argued that given a $\mathrm{L}^1[\circ]$-formula $\psi$, one simply specializes the "$X$" in FUS-Ax to the set of $\psi$'s. Yet, from a strictly axiomatic point of view it is not guaranteed that this set exists. A *comprehension principle* has to be postulated in order to obtain such sets. That is, one has to adopt (the universal closures of) formulas of the form

$$\exists X \forall z\,(Xz \leftrightarrow \psi(z)),$$

where $\psi$ is from $\mathrm{L}^2[\circ]$ and $X$ does not occur in $\psi$, as further axioms for our 2nd order theories.

Sometimes such formulas are regarded as logical truths.[33] Indeed, when only s2-structures are admitted to interpret $\mathrm{L}^2[\circ]$, the additional postulation of comprehension becomes redundant, because it is true in all such structures (cf. Lemma 8). But it need not hold in g2-structures. Therefore, the assumption of comprehension principles should not be left implicit.

Let me add here that there is a further principle which is often taken to be a logical truth: it is a version of a *Leibniz principle*. More precisely:

LEIB $\forall yz\,(\forall X(Xy \leftrightarrow Xz) \longrightarrow y = z)$.[34]

LEIB will not be included in the axiom-sets taken into account here—not because I doubt it, but because it will be a consequence of each of them. I will deal with two versions of 2nd order comprehension-schemata in this paper, however. One is the "full" 2nd order comprehension-schema $\mathrm{Comp}^2$), the other is its restriction to 2nd order formulas containing no bound 2nd order variables ($\mathrm{Comp}^1$). A precise rendering of these

[31] Historically, this order is not accurate. I have already sketched in the introduction that the use of 2nd order languages and of FUS-Ax came first: in the case of Leonard and Goodman (1940). As far as I know, Goodman (1951) was the first publication where a 1st order language was put forward as the right framework in which to formulate calculi of individuals and where, in particular, $\mathrm{FUS}^1$ was suggested as preferable to FUS-Ax. Of course, other researches quickly followed Goodman's lead; see, e.g., Martin (1958), a few papers in *Nous* from the late 1960s and Eberle (1970).

[32] I prefer axioms to be sentences; it usually simplifies the metalogical reasoning.

[33] That is, logical truths of 2nd order logic, if one wants to use that term at all.

[34] Recall that I suppose "$=$" to be axiomatized by reflexivity and substitutivity for each formula of $\mathrm{L}^2[\circ]$.

schemata may be given along the lines of the presentation of the fusion schemata. Thus:

Let $\psi$ be a formula in $\mathrm{L}^2[\circ]$ which does not contain the variable $X$; then set

$$\mathrm{Comp}_\psi := \text{(the universal closure of)} \exists X \forall z\, (Xz \leftrightarrow \psi(z)).$$
$$\mathrm{Comp}^2 := \{\mathrm{Comp}_\psi \mid \psi \text{ is an } X\text{-free } \mathrm{L}^2[\circ]\text{-formula}\}, \text{ and}$$
$$\mathrm{Comp}^1 := \{\mathrm{Comp}_\psi \mid \psi \text{ is an } X\text{-free } \mathrm{L}^2[\circ]\text{-formula containing no bound 2nd order variables}\}.$$

Since the step from $\mathrm{FUS}^1$ to FUS-Ax provides the crucial connection between the calculi of individuals in $\mathrm{L}^1[\circ]$ and their variants in $\mathrm{L}^2[\circ]$, it would be pointless to consider $\mathrm{L}^1[\circ]$-theories which do not prove $\mathrm{FUS}^1$ or 2nd order theories which do not contain FUS-Ax (which should be regarded as the weakest 2nd order extension of $\mathrm{FUS}^1$).[35] In view of the options taken into account here as 2nd order additions to CI[36]—i.e., FUS-Ax or $\mathrm{FUS}^2$, and $\mathrm{Comp}^1$ or $\mathrm{Comp}^2$—we obtain four different axiom sets in $\mathrm{L}^2[\circ]$ which may qualify as the 2nd order analogues of Ax(CI):

| | |
|---|---|
| $\mathrm{Ax}^2(\mathrm{CI})_1$ | Ax(CI), FUS-Ax, $\mathrm{Comp}^1$; |
| $\mathrm{Ax}^2(\mathrm{CI})_2$ | Ax(CI), FUS-Ax, $\mathrm{Comp}^2$; |
| $\mathrm{Ax}^2(\mathrm{CI})_3$ | Ax(CI), $\mathrm{FUS}^2$, $\mathrm{Comp}^1$; |
| $\mathrm{Ax}^2(\mathrm{CI})_4$ | Ax(CI), $\mathrm{FUS}^2$, $\mathrm{Comp}^2$. |

In this paper, I have been somewhat pedantic (in particular notationally) both in distinguishing between a set $\Sigma$ of sentences and "the" theory induced by it, and in distinguishing between a theory $T$ and a set of axioms $\mathrm{Ax}(T)$ for it. It is granted that if $\Sigma$ is a set of 1st order sentences, it is often overdone to pay a particular attention to the first distinction. For the theory induced by $\Sigma$ will be chosen as $\overline{\Sigma}$ and will thus be uniquely determined by $\Sigma$. But if $\Sigma$ is stated in a 2nd order language, we have at least two reasonable ways to extend it to a theory: by closing under $g2$- or under $s2$-consequence. Now we have already seen that $\Sigma$ itself, $Cn^{g2}(\Sigma)$ and $Cn^{s2}(\Sigma)$ can all be different from each other. Thus, talk of *the* induced theory is, in general, inappropriate; and in addition to that, a clear specification of the set of sentences considered is not superfluous.

If $\Sigma$ is one of the above mentioned four sets of axioms, we should not assume from the outset that $Cn^{g2}(\Sigma) = Cn^{s2}(\Sigma)$. That's the topic of the first distinction. The second one rest on a different, more trivial observation (which applies equally to 1st and 2nd order theories): one and the same theory $T$ may be axiomatized by different sets of axioms. In our case, if $\Sigma$ and $\Sigma'$ are two of the above axiom-sets, $Cn^{g2}(\Sigma) = Cn^{g2}(\Sigma')$ may nonetheless be the case.[37] Since ultimately I take theories to be the main objects of

[35] This gives the main motive for dealing only with theories which contain $\mathrm{FUS}^1$ in this paper.

[36] The investigation of just these axioms is inspired by the research done on 2nd order theories of arithmetic. Cf. Simpson (1999).

[37] Actually, it can also happen that $\Sigma \neq \Sigma'$ and $Cn^{g2}(\Sigma) = Cn^{s2}(\Sigma')$.

my investigation, let me first address the second distinction and see whether (relative to a fixed type of semantics) different axiom-sets do deliver the same theories.

In fact, one of the axiom-sets $\mathrm{Ax}^2(\mathrm{CI})_2$ to $\mathrm{Ax}^2(\mathrm{CI})_4$ is redundant:

**Lemma 6.** *Let $\Sigma$ be a set of $\mathrm{L}^2[\circ]$-sentences, $\psi$ be a $\mathrm{L}^2[\circ]$-sentence. Then:*

$$\Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_2 \models^{g2} \psi \iff \Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_4 \models^{g2} \psi.$$

*Proof.* "$\Longrightarrow$" Each instance of $\mathrm{FUS}^2$ can be obtained by FUS-Ax applied to the set of $\psi$'s, where that set is got from $\mathrm{Comp}^2$.

In view of this lemma, let's set (if $\Sigma$ is a set of $\mathrm{L}^2[\circ]$-sentences):

$$\begin{aligned}
\Sigma^{g2+} &:= \{\psi \mid \psi \text{ is a } \mathrm{L}^2[\circ]\text{-sentence} \wedge \Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_2 \models^{g2} \psi\} \\
&\qquad\qquad (= Cn^{g2}(\Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_2)), \\
\Sigma_0^{g2} &:= \{\psi \mid \psi \text{ is a } \mathrm{L}^2[\circ]\text{-sentence} \wedge \Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_1 \models^{g2} \psi\} \\
&\qquad\qquad (= Cn^{g2}(\Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_1)), \\
\Sigma^{g2} &:= \{\psi \mid \psi \text{ is a } \mathrm{L}^2[\circ]\text{-sentence} \wedge \Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_3 \models^{g2} \psi\} \\
&\qquad\qquad (= Cn^{g2}(\Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_3)).
\end{aligned}$$

As regards the relation between these theories, we obviously have some inclusions—but, in general, not more than these:

**Lemma 7.** *(i) If $\Sigma$ is a set of $\mathrm{L}^2[\circ]$-sentences, $\Sigma_0^{g2} \subseteq \Sigma^{g2} \subseteq \Sigma^{g2+}$.*
*(ii) If $\Sigma$ is a set of $\mathrm{L}^1[\circ]$-sentences, then $\Sigma + \mathrm{CI} + \mathrm{FUS}^1 \subseteq \Sigma_0^{g2}$.*

*Proof.* (ii) By definition, $\mathrm{Ax(CI)} \subseteq \Sigma_0^{g2}$. Moreover, each instance of $\mathrm{FUS}^1$ can be obtained from $\mathrm{Comp}^1$ and FUS-Ax along the lines mentioned in the proof of Lemma 6.

In the case of $s2$-structures, we can simply replace "$g2$" by "$s2$" throughout the above definitions and thereby obtain $\Sigma^{s2+}$, $\Sigma_0^{s2}$ and $\Sigma^{s2}$. Luckily, the relation between these theories is much simpler than that between the $g2$-versions. For when it comes to $s2$-structures, there is this well-known lemma:

**Lemma 8.** *For each $M \neq \emptyset$, $\langle M, \wp(M), \circ^M\rangle \models \mathrm{Comp}^2$.*

Therefore, if $\mathcal{M}$ is a $s2$-model of $\mathrm{Ax}^2(\mathrm{CI})_1$, it is a $s2$-model of $\mathrm{Ax}^2(\mathrm{CI})_4$, too; and $\Sigma_0^{s2} = \Sigma^{s2} = \Sigma^{s2+}$. Actually, since already $\{\mathrm{O}, \text{FUS-Ax}\} \cup \mathrm{Comp}^1 \models^{g2}$ SUM, NEG, we have

$$\Sigma^{s2} = \{\psi \mid \psi \text{ is a sentence from } \mathrm{L}^2[\circ] \;\wedge\; \Sigma \cup \{\mathrm{O}, \text{FUS-Ax}\} \models^{s2} \psi\};$$

I will take this equality as the "official" definition for $\Sigma^{s2}$.

**Lemma 9.** *If* $\Sigma$ *is a set of* $\mathrm{L}^2[\circ]$*-sentences,* $\Sigma^{g2+} \subseteq \Sigma^{s2}$.

*Proof.* If $\mathcal{M}$ is a $s2$-structure satisfying $\Sigma$, O and FUS-Ax, it is a $g2$-model of $\mathrm{Ax}^2(\mathrm{CI})_4$.

Let me finally come back to the task of giving an explication of "$T$ is a 2nd order calculus of individuals" for theories $T$ in $\mathrm{L}^2[\circ]$. I have to be content with a few admittedly superficial remarks on the limits of what should be admissible as its *explicans*, using the foregoing considerations as a background. What I would like to have is:

(I) $T$ is a 2nd order calculus of individuals: $\iff$ $T$ is a consistent analogue of a 1st order calculus of individuals.

But this is too vague to stop here. Now, take these specifications of (I):

(I.1) $T$ is a 2nd order calculus of individuals: $\iff$ $T$ is a consistent extension of $\mathrm{Ax}^2(\mathrm{CI})_1$.

(I.2) $T$ is a 2nd order calculus of individuals: $\iff$ there is a 1st order calculus of individuals $S$ such that $T = Cn^{g2}(S \cup \mathrm{Ax}^2(\mathrm{CI})_1)$.

They are precise, but I think they are no adequate renderings of (I). At any rate, I regard them as unacceptable: the *definiens* of (I.1) seems to be is too wide and the *definiens* of (I.2) seems to be too narrow. 2nd order calculi of individuals should be something in between; but for this intuition, I do not have a precise criterion at hand.

### *3. Mereologies and Boolean algebras*

In order to obtain more interesting and specific metatheorems about theories of the type $\Sigma_0^{g2}$, $\Sigma^{g2}$, $\Sigma^{g2+}$ and $\Sigma^{s2}$, I find it useful to employ a certain correspondence between models of CI and Boolean algebras that was known already to Tarski. In what follows, this correspondence is extended to 2nd order structures.

For convenience, I take a Boolean algebra to be a structure $\mathcal{B}$ of the form $\langle B, \leq^B, 0^B, 1^B\rangle$ (i.e., a Boolean lattice rather than the more common $\langle B, \sqcap^B, \sqcup^B, -^B, 0^B, 1^B\rangle$; see Halmos (1963)) where $\leq^B$ is a 2-place relation on $B$ and $0^B$ and $1^B$ are different distinguished elements of $B$. $\mathrm{L}^1[\mathrm{BA}]$ is the 1st order language of the appropriate signature, containing the 2-place relation sign "$\leq$" and the constants "0" and "1". Moreover, let $\mathrm{Ax}(\mathrm{BA})$ be some (finite) axiomatization of the 1st order theory BA of Boolean algebras in this language.

$\mathrm{L}^1[\mathrm{BA}]$ may be extended to a monadic 2nd order language $\mathrm{L}^2[\mathrm{BA}]$ just as $\mathrm{L}^1[\circ]$ has been extended to $\mathrm{L}^2[\circ]$. And $g2$- and $s2$-structures appropriate to $\mathrm{L}^2[\mathrm{BA}]$ may be defined similarly to the $g2$- and $s2$-structures for $\mathrm{L}^2[\circ]$. It is convenient to fix a notation for all these structures: for $\langle M, \circ^M\rangle$ of the signature of $\mathrm{L}^1[\circ]$ I write "$\mathcal{M}$",[38] for $\langle B, \leq^B,$

[38] I identify, as it is common, structures $\langle M, \circ^M\rangle$ with those of the form $\langle M, \mathcal{I}\rangle$, given that $\mathcal{I}(\circ) = \circ^M$ (similarly for other languages).

$0^B, 1^B\rangle$ of the signature of $\mathrm{L}^1[\mathrm{BA}]$ I write "$\mathcal{B}$". Then, $\mathcal{M}^2 := \langle M, \wp(M), \circ^M\rangle$ is a $s2$-structure for $\mathrm{L}^2[\circ]$ and $\mathcal{B}^2 := \langle B, \wp(B), \leq^B, 0^B, 1^B\rangle$ is a $s2$-structure for $\mathrm{L}^2[\mathrm{BA}]$.

Now we have two mappings between $s2$-structures for $\mathrm{L}^2[\circ]$ and $s2$-structures for $\mathrm{L}^2[\mathrm{BA}]$.

**Definition 10.** *For a $s2$-structure $\mathcal{M}^2$ for $\mathrm{L}^2[\circ]$ and $n \notin M$, set $(\mathcal{M}^2)^{+n} := \langle M^{+n}, \wp(M^{+n}), \leq^{M+n}, n, 1^{M+n}\rangle$, with*

$$M^{+n} := M \cup \{n\},$$
$$a \leq^{M+n} b :\Longleftrightarrow (a \in M \wedge b \in M \wedge a \sqsubseteq^M b) \vee a = n \text{ (for } a, b \in M^{+n}\text{)},$$
$$1^{M+n} := \text{the element } a \text{ of } M \text{ such that } \forall b \in M\ (b \sqsubseteq^M a).^{39}$$

**Definition 11.** *For a $s2$-structure $\mathcal{B}^2$ for $\mathrm{L}^2[\mathrm{BA}]$, set $(\mathcal{B}^2)^- := \langle B^-, \wp(B^-), \circ^{B-}\rangle$,[40] with*

$$B^- := B \setminus \{0^B\},$$
$$a \circ^{B-} b :\Longleftrightarrow \exists c \in B\ (c \leq^B a \wedge c \leq^B b \wedge c \neq 0^B) \quad \text{(for } a, b \in B^-\text{)}.$$

**Lemma 12.** *(i) If $\mathcal{B}^2 \models \mathrm{Ax(BA)}$, then $(\mathcal{B}^2)^- \models \mathrm{Ax(CI)}$.*
*(ii) If $\mathcal{M}^2 \models \mathrm{Ax(CI)}$ and $n \notin M$, then $(\mathcal{M}^2)^{+n} \models \mathrm{Ax(BA)}$.*

These model-constructions are now used to establish a correspondence between $\mathrm{L}^2[\mathrm{BA}]$ and $\mathrm{L}^2[\circ]$, too. To be more explicit, there is a function $\mathcal{J}$ from the set of $\mathrm{L}^2[\circ]$-formulas to the set of $\mathrm{L}^2[\mathrm{BA}]$-formulas which can be inductively defined as follows:[41]

$\mathcal{J}(\ulcorner s \circ t\urcorner) = \ulcorner\exists x\ (x \leq s \wedge x \leq t \wedge x \neq 0)\urcorner$,
$\mathcal{J}(\ulcorner Xs\urcorner) = \ulcorner Xs \wedge \neg X0\urcorner$,
$\mathcal{J}$ commutes with the negation sign and the conditional,
$\mathcal{J}(\ulcorner\forall x\varphi\urcorner) = \ulcorner\forall x\ (x \neq 0 \to \mathcal{J}(\varphi))\urcorner$,
$\mathcal{J}(\ulcorner\forall X\varphi\urcorner) = \ulcorner\forall X\ (\neg X0 \to \mathcal{J}(\varphi))\urcorner$.[42]

**Lemma 13.** *If $\mathcal{B}^2$ is a $s2$-structure for $\mathrm{L}^2[\mathrm{BA}]$ and if $\beta$ is an assignment over $(B^2)^-$, then*

$$\forall\psi \in L^2[\circ]\ ((\mathcal{B}^2)^-, \beta \models \psi \iff \mathcal{B}^2, \beta \models \mathcal{J}(\psi)).$$

*Proof.* Induction on the built-up of $\psi$ (from $\mathrm{L}^2[\circ]$).

39 The notation of (definitional extensions of) $\mathrm{L}^2[\circ]$ is supposed to be transferred to the metalanguage.
40 Note that $B^- \neq \emptyset$.
41 I use quasi quotation only at some important places.
42 Officially, relative interpretations map 1st order formulas to 1st order formulas. But $\mathcal{J}$ is a good choice for what may be called a "2nd order interpretation" (for more on this topic see Krajewski (1974)). I think that for such mappings, it is worth considering as a general criterion that if $\delta(x)$ is the relativizing formula for the 1st order quantifiers, the one for the 2nd order quantifiers should be $\ulcorner\forall x\ (Xx \to \delta(x))\urcorner$. Now in our case, where $\delta(x) = \ulcorner x \neq 0\urcorner$, this amounts to the formula $\ulcorner\forall x\ (Xx \to x \neq 0)\urcorner$; but (up to logical equivalence) this is just $\ulcorner\neg X0\urcorner$.

- $\psi \equiv y \circ z$ : then $(\mathcal{B}^2)^-, \beta \models y \circ z \iff \beta(y) \circ^{B-} \beta(z)$ since $\beta(y), \beta(z) \in B^-$

$$\iff \exists c \in B(c \leq^B \beta(y) \wedge c \leq^B \beta(z) \wedge c \neq 0^B)$$
$$\iff \mathcal{B}^2, \beta \models \exists x\,(x \leq y \wedge x \leq z \wedge x \neq 0)$$
$$\iff \mathcal{B}^2, \beta \models \mathcal{J}(y \circ z).$$

- $\psi \equiv Xs$ : then $(\mathcal{B}^2)^-, \beta \models Xs \iff s^{(\mathcal{B}^2)^-,\beta} \in \beta(X) \wedge \beta(X) \in \wp(B^-)$

$$\iff \beta(s) \in \beta(X) \wedge \beta(X) \in \wp(B) \wedge 0^B \notin \beta(X)$$
$$\iff \mathcal{B}^2, \beta \models Xs \wedge \neg X0$$
$$\iff \mathcal{B}^2, \beta \models \mathcal{J}(\psi).$$

- the cases of the propositional operators are trivial.
- $\psi \equiv \forall x \varphi$ : then $(\mathcal{B}^2)^-, \beta \models \psi \iff \forall a(a \in B^- \Longrightarrow (\mathcal{B}^2)^-, \beta(x:a) \models \varphi)$

$$\iff \forall a\,(a \in B \wedge a \neq 0^B \Longrightarrow (\mathcal{B}^2)^-, \beta(x:a) \models \varphi)$$
$$\iff \forall a\,(a \in B \wedge a \neq 0^B \Longrightarrow \mathcal{B}^2, \beta(x:a) \models \mathcal{J}(\varphi)) \quad \text{(by I. H.)}$$
$$\iff \forall a\,(a \in B \Longrightarrow (\mathcal{B}^2, \beta(x:a) \models x \neq 0 \Longrightarrow \mathcal{B}^2, \beta(x:a) \models \mathcal{J}(\varphi)))$$
$$\iff \forall a\,(a \in B \Longrightarrow \mathcal{B}^2, \beta(x:a) \models x \neq 0 \rightarrow \mathcal{J}(\varphi))$$
$$\iff \mathcal{B}^2, \beta \models \forall x(x \neq 0 \rightarrow \mathcal{J}(\varphi))$$
$$\iff \mathcal{B}^2, \beta \models \mathcal{J}(\psi).$$

- $\psi \equiv \forall X \varphi$ : then $(\mathcal{B}^2)^-, \beta \models \psi$

$$\iff \forall C\,(C \subseteq B^- \Longrightarrow (\mathcal{B}^2)^-, \beta(X:C) \models \varphi)$$
$$\iff \forall C\,(C \subseteq B \wedge 0^B \notin C \Longrightarrow (\mathcal{B}^2)^-, \beta(X:C) \models \varphi)$$
$$\iff \forall C\,(C \subseteq B \wedge 0^B \notin C \Longrightarrow \mathcal{B}^2, \beta(X:C) \models \mathcal{J}(\varphi)) \quad \text{(by I.H.)}$$
$$\iff \forall C\,(C \subseteq B \Longrightarrow (\mathcal{B}^2, \beta(X:C) \models \neg X0 \Longrightarrow \mathcal{B}^2, \beta(X:C) \models \mathcal{J}(\varphi)))$$
$$\iff \forall C\,(C \subseteq B \Longrightarrow \mathcal{B}^2, \beta(X:C) \models \neg X0 \rightarrow \mathcal{J}(\varphi))$$
$$\iff \mathcal{B}^2, \beta \models \forall X(\neg X0 \rightarrow \mathcal{J}(\varphi))$$
$$\iff \mathcal{B}^2, \beta \models \mathcal{J}(\psi).$$

Lemma 13 can be used to establish, loosely speaking, the equivalence of complete Boolean algebras with $s2$-structures satisfying FUS-Ax: see Lemma 14 and Lemma 16.

**Lemma 14.** *If $\mathcal{B}$ is a complete Boolean algebra, then* $(\mathcal{B}^2)^- \models$ FUS-Ax.[43]

*Proof.* Let $\beta$ be a variable-assignment over $(\mathcal{B}^2)^-$ and assume $(\mathcal{B}^2)^-, \beta \models \exists x\, Xx$.

By Lemma 7, $\mathcal{B}^2, \beta \models \mathcal{J}(\exists x\, Xx)$, i.e., $\mathcal{B}^2, \beta \models \exists x(x \neq 0 \wedge \mathcal{J}(Xx))$. Therefore,

$$D := \{a \in B \mid a \neq 0^B \wedge \mathcal{B}^2, \beta(x:a) \models \mathcal{J}(Xx)\}$$

[43] A Boolean algebra $\mathcal{B}$ is complete iff for each nonempty subset $\Omega$ of $B$, there is a $b \in B$ which is the supremum, i.e., the least upper bound, of $A$.

is a non-empty subset of $B$. Now, by the assumed completeness of $\mathcal{B}$, an element of $B$ exists which is the supremum of $D$; let's call ist "$\bigvee D$". Since $\bigvee D \neq 0^B$, actually $\bigvee D \in B^-$.

In order to show the lemma, it then suffices to establish the following

*Claim.* $(\mathcal{B}^2)^-, \beta(z : \bigvee D) \models \forall y\, (y \circ z \longleftrightarrow \exists x(Xx \wedge y \circ x))$.

*Proof.* Let $b \in B^-$. Reasoning with complete Boolean algebras we obtain

$$\begin{aligned}
&\exists c \in B\, (c \leq^B b \wedge c \leq^B \bigvee D \wedge c \neq 0^B)\\
&\Longrightarrow \exists c \in B\, (c \leq^B b \wedge \exists a \in D \exists d \in B\, (d \leq^B c \wedge d \leq^B a \wedge d \neq 0^B) \wedge c \neq 0^B)\\
&\Longrightarrow \exists a\, (a \in D \wedge \exists d \in B\, (d \leq^B b \wedge d \leq^B a \wedge d \neq 0^B))\\
&\Longrightarrow \exists c \in B\, (c \leq^B b \wedge c \leq^B \bigvee D \wedge c \neq 0^B),
\end{aligned}$$

that is

$$\exists c \in B\, (c \leq^B b \wedge c \leq^B \bigvee D \wedge c \neq 0^B) \iff \exists a\, (a \in D \wedge b \circ^{B^-} a).$$

Therefore,

$$\begin{aligned}
(\mathcal{B}^2)^-, \beta(z : \bigvee D)(y : b) \models y \circ z &\iff b \circ^{B^-} \bigvee D\\
&\iff \exists a\, (a \in D \wedge b \circ^{B^-} a).
\end{aligned}$$

But we also have (since "$z$" is new, and by using Lemma 13 for the third equivalence)

$$\begin{aligned}
&(\mathcal{B}^2)^-, \beta(z : \bigvee D)(y : b) \models \exists x\, (Xx \wedge y \circ x)\\
&\qquad \iff \exists a\, (a \in B^- \wedge (\mathcal{B}^2)^-, \beta(y : b)(x : a) \models Xx \wedge y \circ x)\\
&\qquad \iff \exists a\, (a \in B \wedge a \neq 0^B \wedge b \circ^{B^-} a \wedge (\mathcal{B}^2)^-, \beta(x : a) \models Xx)\\
&\qquad \iff \exists a\, (a \in B \wedge a \neq 0^B \wedge b \circ^{B^-} a \wedge \mathcal{B}^2, \beta(x : a) \models \mathcal{J}(Xx))\\
&\qquad \iff \exists a\, (a \in D \wedge b \circ^{B^-} a).
\end{aligned}$$

The claim follows.

**Corollary 15.** *If $\mathcal{B}$ is a complete Boolean algebra, then* $(\mathcal{B}^2)^- \models \mathrm{Ax(CI)}, \mathrm{Comp}^2$, FUS-Ax.

The converse of Lemma 14 can also be shown:

**Lemma 16.** *If $\mathcal{M}^2(:= \langle M, \wp(M), \circ^M\rangle)$ satisfies* $\mathrm{Ax(CI)}$, FUS-Ax *and if $n \notin M$, then $\mathcal{M}^{+n}(:= \langle M^{+n}, \leq^{M+n}, n, 1^{M+n}\rangle)$ is a complete Boolean algebra.*

*Proof.* By Lemma 12(ii), $\mathcal{M}^{+n}$ is a Boolean algebra. For completeness, let $A \subseteq M^{+n}$ be nonempty.

If $A = \{n\}$, its supremum (in $\mathcal{M}^{+n}$) equals $n$.

If $A \neq \{n\}$, consider $C := A \setminus \{n\}$, which is a nonempty element of $\wp(M)$: that is, if $\beta$ is an assignment over $\mathcal{M}^2$,

$$\mathcal{M}^2, \beta(X : C) \models \exists x\, Xx.$$

Now since $\mathcal{M}^2 \models$ Ax(CI), FUS-Ax, for some $e \in M$

$$\mathcal{M}^2, \beta(X : C)(z : e) \models \forall y\, (y \circ z \longleftrightarrow \exists x\, (Xx \wedge y \circ x)) \qquad (*)$$

$e$ is the supremum of $A$ in $\mathcal{M}^{+n}$.

For *first*, $\forall a\, (a \in A \Rightarrow a \leq^{M+n} e)$: if $a = n$, this is trivial; and if $a \in C$, $a \leq^{M+n} e$ holds by (∗).

And *second*, assume $\forall a\, (a \in A \Rightarrow a \leq^{M+n} e')$ with $e' \in M^{+n}$; then $\forall a\, (a \in C \Rightarrow a \leq^{M+n} e')$ and $e' \neq n$, i.e., $e' \in M$. Thus, by (∗), $e \sqsubseteq^M e'$, whence by definition, $e \leq^{M+n} e'$, too.

## 4. *2nd order extensions of* CI*: some metatheorems*

In the light of Theorem A from section 1.2, the "strongest" result one could hope for here is that in $\mathrm{L}^2[\circ]$, the maximal-consistent extensions of $\emptyset_0^{g2}$ are exactly the $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ and the $(\mathrm{Ax}(\mathrm{MCI}_{n+1}))^{s2}$ $(n \in \mathbb{N})$, $(\mathrm{Ax}(\mathrm{ACI}_\infty))^{s2}$, $(\mathrm{Ax}(\mathrm{FCI}))^{s2}$ and $(\mathrm{Ax}(\mathrm{MCI}_\infty))^{s2}$. It will turn out, however, that this is not the case: see section 4.3. Actually, the situation seems to be much more complicated than in the 1st order setting. More modestly, one could ask whether those theories have at least $s2$-models. Here the answer is positive: see section 4.1. Sections 4.2 and 4.4 are concerned with the relation between 1st order calculi of individuals and 2nd order extensions of $\emptyset_0^{g2}$; in particular, they contain the description of several maximal-consistent examples of the latter.

### 4.1. $s2$-*satisfiability*

**Lemma 17.** *Each of the* $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ *and the* $(\mathrm{Ax}(\mathrm{MCI}_{n+1}))^{s2}$ $(n \in \mathbb{N})$, $(\mathrm{Ax}(\mathrm{ACI}_\infty))^{s2}$, $(\mathrm{Ax}(\mathrm{FCI}))^{s2}$ *and* $(\mathrm{Ax}(\mathrm{MCI}_\infty))^{s2}$ *has a* $s2$*-model.*

*Proof.* Let Ax be any of the axiom-sets $\mathrm{Ax}(\mathrm{ACI}_{n+1})$, $\mathrm{Ax}(\mathrm{ACI}_\infty)$, Ax(FCI), $\mathrm{Ax}(\mathrm{MCI}_{n+1})$ and $\mathrm{Ax}(\mathrm{MCI}_\infty)$.

In order to get the desired result, it suffices by Corollary 15 to show:

*Claim.* There is a complete Boolean algebra $\mathcal{B}$ such that $(\mathcal{B}^2)^- \models$ Ax.

*Proof.* For $\mathrm{Ax}(\mathrm{ACI}_{n+1})$, $\mathrm{Ax}(\mathrm{ACI}_\infty)$ and Ax(FCI), it is easy to find these Boolean algebras:

- $\mathrm{Ax}(\mathrm{ACI}_{n+1})$: set $\mathcal{B} := \langle \wp(\{0, \ldots, n\}), \subseteq, \emptyset, \{0, \ldots, n\} \rangle$.

- $\mathrm{Ax}(\mathrm{ACI}_\infty)$: set $\mathcal{B} := \langle \wp(\mathbb{N}), \subseteq, \emptyset, \mathbb{N}\rangle$.
- $\mathrm{Ax}(\mathrm{FCI})$: set $\mathcal{B} := \langle RO(\mathbb{R}), \subseteq, \emptyset, \mathbb{R}\rangle$. $RO(\mathbb{R})$ is the set of the regular open subsets of $\mathbb{R}$ (with its usual topology). It is well known that $\mathcal{B}$ is a complete Boolean algebra —see Halmos (1963). Moreover, it is atomfree: therefore, $(\mathcal{B}^2)^- \models \mathrm{AF}$.

For $\mathrm{Ax}(\mathrm{MCI}_{n+1})$ and $\mathrm{Ax}(\mathrm{MCI}_\infty)$, products of the above mentioned Boolean algebras do their jobs:

- $\mathrm{Ax}(\mathrm{MCI}_{n+1})$: set $\mathcal{B} := \langle \wp(\{0,\dots,n\}), \subseteq, \emptyset, \{0,\dots,n\}\rangle \times \langle RO(\mathbb{R}), \subseteq, \emptyset, \mathbb{R}\rangle$.
- $\mathrm{Ax}(\mathrm{MCI}_\infty)$: set $\mathcal{B} := \langle \wp(\mathbb{N}), \subseteq, \emptyset, \mathbb{N}\rangle \times \langle RO(\mathbb{R}), \subseteq, \emptyset, \mathbb{R}\rangle$.

**Corollary 18.** *For each (1st order) model $\mathcal{M}(= \langle M, \circ^M\rangle)$ of $\mathrm{CI} + \mathrm{FUS}^1$, there is a complete Boolean algebra $\langle B, \leq^B, 0^B, 1^B\rangle$ such that $\langle B^-, \circ^{B^-}\rangle \equiv \mathcal{M}$.*

*Proof.* Consider $\mathrm{Th}(\mathcal{M})$ (i.e., the set of $\mathrm{L}^1[\circ]$-sentences true in $\mathcal{M}$); it is a maximally consistent extension of $\mathrm{CI} + \mathrm{FUS}^1$ (in $\mathrm{L}^1[\circ]$), hence by Theorem A equal to one of the theories $\mathrm{ACI}_{n+1}$, $\mathrm{MCI}_{n+1}$ ($n \in \mathbb{N}$), $\mathrm{ACI}_\infty$, FCI or $\mathrm{MCI}_\infty + \mathrm{FUS}^1$. Therefore, by the *claim* from Lemma 17, there is a complete Boolean algebra $\mathcal{B}(= \langle B, \leq^B, 0^B, 1^B\rangle)$ such that $(\mathcal{B}^2)^- \models \mathrm{Th}(\mathcal{M})$. Now $\mathrm{Th}(\mathcal{M})$ contains only 1st order sentences; therefore,

$$\langle B^-, \circ^{B^-}\rangle \models \mathrm{Th}(\mathcal{M}),$$

too. But then $\langle B^-, \circ^{B^-}\rangle \equiv \mathcal{M}$.

### 4.2. *Finite $g2$- and $s2$-models*

For finite structures, being a $g2$-structure and being a $s2$-structure amounts to the same.

**Lemma 19.** *(i) If $\langle M, \Omega, \circ^M\rangle \models \mathrm{Comp}^1$ and $M$ is finite, then $\Omega = \wp(M)$.*

*(ii) If all $g2$-models of $\Sigma \cup \mathrm{Ax}^2(\mathrm{CI})_1$ are finite, then $\Sigma_0^{g2} = \Sigma^{g2} = \Sigma^{g2+} = \Sigma^{s2}$.*

*Proof.* (i) Let $C \subseteq M$, i.e. $C = \{a_1, \dots, a_n\}$, with $a_1, \dots, a_n \in M$. Now, since $\langle M, \Omega, \circ^M\rangle \models \mathrm{Comp}^1$, we have in particular (for arbitrary assignments $\beta$ over $M$)

$$\langle M, \Omega, \circ^M\rangle, \beta \models \forall x_1 \dots x_n \exists X \forall y\, (Xy \leftrightarrow y = x_1 \vee \dots \vee y = x_n).$$

Therefore, $\langle M, \Omega, \circ^M\rangle, \beta(x_1 : a_1) \dots (x_n : a_n) \models \exists X \forall y\, (Xy \leftrightarrow y = x_1 \vee \dots \vee y = x_n)$, whence

$$\exists B \in \Omega \forall r \in M\, (r \in B \iff r = a_1 \vee \dots \vee r = a_n).$$

That is, $C = \{a_1, \dots, a_n\} = B$, and since $B \in \Omega$, $C \in \Omega$ follows.

This shows $\wp(M) \subseteq \Omega$.

(ii) It suffices to show "$\Sigma^{s2} \subseteq \Sigma_0^{g2}$". Thus, assume $(*)$ $\Sigma \cup \{\mathrm{O}, \text{FUS-Ax}\} \models^{s2} \psi$ and let $\langle M, \Omega, \circ^M\rangle$ be a $g2$-structure satisfying $\mathrm{Ax}^2(\mathrm{CI})_1$ and $\Sigma$. By assumption, $M$ is finite; therefore, by (i), $\Omega = \wp(M)$, and $\langle M, \Omega, \circ^M\rangle$ is actually a $s2$-structure. Moreover, $\langle M, \Omega, \circ^M\rangle$ satisfies $\Sigma$, O and FUS-Ax. Therefore, $(*)$ implies that $\langle M, \Omega, \circ^M\rangle \models \psi$.

**Theorem 20.** *For each* $n \in \mathbb{N}$,

*(i)* $\mathrm{Ax}(\mathrm{ACI}_{n+1})$ *is* $s2$*- and* $g2$*-categorical, and* $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ *is maximal-consistent.*

*(ii)* $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{g2}_0 = (\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{g2} = (\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{g2+} = (\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$
$= \mathrm{Th}^2(\langle(\wp(\{0,\ldots,n\}))^-, \wp((\wp(\{0,\ldots,n\}))^-), \cap\lceil(\wp(\{0,\ldots,n\}))^-\rangle)$.

*(iii)* $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ *is decidable.*

*Proof.* (i) All $s2$-models of $\mathrm{Ax}(\mathrm{ACI}_{n+1})$ have exactly $n+1$ atoms and are isomorphic with each other. Moreover, each $g2$-model of $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ is finite and satisfies $\mathrm{Comp}^1$; thus, by Lemma 19(i), the $g2$- and the $s2$-models of $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ are the same objects. Since $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ is $s2$-categorical and consistent, it is maximal-consistent by Lemma 5(iii).

(ii) Since all $g2$-models of $\mathrm{Ax}(\mathrm{ACI}_{n+1})$ are finite and $\langle(\wp(\{0,\ldots,n\}))^-, \wp((\wp(\{0,\ldots,n\}))^-), \cap\lceil(\wp(\{0,\ldots,n\}))^-\rangle$ is one of them, Lemma 3(iv) and 19(ii) imply the claim.

(iii) Being the deductive closure of a recursively enumerable set of axioms, $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{g2}$ is recursively enumerable. Therefore, by (i) and (ii), $(\mathrm{Ax}(\mathrm{ACI}_{n+1}))^{s2}$ is decidable.

### *4.3. On extensions of* $\mathrm{ACI}_\infty$

Certainly, Lemma 5 has a somewhat restricted applicability. For similarly to the 1st order case, if a set $\Sigma$ of 2nd order sentences has infinite models, it cannot be $g2$-categorical (by the Löwenheim-Skolem theorems). Moreover, it is in general more likely that $\Sigma$ is $\kappa$-$s2$-categorical than $\kappa$-$g2$-categorical. But it is only from the latter that we obtain the maximal-consistency of $\Sigma^{g2}$. Nonetheless, at least for theories around $(\mathrm{Ax}(\mathrm{ACI}_\infty))^{s2}$, the concepts from section 2.2 prove to be useful.

To start with, we easily obtain an approximation to categoricity:

**Lemma 21.** $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}\}$ *is* $2^{\aleph_0}$*-*$s2$*-categorical.*

*Proof.* Let $\mathcal{M}^2$ and $\mathcal{M}'^2$ be $s2$-models of $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}\}$ with $|M| = |M'| = 2^{\aleph_0}$. Then, $\mathcal{B}(:= \langle M^{+n}, \leq^{M+n}, n, 1^{M+n}\rangle)$ and $\mathcal{B}'(:= \langle M'^{+n}, \leq^{M'+n}, n, 1^{M'+n}\rangle)$ (with new $n$) are atomistic Boolean algebras which are also complete (by Lemma 16).

Now, we have the following fact concerning Boolean algebras:[44]

*Fact.* If $\mathcal{B}$ is an atomistic, complete Boolean algebra and $At^B$ is the set of atoms of $\mathcal{B}$, then $\mathcal{B} \cong \langle\wp(At^B), \subseteq, \emptyset, At^B\rangle$.

Since $|M^{+n}| = |M'^{+n}|$, we also have $|At^B| = |At^{B'}|$. But then, $\mathcal{B} \cong \mathcal{B}'$. The isomorphism between these structures can first be restricted to $M$, giving an isomorphism between $\langle M, \circ^M\rangle$ and $\langle M', \circ^{M'}\rangle$. And this, in turn, can be extended to an ismorphism between $\mathcal{M}^2$ and $\mathcal{M}'^2$.[45]

[44] See Halmos (1963), Ridder (2002).

[45] If $f : M \longrightarrow M'$ is a bijection, set $g(a) = \{f(k) \mid k \in a\}$ for $a \subseteq M$: $g$ is a bijection from $\wp(M)$ to $\wp(M')$.

Yet, this lemma does not imply the maximal-consistency of $(\mathrm{Ax}(\mathrm{ACI}_\infty))^{s2}$: for not all $s2$-models of $(\mathrm{Ax}(\mathrm{ACI}_\infty))^{s2}$ need have cardinality $2^{\aleph_0}$. But maybe 2nd order sentences can be added to that theory which force their models to have that cardinality?—It turns out that these sentences can already be found in Lewis (1991). Thus, the following definitions and lemmata are mainly taken from or inspired by this book.[46]

**Definition 22.** *(i)* $x = \bigvee X :\longleftrightarrow \forall y\,(y \circ x \leftrightarrow \exists z(Xz \wedge y \circ z))$,

*(ii)* $1 = \bigvee X :\longleftrightarrow \exists x\,(x = \bigvee X \wedge \forall y\, y \sqsubseteq x)$,

*(iii)* $\inf(x) :\longleftrightarrow \exists X\,(\exists y\, Xy \wedge x = \bigvee X \wedge \forall y(Xy \rightarrow \exists z(Xz \wedge y \sqsubset z)))$,

*(iv)* $\mathrm{large}(x) :\longleftrightarrow \exists X\,[\exists y\, Xy \wedge \forall yz(Xy \wedge Xz \wedge y \circ z \rightarrow y = z) \wedge 1 = \bigvee X$
$\wedge \forall y(Xy \rightarrow \exists_1 z(At(z) \wedge z \sqsubseteq y \wedge z \sqsubseteq x) \wedge \exists_{\leq 2} z(At(z) \wedge z \sqsubseteq y))]$.

(CountAt)$\forall x\,(\inf(x) \longrightarrow \mathrm{large}(x))$.

**Lemma 23.** *Let* $\mathcal{M}$ *be a* $g2$*-structure with* $\mathcal{M} \models \mathrm{Ax(CI)}$. *Then*

*(i)* $\mathcal{M}, \beta \models \inf(x) \Longrightarrow \{a \in M \mid a \sqsubseteq^M \beta(x)\}$ *is infinite,*

*(ii)* $\mathcal{M} \models \exists x\ \inf(x) \Longrightarrow M$ *is infinite.*

*Proof.* (i) If $\mathcal{M}, \beta \models \inf(x)$, then for some nonempty $B \subseteq M$ we have $\beta(x) = \bigvee^M B$,[47] and $\forall r \in B \exists s \in B\; r \sqsubset^M s$. Thus, $B$ is infinite; and since $B \subseteq \{a \in M \mid a \sqsubseteq^M \beta(x)\}$, $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$ is infinite, too.

(ii) If $\mathcal{M} \models \exists x\ \inf(x)$, then by (i) $\exists b\,(b \in M \wedge \{a \in M \mid a \sqsubseteq^M b\}$ is infinite), whence $M$ is infinite.

**Lemma 24.** *Let* $\mathcal{M}$ *be a* $s2$*-structure with* $\mathcal{M} \models \mathrm{Ax(ACI)}$, FUS-Ax. *Then*

*(i)* $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$ *is infinite* $\Longrightarrow \mathcal{M}, \beta \models \inf(x)$,

*(ii)* $M$ *is infinite* $\Longrightarrow \mathcal{M} \models \exists x\ \inf(x)$.

*Proof.* (i) Let $B := \{b \in M \mid \exists a_1 \ldots a_k\,(a_1, \ldots, a_k$ are atoms$^M \wedge b = \bigvee^M\{a_1, \ldots, a_k\} \wedge b \sqsubseteq^M \beta(x))\}$.

Then $B \in \wp(M)$, and since $\mathcal{M}$ is atomistic, $\emptyset \neq B$. Moreover, since $\mathcal{M}$ is a s2-structure which satisfies FUS-Ax, $\bigvee^M B$ exists and is an element of $M$. In addition, we have $\beta(x) = \bigvee^M B$: for if $c \circ^M \beta(x)$, then for some atom$^M$ $a$, $a \sqsubseteq^M c \wedge a \sqsubseteq^M \beta(x)$; therefore, $\bigvee^M\{a\} \in B$, whence $a \in B$ and $a \sqsubseteq^M \bigvee^M B$—which implies $c \circ^M \bigvee^M B$.

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[46] As for the formal details, I sometimes follow the presentation from Ridder (2002).

[47] For $\beta(X) = B$ we have

$$\begin{aligned}
\mathcal{M}, \beta \models x = \bigvee X \iff\ & \mathcal{M}, \beta \models \forall y\,(y \circ x \leftrightarrow \exists z(Xz \wedge y \circ z)) \\
\iff\ & \forall b \in M\,(b \circ^M \beta(x) \Leftrightarrow \exists c \in M\,(c \in B \wedge b \circ^M c)) \\
\iff\ & \forall b \in M\,(b \circ^M \beta(x) \Leftrightarrow \exists c\,(c \in B \wedge b \circ^M c)),
\end{aligned}$$

which is abbreviated by "$\beta(x) = \bigvee^M B$".

Finally, let $r \in B$; then $r$ is a finite fusion$^M$ of atoms$^M$ $a_1, \ldots, a_k$ with $r \sqsubseteq^M \beta(x)$. Now, since $\{a \in M \mid a \sqsubseteq^M \beta(x)\}$ is supposed to be infinite, $r = \beta(x)$ cannot be the case. Thus, there is an atom$^M$ $a$ such that $a \sqsubseteq^M \beta(x)$ and $a$ is disjoint$^M$ from $r$ (again, because of atomicity). Taking $s := \bigvee^M \{a_1, \ldots, a_k, a\}$ provides for a finite fusion$^M$ of atoms$^M$ which is a part$^M$ of $\beta(x)$, and therefore, for an element $s \in B$ such that $r \sqsubset^M s$.

(ii) Since $\mathcal{M} \models$ CI, there is a maximal element $1^M$ in $M$, i.e., an object $b \in M$ such that $\forall a\,(a \in M \Rightarrow a \sqsubseteq^M b)$. Thus, $M = \{a \in M \mid a \sqsubseteq^M 1^M\}$. Now, if $M$ is infinite, $\{a \in M \mid a \sqsubseteq^M 1^M\}$ is infinite, and $\mathcal{M} \models \exists x\ \mathrm{inf}(x)$ follows by (i).

**Lemma 25.** *Let* $\mathcal{M}$ *be a* $s2$*-structure with* $\mathcal{M} \models \mathrm{Ax}(\mathrm{ACI}_\infty)$*,* FUS-Ax. *Then*

*(i)* $\mathcal{M}, \beta \models \mathrm{large}(x) \iff |\{a \in M \mid a$ *is an atom*$^M\}| = |\{a \in M \mid a$ *is an atom*$^M \wedge a \sqsubseteq^M \beta(x)\}|$.

*(ii)* $\mathcal{M} \models (\mathrm{CountAt}) \iff \{a \in M \mid a$ *is an atom*$^M\}$ *is countable (and infinite).*[48]

*Proof.*[49] (ii) "$\Longrightarrow$" Let $A \subseteq M$ be a countably infinite set of atoms$^M$; such a set exists because, by assumption, $\mathcal{M} \models \mathrm{Ax}(\mathrm{ACI}_\infty)$. Furthermore, $\bigvee^M A \in M$, since $\mathcal{M}$ is a s2-structure which satisfies FUS-Ax. A consequence is $(*)$:

$$A = \{a \in M \mid a \text{ is an atom}^M \wedge a \sqsubseteq^M \bigvee^M A\}.$$

Now, assume $\mathcal{M}, \beta \models (\mathrm{CountAt})$; then $\mathcal{M}, \beta(x : \bigvee^M A) \models \mathrm{inf}(x) \longrightarrow \mathrm{large}(x)$. And by Lemma 24(i), the choice of $A$ and $(*)$, $\mathcal{M}, \beta(x : \bigvee^M A) \models \mathrm{inf}(x)$. Thus, it follows that $\mathcal{M}, \beta(x : \bigvee^M A) \models \mathrm{large}(x)$. Because of (i) and $(*)$, this yields $|\{a \in M \mid a \text{ is an atom}^M\}| = |\{a \in M \mid a \text{ is an atom}^M \wedge a \sqsubseteq^M \bigvee^M A\}| = |A|$. But $A$ was chosen to be countably infinite. Thus, $\{a \in M \mid a \text{ is an atom}^M\}$ has to be countable (and infinite), too.

"$\Longleftarrow$" Assume that $B := \{a \in M \mid a \text{ is an atom}^M\}$ is countable and infinite. In addition, let $b \in M$ be arbitrary and assume that $\mathcal{M}, \beta(x : b) \models \mathrm{inf}(x)$. Then by Lemma 23(i), $C := \{a \in M \mid a \sqsubseteq^M b\}$ is infinite.

Now, consider $A := B \cap C$, i.e., $= \{a \in M \mid a \text{ is an atom}^M \wedge a \sqsubseteq^M b\}$.

$A \neq \emptyset$, since $\mathcal{M} \models$ Ax(ACI). And for the same reason, $A$ is not finite: for if it were, $b$ would be the fusion$^M$ of finitely many atoms$^M$ $a_1, \ldots, a_k$. But then, $C$ would have only finitely many elements, too ($2^k - 1$, that is); contradiction. Thus, $A$ is infinite, and we have $|\{a \in M \mid a \text{ is an atom}^M\}| = |\{a \in M \mid a \text{ is an atom}^M \wedge a \sqsubseteq^M b\}|$.

By (i), it follows that $\mathcal{M}, \beta(x : b) \models \mathrm{large}(x)$.

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[48] Let me note that the $\mathrm{L}^2[\circ]$-formula put forward as a formalization of "there are countably many atoms" in Lewis (1991) and Ridder (2002) is not (CountAt), but the more complicated
(CountAt$'$)$\forall Xx\,(x = \bigvee X \wedge \forall y(Xy \rightarrow At(y)) \wedge \mathrm{inf}(x) \longrightarrow \mathrm{large}(x))$.
The following can however be shown: Ax(ACI) $\models^{g2}$ (CountAt) $\longleftrightarrow$ (CountAt$'$).

[49] I skip the proof of (i). A few hints can be found in Lewis (1991) and Ridder (2002).

Let's put these sentences of $\mathrm{L}^2[\circ]$ to use.

**Theorem 26.** *(i) All $s2$-models of* $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}, (\text{CountAt})\}$ *have cardinality* $2^{\aleph_0}$.

*(ii)* $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}, (\text{CountAt})\}$ *is $s2$-categorical.*

*(iii)* $(\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{(\text{CountAt})\})^{s2}$ *is maximal-consistent.*

*(iv)* $(\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{(\text{CountAt})\})^{s2} = \mathrm{Th}^2(\langle(\wp(\mathbb{N}))^-, \wp((\wp(\mathbb{N}))^-), \cap\lceil(\wp(\mathbb{N}))^-\rangle)$.

*Proof.* (i) Let $\mathcal{M}$ be a $s2$-structure satisfying $\mathrm{Ax}(\mathrm{ACI}_\infty)$, FUS-Ax, (CountAt). By Lemma 25(ii), $M$ has $\aleph_0$ atoms. Therefore (employing the fact from the proof of Lemma 21), $|M| = 2^{\aleph_0}$.

(ii) Because of (i) and Lemma 21.

(iii) Because of (ii) and Lemma 5(iii).

(iv) $\langle(\wp(\mathbb{N}))^-, \wp((\wp(\mathbb{N}))^-), \cap\lceil(\wp(\mathbb{N}))^-\rangle$ is a $s2$-structure satisfying $\mathrm{Ax}(\mathrm{ACI}_\infty)$, FUS-Ax, (CountAt). The claim follows from (iii) and Lemma 3(iii).

Furthermore, we have

**Lemma 27.** *(i)* $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}\} \models^{s2} \exists x\ \mathrm{inf}(x)$.

*(ii) "$\exists x\ \mathrm{inf}(x)$" $\notin (\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{(\text{CountAt})\})^{g2+}$, "$\neg\exists x\ \mathrm{inf}(x)$" $\notin (\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{(\text{CountAt})\})^{g2+}$.*

*(iii)* $(\mathrm{Ax}(\mathrm{ACI}_\infty))^{g2+} \subset (\mathrm{Ax}(\mathrm{ACI}_\infty))^{s2}$, $(\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{(\text{CountAt})\})^{g2+} \subset (\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{(\text{CountAt})\})^{s2}$.

*Proof.* (i) If $\mathcal{M}$ is a $s2$-structure with $\mathcal{M} \models \mathrm{Ax}(\mathrm{ACI}_\infty)$, then $M$ is infinite. Thus, since also $\mathcal{M} \models$ FUS-Ax, it follows by Lemma 24(ii) that $\mathcal{M} \models \exists x\ \mathrm{inf}(x)$.

(ii) For assume the contrary: i.e., $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \mathrm{Ax}^2(\mathrm{CI})_2 \cup \{(\text{CountAt})\} \models^{g2} \exists x\ \mathrm{inf}(x)$. Then by the completeness theorem for $g2$-consequence, $\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \mathrm{Ax}^2(\mathrm{CI})_2 \cup \{(\text{CountAt})\} \vdash \exists x\ \mathrm{inf}(x)$. But in this case, "$\exists x\ \mathrm{inf}(x)$" would be derivable from a finite subset of this set of axioms, whence from $\Sigma := \mathrm{Ax}(\mathrm{ACI}_{n+1}) \cup \mathrm{Ax}^2(\mathrm{CI})_2 \cup \{(\text{CountAt})\}$ (for some $n \in \mathbb{N}$).

Now take a $s2$-structure $\langle M, \wp(M), \circ^M\rangle$ which is a model of $\mathrm{Ax}(\mathrm{ACI}_{n+1})$. It also satisfies $\mathrm{Ax}^2(\mathrm{CI})_2$ (by Theorem 20). In addition, since $M$ must be finite, $\langle M, \wp(M), \circ^M\rangle$ trivially makes (CountAt) true. Thus, $\langle M, \wp(M), \circ^M\rangle \models \Sigma$, which implies $\langle M, \wp(M), \circ^M\rangle \models \exists x\ \mathrm{inf}(x)$. Yet by Lemma 23(ii), this yields that $M$ must be infinite. Contradiction.

(iii) By Lemma 27(ii) and because by Lemma 24(ii), each $s2$-structure which satisfies $\mathrm{Ax}(\mathrm{ACI}_\infty)$ is infinite and therefore a model of "$\exists x\ \mathrm{inf}(x)$".

Let's conclude this subsection with counterexamples to some of the more noteworthy modifications and strenghtenings of Lemma 5. Here are the candidates (for consistent sets $\Sigma$ of $\mathrm{L}^2[\circ]$-sentences):

(iii.1) If $\Sigma$ is $s2$-categorical and $Cn^{s2}(\Sigma)$ is consistent, then $Cn^{g2}(\Sigma)$ is maximal-consistent.

(iii.2) If $\Sigma$ is $s2$-categorical and $Cn^{g2}(\Sigma)$ is consistent, then $Cn^{s2}(\Sigma)$ is maximal-consistent.

(v.1) Let $\kappa$ be an infinite cardinal. If $\Sigma$ is $\kappa$-$s2$-categorical and has only infinite $s2$-models, and if $Cn^{s2}(\Sigma)$ is consistent, then $Cn^{s2}(\Sigma)$ is maximal-consistent.

(iii.1) is refuted by $\Sigma := \mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}, (\text{CountAt})\}$ (by Theorem 26(ii), (iii) and Lemma 27(ii)). (iii.2) is refuted by $\Sigma := \mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}, (\text{CountAt}), \neg\exists x\ \inf(x)\}$ (by Theorem 26(ii), Lemma 27(ii) and since $Cn^{s2}(\Sigma)$ fails to be consistent (see Lemma 27(iii))).[50] (v.1) is refuted by $\Sigma := \mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}\}$; for although $\Sigma^{s2}$ is $2^{\aleph_0}$-$s2$-categorical (by Lemma 21) and consistent, it neither contains (CountAt) (because it has models with uncountably many atoms) nor $\neg$(CountAt).

### *4.4. Conservative extensions*

Let me close the metalogical part of this paper with a further topic: what is the strength of the 1st order portions of the 2nd order variants of calculi of individuals? This is connected with the theme *conservativity*. In our context, "conservativity" may be explained as follows:

**Definition 28.** *Let $T$ be a set of* $\mathrm{L}^2[\circ]$*-sentences such that $T^{g2} \subseteq T$ and $S$ be a theory in* $\mathrm{L}^1[\circ]$*. Then: $T$ is a conservative extension of $S$: $\iff$ $S \subseteq T$ and for all* $\mathrm{L}^1[\circ]$*-sentences $\psi$ ($\psi \in T \iff \psi \in S$).*

Since the $\mathrm{ACI}_{n+1}$ and the $\mathrm{MCI}_{n+1}$, $\mathrm{ACI}_\infty$, FCI and $\mathrm{MCI}_\infty + \mathrm{FUS}^1$ are maximally consistent (in $\mathrm{L}^1[\circ]$), it is clear that for each of them, each consistent extension in $\mathrm{L}^2[\circ]$ is conservative over that very $\mathrm{L}^1[\circ]$-theory. This, however, is a quite weak claim.

Now, there is a general method for obtaining stronger conservativity results which can be applied also here:[51] Start with a 1st order structure $\mathcal{M}(= \langle M, \circ^M \rangle)$ and extend it to the $g2$-structure $\langle M, \mathrm{Def}^1(M), \circ^M \rangle$, where $\mathrm{Def}^1(M)$ contains the subsets of $M$ which are *parametrically definable* over $\mathcal{M}$. It can be shown that $\mathcal{M}^{\mathrm{Def}} \models \mathrm{Comp}^1$, and that if $\mathcal{M} \models \mathrm{FUS}^1$, then $\mathcal{M}^{\mathrm{Def}} \models$ FUS-Ax. Eventually, this yields this conservativity claim:

**Lemma.** *If $\Sigma$ is a set of* $\mathrm{L}^1[\circ]$*-sentences, then $\Sigma_0^{g2}$ is a conservative extension of* $\Sigma + \mathrm{CI} + \mathrm{FUS}^1$.

But here, with Corollary 18 at hand, a much stronger conservativity result can be established.

**Theorem 29.** *If $\Sigma$ is a set of* $\mathrm{L}^1[\circ]$*-sentences, then $\Sigma^{s2}$ is a conservative extension of* $\Sigma + \mathrm{CI} + \mathrm{FUS}^1$.

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[50] Employing these examples, it can even be shown that:
If $\Sigma$ is $s2$-categorical and $Cn^{s2}(\Sigma)$ is maximal-consistent, then $Cn^{g2}(\Sigma)$ is maximal-consistent.
If $\Sigma$ is $s2$-categorical and $Cn^{g2}(\Sigma)$ is maximal-consistent, then $Cn^{s2}(\Sigma)$ is maximal-consistent.
are false.

[51] See Leivant (1994) and Simpson (1999) for more on this.

*Proof. First*, by Lemma 7(ii), $\Sigma + \mathrm{CI} + \mathrm{FUS}^1 \subseteq \Sigma^{s2}$.

*Second*, let $\varphi$ be a $\mathrm{L}^1[\circ]$-sentence and assume $\varphi \in \Sigma^{s2}$, i.e., (i) $\Sigma \cup \{\mathrm{O}, \text{FUS-Ax}\} \models^{s2} \varphi$; furthermore, assume (ii) $\langle M, \circ^M \rangle \models \Sigma + \mathrm{CI} + \mathrm{FUS}^1$: "$\langle M, \circ^M \rangle \models \varphi$" has to be shown.

Because of (ii) and Corollary 18, there is a complete Boolean algebra $\langle B, \leq^B, 0^B, 1^B \rangle$ such that

(iii) $\langle M, \circ^M \rangle \equiv \langle B^-, \circ^{B^-} \rangle$,

(iv) $\langle B^-, \circ^{B^-} \rangle \models \Sigma$.

By (iv), since $\Sigma$ is a set of $\mathrm{L}^1[\circ]$-sentences, we have $\langle B^-, \wp(B^-), \circ^{B^-} \rangle \models \Sigma$; and by Lemma 14, $\langle B^-, \wp(B^-), \circ^{B^-} \rangle \models$ FUS-Ax. Therefore, (i) yields that $\langle B^-, \wp(B^-), \circ^{B^-} \rangle \models \varphi$.

Now $\varphi$ is a $\mathrm{L}^1[\circ]$-sentence, whence also $\langle B^-, \circ^{B^-} \rangle \models \varphi$. Together with (iii), this implies the desired result.

**Corollary 30.** *If $\Sigma$ is a set of $\mathrm{L}^1[\circ]$-sentences, then $\Sigma_0^{g2}$, $\Sigma^{g2}$, $\Sigma^{g2+}$ and $\Sigma^{s2}$ contain the same sentences from $\mathrm{L}^1[\circ]$.*

*Proof.* By Theorem 29 and Lemma 7(ii).

I stop the investigation of the 2nd order extensions of $\mathrm{CI} + \mathrm{FUS}^1$ here. There are two natural next steps. The first would be an analysis of the 2nd order analogues of FCI, the $\mathrm{MCI}_{n+1}$ ($n \in \mathbb{N}$) and $\mathrm{MCI}_\infty + \mathrm{FUS}^1$. I do not have results comparable to the ones just presented for the 2nd order extensions of the $\mathrm{ACI}_{n+1} (n \in \mathbb{N})$ and of $\mathrm{ACI}_\infty$, however: what is missing are $\mathrm{L}^2[\circ]$-formulas "expressing" cardinalities similar to the ones stated in Definition 22. The other step would be to find out more about the relation between $(\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}, (\text{CountAt})\})^{s2}$ and, e.g., $(\mathrm{Ax}(\mathrm{ACI}_\infty) \cup \{\text{FUS-Ax}, (\text{CountAt})\})^{g2}$. That should be possible by employing the methods of Weese (cf. Weese (1989)); but this is beyond the scope of this paper.[52]

## 5. *Field and* the complete logic of the part/whole relation

To close this paper, let me return to one of the motivations for writing it—to the question how the expressions "the complete logic of the part/whole relation" and "the complete logic of Goodmanian sums" are understood in Field (1980) or could be understood.[53] Let it be clear from the outset that there are no explicit explanations (e.g., definitions) for these phrases in that book. Rather, Field simply *uses* them. It will therefore be no surprise that the subsequent thoughts about a proper Fieldian analysis of them will turn out to be rather conjectural.

[52] The conjecture would be that the first theory is much more complex from a recursion-theoretic viewpoint and much stronger with respect to relative interpretability than the second.

[53] Let me stress that I am not concerned with Field's later writings on this topic; see, e.g., Field (1989).

As a starter, here is a reminder how Field introduces the expressions "the complete logic of the part/whole relation" and "the complete logic of Goodmanian sums".

On p. 25 ff., Field presents a theory (of geometry) which he sometimes calls *Hilbert's formulation of the Euclidean theory of space and time*; let me abbreviate this by "H". H is formulated in a monadic 2nd order language $\mathrm{L}^2[B, \mathrm{Cong}]$ which has a vocabulary consisting of two non-logical predicates: the 3-place "$B$" ("$Bxyz$" is read "$y$ is between $x$ and $z$") and the 4-place "Cong" ("$\mathrm{Cong}(a, b, x, y)$" is read "the distance between $a$ and $b$ is identical with the distance between $x$ and $y$"). On p. 37 ff., Field addresses the problem of interpreting $\mathrm{L}^2[B, \mathrm{Cong}]$—which, being a 2nd order language, seems to be committed to sets—in a nominalistically admissible way. What he suggests is: interpret the 2nd order variables as having *regions* as their values. In Field's words (1980, p. 37):

(1) So we can regard the second-order quantifiers in Hilbert's theory as ranging over regions.[54]

It is a few lines after (1) that we encounter the first occurence of "the complete logic of the part/whole relation" and "the complete logic of Goodmanian sums" Field (1980, p.38):

(2) It [i.e., H] does, admittedly, have a logic that one might find objectionable:it involves what might be called the *the complete logic of the part/whole relation*, or *the complete logic of Goodmanian sums*, and this is not a recursively axiomatizable logic.

In what follows, I will concentrate on the phrase "the complete logic of the part/whole relation". My approach is to discuss its components "logic", "complete", "complete logic" and also "part/whole relation" in the hope that this will lead to an interpretation of the entire phrase.

To start with, it is remarkable that there seems to be no *explicans* of "logic" which is both precise, generally applicable and intuitively convincing and which, moreover, is widly accepted. "$x$ is a formal language" and "$y$ is a logical truth from $x$"—these are the predicates which are well-understood and which have received convincing explications (though opinions may differ as to which ones are most appropriate). But the word "logic"

[54] Field continues with

($*$) If we write Hilbert's theory in this way, then the quantifiers (both first-order and second-order) range *only* over regions of space.

Yet this is somewhat peculiar. *First*, the phrase "in this way" may be interpreted as referring to (1). But in (1), the question of how to *write* H is not addressed. Rather, it deals with how to *interpret* H or $\mathrm{L}^2[B, Cong]$. *Second*, the expression "If" beginning ($*$) may be understood as conveying that at a certain point, Field had envisaged writing a version of H which is not formulated in $\mathrm{L}^2[B, Cong]$; but this he never did in Field (1980).

For later use, here is a suggestion: translate $\mathrm{L}^2[B, Cong]$ *via* a mapping $\mathcal{I}$ into the 1st order language $\mathrm{L}^1[B, Cong, \circ]$ with the additional predicate "$\circ$" such that, among others, $\mathcal{I}(\ulcorner\forall x\psi\urcorner) = \ulcorner\forall x\ (At(x) \to \mathcal{I}(\psi))\urcorner$ and $\mathcal{I}(\ulcorner\forall X\psi\urcorner) = \ulcorner\forall x\, \mathcal{I}(\psi)\urcorner$ (alternatively, the predicate "region" could belong to the 1st order language, and $\ulcorner\forall X\psi\urcorner$ could be translated as $\ulcorner\forall x\, region(x) \to \mathcal{I}(\psi))\urcorner$). As applied to the theory H, this leads to a 1st order theory $\mathrm{H}^1$ which really is sort of a rewritten variant of H.

typically remains simply unexplained. Moreover, if it is understood at all, it is usually vague or ambiguous. Thus, given a formal language L, for example, it may be understood as the set of logical truths from L, or as that language itself (for example, its vocabulary or its set of formulas or its set of formation rules); it could be a consequence relation, in particular one which has the set of logical truths (of L) as the set of consequences of the empty set. Construed more liberally, a logic could be an arbitrary class of structures (of a signature appropriate to L) and, therefore, a theory in L which does not contain only logical truths. Finally, it may be a complex of the items just suggested—or something else.

Since "logic" is also not explained in Field (1980) it can only be guessed which of these conceptions Field had in mind when he employed the words "the complete logic of the part/whole relation". Yet, I think that his additional claim in (2), i.e., that it "is not a recursively axiomatizable logic", sheds some light on his understanding. For of all the interpretations of "logic" mentioned above, the one that takes a logic to be a set of sentences is that where it makes most sense to ask if a logic could be recursively axiomatizable. In this case, the first choice for a or the logic relative to $\mathrm{L}^2[B, \mathrm{Cong}]$ would be the set of logical truths which are formulated in $\mathrm{L}^2[B, \mathrm{Cong}]$. But recall that we have at least two ways to define this set: as "the set $PL^{g2}$ of all $\mathrm{L}^2[B, \mathrm{Cong}]$-sentences true in all $g2$-structures appropriate to $\mathrm{L}^2[B, \mathrm{Cong}]$", but also as "the set $PL^{s2}$ of all $\mathrm{L}^2[B, \mathrm{Cong}]$-sentences true in all $s2$-structures appropriate to $\mathrm{L}^2[B, \mathrm{Cong}]$".

With this as a background, let me now turn to "complete". I think that it is most plausible to reason as follows: By the completeness theorem for 2nd order logic provided with $g2$-structures, $PL^{g2}$ *is* recursively enumerable. Thus, when it is construed as the set of logical truths (in $\mathrm{L}^2[B, \mathrm{Cong}]$), *the complete logic of the part/whole relation* should be regarded rather as $PL^{s2}$ than as $PL^{g2}$. Yet, if it is not the recursive enumerability but the completeness which is emphasized, the opposite is suggested: the completeness theorem we have for $PL^{g2}$, but not for $PL^{s2}$. Now, *the complete logic of the part/whole relation* should be regarded rather as $PL^{g2}$ than as $PL^{s2}$.

Is there a sense in which $PL^{s2}$ is, but $PL^{g2}$ fails to be complete? The one idea[55] that comes to my mind is: $PL^{s2}$ contains each instance of Comp$^2$—the "full" or "complete" 2nd order comprehension schema—but $PL^{g2}$ doesn't.

These considerations on the phrase "the complete logic of the part/whole relation", however, share the weakness of ignoring its "part/whole relation"-part. For whatever "logic" might mean, I take it that a logic is intimately connected with a language or a vocabulary; and Field tells us that H has or involves the complete logic of *the part/whole relation*. This, then, suggests that (2) should have the following consequence:

(A) The part-of relation is expressible in H's language, i.e., in $\mathrm{L}^2[B, \mathrm{Cong}]$.

Yet, as far as I can tell, this is simply not the case:[56] $\mathrm{L}^2[B, Cong]$ does not contain "$\sqsubseteq$"; it is far from evident whether "$\sqsubseteq$" could be defined in an appropriate way in H;

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[55] $PL^{s2}$ may be claimed to be complete in the sense that it contains each logical truth (of $\mathrm{L}^2[B, \mathrm{Cong}]$), but the same could be said about $PL^{g2}$: that depends, of course, on what one understands by "logical truth". Furthermore, $PL^{g2}$ and $PL^{s2}$ fail to be maximal-consistent and are therefore, in this sense, both incomplete.

[56] Note that the theory $\mathrm{H}^1$ suggested above *is* stated in a language in which "$\sqsubseteq$" is expressible. But there we have a crucial difference between H and $\mathrm{H}^1$.

and at this point of his book, Field does not show any interest in such a definition or in enlarging the vocabulary of $\mathrm{L}^2[B, Cong]$ by "$\sqsubseteq$".[57] Whatever the precise meaning of (2) may be, it seems to be trivially false under this interpretation of Field's text.

In principle, I see two ways out of the difficulty addressed with (A). The first way is to deal with an extension of $\mathrm{L}^2[B, \mathrm{Cong}]$ by "$\sqsubseteq$" or, say, "$\circ$", instead of $\mathrm{L}^2[B, \mathrm{Cong}]$; the second way is to replace (A) by the requirement that the part-of relation should by expressible in a metalanguage of $\mathrm{L}^2[B, \mathrm{Cong}]$. Now, it may well be that Field adopted one of these strategies in Field (1980). Here are some details.

*First strategy:*

In a later passage from Field (1980, p.99), Field actually takes $\mathrm{L}^2[B, \mathrm{Cong}]$ to be extended by "$\circ$" to a monadic 2nd order language (which I call "$\mathrm{L}^2$"). When asked what the *complete logic of the part/whole relation* could be with respect to $\mathrm{L}^2$, however, we still do not have a definite answer. Similarly to the case $\mathrm{L}^2[B, \mathrm{Cong}]$, it could be $PL^{g2*}$, the set of all $\mathrm{L}^2$-sentences true in all $g2$-structures appropriate to $\mathrm{L}^2$, but also $PL^{s2*}$, the set of all $\mathrm{L}^2$-sentences true in all $s2$-structures appropriate to $\mathrm{L}^2$. Now surely both $PL^{g2*}$ and $PL^{s2*}$ contain sentences involving "$\circ$" or "$\sqsubseteq$"; but it is not at all settled that any of these sets should be regarded as the complete logic of the *part/whole relation*. For in none of these sets, "$\circ$" or "$\sqsubseteq$" plays an assential role: in particular, $PL^{g2*}$ and $PL^{s2*}$ do not suggest, let alone determine, that "$\sqsubseteq$" should be read or interpreted as *part-of.*

As a matter of fact, now that we have "$\circ$" as part of the formal language, other options for an explication of "the complete logic of the part/whole relation" can be taken into account.[58] The ones presented here are inspired by occurences of "completeness" which can be found in arithmetical contexts.

(i) It is not uncommon to use the predicate "the complete arithmetic" for the set of all sentences from $\mathrm{L}^2[\mathrm{PA}]$ which are "true in the standard model". Similarly, *the complete logic of the part/whole relation* could be understood as the set of all truths in $\mathrm{L}^2$ about the part-whole relation; let's call it $\mathrm{Th}(\sqsubseteq)$. I think that this suggestion, however, has a rather obvious shortcoming: $\mathrm{Th}(\sqsubseteq)$ is not well-defined; I have not the slightest idea what it could be and doubt that anybody else has; and calculi of individuals were not invented for having just "one intended model".

(ii) Sometimes, the induction schema from $\mathrm{L}^2[\mathrm{PA}]$ is regarded as expressing not the *full* induction principle, but only its 1st order approximation. Full induction is supposed to be formalized as an induction axiom in $\mathrm{L}^2[\mathrm{PA}]$. Transferred to our situation, this suggests to accept FUS-Ax instead of $\mathrm{FUS}^1$ when we want to have the full—or: *complete—logic of Goodmanian sums*.

Let me sum up: When construed as a set of sentences in $\mathrm{L}^2[B, \mathrm{Cong}]$, the *complete logic of the part/whole relation* may be $PL^{s2}$ or $PL^{g2}$; but in this case, (A) is violated. When regarded as a set of sentences in $\mathrm{L}^2$, the *complete logic of the part/whole relation* may be $PL^{s2*}$ or $PL^{g2*}$, but also, for example, the set of truths (in $\mathrm{L}^2$) about the part-of-relation or a theory (in $\mathrm{L}^2$) which contains FUS-Ax.

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[57] Field mentions that apart from "$B$" and "$Cong$", further expressions may belong to $\mathrm{L}^2[B, Cong]$: e.g., "is a point". But he nowhere envisages "$\circ$" or "$\sqsubseteq$" to be one of them.

[58] Note that they are not applicable to $\mathrm{L}^2[B, \mathrm{Cong}]$, since "$\circ$" is missing in that language.

This last-mentioned option points to a different approach to (2), which I will briefly address before presenting the second strategy. Its main idea is that instead of asking what "the complete logic of the part/whole relation" could mean, one should try to understand the full

($*$) H involves the *the complete logic of the part/whole relation*.

As I see it, the considerations just put forward are useful here, too. Drawing on them, here are some suggestions for an interpretation of ($*$): H is closed under $s2$-consequence;[59] H contains FUS-Ax; H contains each instance of Comp$^2$; H contains each truth (in L$^2$) about the part-of-relation.

Thus, here again (2) seems to be compatible with several nonequivalent interpretations. But let me finally come to the second strategy and, herewith, to a new perspective.

*Second strategy:*

The starting point is the thought that perhaps all that Field wanted to convey with (2) is that he preferred to interpret L$^2[B, \mathrm{Cong}]$ by using only $s2$-structures. As a matter of fact, already before he introduced the problematic expressions discussed here, he had considered just $s2$-structures as possible models for L$^2[B, \mathrm{Cong}]$: see Field (1980, pp. 25–26). Furthermore, throughout his entire book, Field wants the 2nd order theories he deals with to be categorical; and this he will no get if he interprets them *via* arbitrary $g2$-structures.

Once more, this reading of Field (1980) has an obvious weakness: Interpreting a language L only *via* $s2$-structures has nothing specific to do with the part-of relation. *First*, a language L which is interpreted solely *via* $s2$-structures need not contain means to express *part-of*.[60] *Second*, even L$^2$ need not be interpreted admitting only $s2$-structures: each $g2$-structure of the right signature *could* be allowed as a possible model for L$^2$.

So where does an interplay between $s2$-structures and the part-of-relation enter the stage? In order to find out, let's go back to (1) and ($*$), i.e., to the context in Field (1980) preceeding the introduction of the "the complete logic of the part/whole relation"-talk. At this point, it may be that Field employs the *complete logic of the part/whole relation* as a means to specify "$x$ is a region". But as I have already pointed out, this formula is not formulated in L$^2[B, \mathrm{Cong}]$. Rather, the predicate "is a region" belongs to some metalanguage ML of L$^2[B, \mathrm{Cong}]$: it is used to apply to structures in which expressions from L$^2[B, Cong]$ can be evaluated, but *it* is not evaluated in such structures.

Since I understand Field's use of the predicate "is a region" as being exchangeable by "is a part of", I now read (2) as implying

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[59] Under this interpretation of (2), I take it to be inappropriate to construe "the complete logic of the part/whole relation is not recursively axiomatizable" as "the set of logical truths (of one of the above mentioned languages L) is not recursively enumerable". Rather, I prefer to think of it as being adequately explained by "the of sentences from L which are consequences of $\Sigma$ is not recursively enumerable in $\Sigma$" (where $\Sigma$ is a suitable set of sentences from L). Now since $Cn^{g2}(\Sigma)$ is always recursively enumerable in $\Sigma$, closure of H under $g2$-consequence would not be relevant.

[60] If L = L$^2[B, \mathrm{Cong}]$, this objection is reminiscent of (A).

(B) The part-of relation is expressible in a metalanguage ML of H's language,

—and (A) no longer poses a threat.

Statements such as (1) make it, moreover, plausible that ML should be regarded as a semantical metalanguage of $\mathrm{L}^2[B, \mathrm{Cong}]$: to ML, sentences should belong which express that certain expressions of $\mathrm{L}^2[B, \mathrm{Cong}]$ are interpreted by certain entities. Now, typically it is set theoretical languages which are used for this role. After all, in order to get the semantic machinery going, one is used to employ set theory (at the metalevel) as a means to obtain sets as extensions of predicates and to define the satisfaction relation. In particular, phrases such as "value-assignment $\beta$ assigns the set $\beta(X)$ to the 2nd order variable $X$" are meaningful in this context. But for Field, these formulations are to be avoided. Rather, in ML, it has to be possible to formulate sentences like "value-assignment $\beta$ assigns the region $\beta(X)$ to the 2nd order variable $X$": think of (1)—this is just the crucial idea.

Let me emphasize that from this point of view, the possible models for $\mathrm{L}^2[B, \mathrm{Cong}]$ are, strictly speaking, neither $s2$- nor $g2$-structures. Rather, they are of the form $\langle A, \Lambda, \mathcal{I}\rangle$, where $\Lambda$ is the set of *all* regions in $A$ or, alternatively, a *nonempty subset* of the set of all regions in $A$. This leads to structures which may be called $n$-$s2$- and, alternatively, $n$-$g2$-structures. Now, (2) can be understood as expressing that Field preferred to interpret $\mathrm{L}^2[B, \mathrm{Cong}]$ by using only $n$-$s2$-structures.

At first sight, this may seem to be a far-fetched interpretation of Field's remarks on "the complete logic of the part/whole relation". From a systematic point of view, though, it has its strenghts: it solves the problem given by (A); it fits to claims like (1); and it delivers the categoricity results Field aims at, but is not just the (trivial) commitment to a $s2$-semantics.

When it comes to the textual evidence provided by Field (1980), the distinction between what I have called $\mathrm{L}^2[B, \mathrm{Cong}]$ and ML is, as far as I can tell, not explicitly made.[61] It is therefore difficult to pin down whether (A) or (B) is closer to Field's intentions. What may be more important, though, is that the second strategy also does not really give an answer as to what *the complete logic of the part/whole relation* could be. On the one side, if it is a set $\Sigma$ of sentences from $\mathrm{L}^2[B, \mathrm{Cong}]$, presumably it is one which has a $n$-$s2$-model. But this is compatible, for example, with all of the suggestions put forward for such a $\Sigma$ at the end of the first strategy. On the other side, if it is a set of sentences from ML, Field (1980) does not give the slightest hint which one. And if *the complete logic of the part/whole relation* is not set of sentences at all, I am completely lost.

With these—admittedly rather inconclusive—thoughts I close the section on Field (1980). Although I still have no convincing interpretation of "the complete logic of the part/whole relation" and "the complete logic of Goodmanian sums" at hand, I do not view this section as a failure. For it should have made clear enough that different—nonequivalent—interpretations of these phrases are admitted by Field's book, and the

[61] That is, almost nothing can be found concerning the specification of ML. We may assume that its vocabulary contains "$\sqsubseteq$". But beyond this, there is, in particular, no information whether ML is a 2nd order language or whether it contains set-theoretical vocabulary. In addition, it may be wondered how theories formulated in ML could be suitable as semantic metatheories for $\mathrm{L}^2[B, \mathrm{Cong}]$ and H.

metalogical concepts and results from the previous sections should have helped in understanding that and how these interpretations differ from each other.

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