On The Plurality of Parts of Classes

Daniel Nolan University of Notre Dame

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The ontological picture underpinning Lewis's *Parts of Classes* (Lewis 1991) has some unusual features. It posits many, many simple, abstract objects that serve to be the subject-matter of set theory. (We require so many, as Lewis points out, since standard set theory is committed to so many sets.) However, when we put the ontology posited by *Parts of Classes* together with the doctrines of Lewis's *On the Plurality of Worlds* (Lewis 1986), two problems surface. The first, to do with the relationship between sets and possible worlds, is perhaps a drawback, but is a result a Lewisian could comfortably accept. However the second problem, concerning how to integrate this ontology with Lewis's understanding of possible worlds, may look more like an inconsistency, though I will argue that we can interpret Lewis consistently here. The second tension is a more serious problem in the combination of Lewis's views, unless it is dealt with. There are two ways to resolve this second tension, each of which goes beyond what Lewis explicitly says in interesting ways. I think Lewis would have been best off extending his system in the second way I will suggest: and indeed, there is some textual evidence that he may have been tempted to extend it in this way as well. This gives Lewis an additional reason to embrace a proper class of worlds and *possibilia*, over and above others explored in the literature.

Lewis's central conjecture in *Parts of Classes* is that "the parts of classes are all and only their subclasses". By "class" Lewis meant things with members: the empty *set* was excluded from Lewis's use of "class", and while he counted all other sets as classes, he also defended the view that there are some classes that were not sets (so-called *proper classes*). From his central conjecture, and the exclusion of the empty set, it follows that unit classes (i.e. classes with exactly one member) are atomic, lacking proper parts altogether. (All of the other classes are fusions of these unit classes.) How many unit classes are there, according to Lewis? As many as there are sets at all, since each set belongs to a unit-class. (For Lewis, proper classes are distinguished by not belonging to classes.) There are thus "proper-class many" atoms in the ontology of Lewis 1991, since there are more sets than the cardinality of any set whatsoever.

1. Wholly Impossible Atoms

Put this together with the commitments of Lewis 1986, and the first problem for the view emerges. Lewis 1986 is committed to there only being a set of possible worlds and a set of possible objects (p 104), so almost all the atoms postulated to be the unit sets in Lewis 1991 must lie outside the possible worlds, in the sense of not being part of those worlds.[1] (A proper class of objects, minus a set of objects, leaves a proper class of objects.) Any objects that exist outside all of the possible worlds must be *impossible*: to be a possible object is to be part of a possible world, according to Lewis. Is this inconsistent, to be committed both to the claim that certain things exist and that it is impossible that such things exist? Not according to Lewis's system. Lewis already admits that there are entities that do not exist in any possible world: since he accepts unrestricted mereological composition, he accepts that objects in different worlds make up fusions that cannot be entirely found in any single world, and that in a sense these trans-world fusions "cannot possibly exist" (Lewis 1986 p 211). That is, no single possible world is a witness to their existence, and there is no world that they are a proper part of (as opposed to parts of them being parts of worlds). So it is not inconsistent for Lewis, given this sense of "possible", to say that there exist objects that do not possibly exist.

However, the atoms postulated for the purposes of mathematics by Lewis 1991 are arguably in a worse position than trans-world fusions. At least the fusions resolve into parts, each of which is part of a world: and the aggregate of all worlds has all of them as parts. The trans-world fusions are "in" the world*s*, at least collectively. However, these atoms postulated by Lewis 1991 must be "completely impossible", as I have put it elsewhere (Nolan 2002 p 156 ftnt 9). They are not parts of any world, and no part of them is part of any world either. (It may well be that there is a different "advanced modalising" sense of possible, in which every existence claim that is true is possibly true—see Lewis 1986 p 6, Divers 1999 p 229–230—but I do not want to wade into any debates about how this might be best understood here.) No doubt Lewis could stipulate that these atoms are possible in some sense, and perhaps intends to with his talk of sets existing "from the standpoint of a world": see Lewis 1983 pp

1 Lewis distinguishes three ways of "being in" a possible world in Lewis 1983 pp 39-40., and 1986 p 96 adds a fourth way involving counterparts that need not concern us here. The first is to be part of the world in question; the second to be partially in a world (i.e. to share a part with a world); and the third is to exist "from the standpoint of a world": in effect, to be one of the things that an inhabitant of a world that shared our ordinary way of talking might correctly talk about as existing. This third way of "being in" was intended, in 1983, to include sets or properties that might not count as part of a given world, such as e.g. the pure sets. Even if we are licensed as counting some entities as being "in" our world without overlapping it, my focus in this paper is on what objects are parts of worlds. For the sake of an idiomatic discussion I will talk as if all and only parts of worlds are "in" those worlds, unless indicated otherwise, though my point concerns Lewis's commitments about what are parts of the possible worlds.

39-40 and footnote 1 above. That would not stop them failing to be possible in the way that possible objects are typically possible in Lewis's system, and would not stop them sharing the "impossibility" that Lewis admits trans-world fusions have. It strikes me that it would be better to have a non-disjunctive account of possibility in terms of possible worlds, if such a thing could be had.[2]

Lewis does not tell us much about these atoms. We do not have answers about what intrinsic nature they have, if any, or what relations they may stand in, if any (Lewis 1991 pp 31–35, 142–3): only that they are the singleton sets of other entities (either individuals or other classes).[3] If we move to the structuralist understanding of set theory set out in the Appendix of Lewis 1991, or in Lewis 1993, then we do not even require that our atoms stand in a distinctive singleton relation. The demand that there be proper-class-many of them, while there are only set-many objects in the possible worlds will remain, however, so this aspect of his view will require that nearly all the atoms postulated will be "completely impossible" in the sense above even when we move to Lewis's structuralist framework.

Another feature of Lewis's system ensures that *all* the atoms needed to be classes are disjoint from the possible worlds, whether or not we move to the structuralism mentioned above. Possible worlds and their contents are treated as *individuals* in Lewis's system: that is, they are the ur-elements and do not themselves have members (Lewis 1986 p 83). Lewis furthermore insists on a *Priority Thesis* (Lewis 1991 p 7): that no class is part of any individual. So in particular, no class can be a part of any possible world. So we are left with the result that there must be proper-class many atoms outside all of the possible worlds, serving as the ontology of class theory even if we go structuralist about the

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2 One referee has suggested that Lewis better preserves the mathematical platonist intuition that numbers, sets, and other such mathematical objects are not parts of concrete reality and are not found in space and time, by holding that they are all disjoint with his possible worlds. That will be an advantage for some of Lewis's way of going. But for other mathematical platonists, it may seem like an undesirable upshot of his attempt to account for possibility with alternative *concrete* cosmoi, when reality contains a non-concrete aspect as well.

3 Lewis also says that we do not know if they have locations, and indeed "haven't a clue" whether they do (Lewis 1991 p 33). This marks a departure from the view he expresses in Lewis 1986 pp 94-96 that sets are located where their members are.

If classes do have spatiotemporal locations, that would make them worldmates with individuals, at least in worlds like ours, and so parts of possible worlds: so given his *Priority Thesis*, that no class is part of any individual, some or all of the possible worlds would fail to be individuals (having parts that are classes). He is also committed to all possible worlds being individuals (Lewis 1986 p 83), which leaves his views in conflict. (At least unless he concedes he does not have a clue whether his own theory is correct.) His own views, by the time of Lewis 1991, committed him to denying sets spatiotemporal location. His implicit commitment to nearly all the singletons being outside all the worlds also requires that most of them lack spatiotemporal locations. I think the Lewis of 1991 would be well-advised to renounce his scepticism about the location of singletons, and instead admit that they all *lack* spatiotemporal location. A more contemporary Lewisian tempted by the more radical revisions suggested towards the end of this paper may wish to revise that commitment again, however.

relationship between those atoms and the entities that they are the singleton-sets of.

2. Are The Mathematical Atoms Worlds After All?

The second problem to be addressed in this paper emerges when we come to consider which things count as worlds. Given the letter of Lewis 1986, it might seem that these atoms must be parts of worlds after all. Lewis defines a *worldmate* relation: his first pass is to say "things are worldmates iff they are spatiotemporally related" (p 71), and then extends this to include as worldmates entities that are "*analogically* spatiotemporal" (pp 75–76), to handle alien possibilities where the connections between entities are not the actual, familiar, spatiotemporal relations. Lewis also says that a world "is a maximal sum: anything that is a worldmate of any part of it is itself a part" (p 69). Furthermore, it is clear from context that these are the only parts of worlds, and nothing further is required to be a world than to be such a maximal sum, since he has taken himself to have given "the unity relation for possible worlds" (p 70).[4]

Lewis also accepts unrestricted composition: for any entities, there is a sum of those entities. (Lewis 1986 p 201, Lewis 1991 p 74). Now, consider two cases for each of these allegedly beyond-wordly mathematical atoms. Either it has some worldmates, or it has no worldmates. In the first case, there will be a sum of it and its worldmates (and their worldmates, etc.), and so it is part of a possible world. In the second case, it has no worldmates: therefore the sum of it alone satisfies the condition "anything that is a worldmate of any part of it is itself a part". It is degenerately maximal under the worldmate relation in this way. So it is a possible world all by itself, and so part of a world (an improper part of itself). But, as pointed out above, these atoms must not be parts of any possible world. We have reached a contradiction.

Let us deal with this apparent inconsistency first. One potential repair is obvious: instead of understanding maximal interrelation in the manner presupposed by the previous paragraph, Lewis could insist that possible worlds are *non-degenerately* maximally interrelated by spatiotemporal, or analogously spatiotemporal, external relations. For example, he could say that every world *w* is a sum with at least one part, and *w* includes all the worldmates of that part, *and that its parts all have worldmates*. A single atom not standing in worldmate relations to anything would not count as a world

4 This account of worldmates would have to be modified were Lewis to accept the existence of immanent universals, as he points out in Lewis 1986 on p 69 and especially on p 208-209.

on this revised definition. This is the natural way to understand the spirit of specifying things as "worldmates": if something is not even its own worldmate, plausibly it is not in any world. I expect Lewis intended that everything that was part of a possible world would stand in spatiotemporal relations, or at least analogously spatiotemporal ones, and that this is how we should read his definition of a possible world.

This definition of worlds need not even rule out worlds of a single mereological atom, since it may be that atoms stand in spatiotemporal relations or analogously spatiotemporal relations *to themselves*. On this proposal it is not trivial that everything is its own worldmate: but nevertheless things that stand in the right kinds of relations to themselves can be their own worldmates. We would need to draw a distinction between being zero distance from oneself and not being in any spatiotemporal relationship to oneself at all, if we wanted some atoms to be their own worldmates but some (indeed, most) to not be: but we should probably want to draw this distinction in any case, if we are to allow it is coherent for something to not be in space and time, since such a thing is not located at all, and so not co-located with itself.

Avoiding the contradiction in this fashion, however, does have an unwelcome consequence for Lewis's system. It will rule out as a possibility that there could be an individual object that did not stand in spatiotemporal relationships, (or analogously spatiotemporal relationships) to anything. On the face of it, there does not seem to be anything metaphysically necessary about there being spatiotemporality. Why couldn't there just be an electron on its own, with charge, spin, but no spatiotemporal features? You might reply that electrons are essentially spatiotemporal, so it would have to have location and perhaps duration. But what about some radically different kind of individual, existing by itself, not in space and time? It does not seem to be essential to being a non-class that something is in space or time (or is in relations analogous to spatio-temporal ones.) There is nothing, on the face of it, incoherent about such a scenario, yet if it does not occur in any possible world, by Lewis's standards it is not possible, at least in the "ordinary" sense.

Counting something which is apparently possible, in the standard sense of metaphysically possible, as being impossible is a mark against this version of Lewis's theory. However, this problem is similar to other kinds of marks against Lewis's theory: Lewis also cannot allow that there could be nothing concrete, and cannot allow, in the ordinary sense of possible, that it is possible for there to be co-existing objects that are not spatiotemporally related to each other (and not analogously-

spatiotemporally related to each other). In each of these other cases Lewis bites the bullet, allowing that these apparent possibilities are not indeed possible, and he considers these as costs worth paying for the attractions of his theory (Lewis 1986 p 71–74). So a Lewisian who refused to countenance the possibility of an entity not standing in spatiotemporal or analogously spatiotemporal relations, even to itself or its own parts, would probably bite the bullet on this in a similar way.

One option for Lewis here, suggested by a referee, would be to allow that atoms standing in no relations could be their own worldmates, but to put a constraint on the worldmate relation so that *non-individual* atoms (i.e. singletons) never counted as their own worldmates. A featureless individual could then be possible, without all the singletons being their own worldmates and thus their own possible worlds. I would be uncomfortable with solving the problem through redefinition like this without an explanation of *why* it makes a difference to the metaphysics of possibility whether a featureless atom is a class or not, though other's tastes may differ. At any rate, I think this sort of solution will be difficult to plausibly implement were we to move to the structuralist approach preferred by Lewis in Lewis 1993, where there would be no intrinsic difference (or difference in natural external relations) between the featureless atoms that played the structural role of singletons and those which would not. While the referee is right that there is an option here, I will turn to revisions I find more satisfying.

3. A Natural Resolution: Worlds Form a Proper Class

Ruling out the mathematical simples as counting as worlds, or indeed being in worlds at all, also retains the strikingly implausible feature of the *Parts of Classes* system mentioned above. Since each of these atoms is not a part of any possible world, it remains completely impossible. The other way of responding to the question of whether these atoms postulated to be the singletons are worlds would be to embrace the claim that each *is* a possible world after all, and that when a thing is not a worldmate of anything else, it is a possible world all by itself. To modify his views in this way, Lewis would need to drop the claim that the possible worlds form a set, and that the possible objects form a set. Once that is done, we can allow that there are proper-class many atomic *possible* worlds, alongside all of the worlds embraced in Lewis 1986. These possible worlds can then serve as the ontology for mathematics. There is now no need to say that those objects are absolutely impossible, since they are just additional possible worlds. As a bonus, we can now recognise as a genuine possibility that something exists

without being spatiotemporally related to anything (nor standing in a relation analogous to spatiotemporal ones). Lewis would need to answer "yes" to the question of whether there are indistinguishable possible worlds, if nearly all of them are featureless atoms, and this was a question he wished to stay neutral on: but giving up neutrality for a good theoretical reason does not seem like a cost.[5]

We face some choices about whether to treat every possible world as an individual. (That is, in this context, a member of a class that is not itself a class.) On the current proposal some are and some are not. If we did want all possible worlds to be individuals, while insisting that all the atoms serving as singletons were in worlds, we could instead adopt a position where some, or all, possible worlds had individual parts and singleton parts. (This would require that some "mixed fusions" of classes and individuals were themselves individuals, contrary to the letter of Lewis 1991 pp 7–8 and Lewis's *Priority Thesis*, but the modification makes little difference to the overall system.) We would also want to tweak Lewis's definition of the null set (pp 10–15) to continue to ensure that it had no classes as parts: perhaps by making it the fusion of all atomic individuals. Further choices may have to be made: does every possible world contain classes? Does each contain all of them (perhaps through trans-world identity), or is the mathematical universe spread out amongst them? These are theoretical choices we can leave to partisans of this kind of view, should there ever be any.

A more radical option also becomes available, once we no longer need proper-class many atoms outside the possible worlds. Instead of accepting the existence of proper-class many additional atoms, whether within or outside of worlds, we could instead allow the more usual inhabitants of possible worlds to provide the material for mathematics, provided only that there are enough of them. If there are proper class many possible electrons, for example, a variation on the structuralism of the appendix of Lewis 1991 or of Lewis 1993 can be employed to let them be the ontology of mathematics, while also preserving their role as individuals (i.e. ur-elements of sets). I have explored one way of developing a view like this, with different motivations: see Nolan 2002 chapter 7 and appendix, and Nolan 2004, Schwarz 2005 and Cowling 2017 ch 7 offer introductions and some philosophical motivations for the system. This way of developing a megethological system requires minor

5 Divers 1994 argues that a Lewisian should reject indistinguishable possible worlds, largely on the grounds of quantitative parsimony. Parsimony arguments are at their strongest when theories are equal, or nearly equal, in other respects. But if a Lewisian theory with many duplicate featureless worlds provides an ontology for mathematics without "completely impossible" objects, while its rival requires nearly all the entities committed to to lie entirely outside the possible worlds, then the former theory plausibly has a theoretical advantage that outweighs any cost in parsimony: especially if the latter theory is arguably just as unparsimonious, only about the number of entities outside possible worlds.

modifications to two of Lewis's principles used to develop his Parts of Classes framework: both the *Division Thesis* and the *Fusion Thesis* must be tweaked. (Nolan 2002 pp 162–3, 195–200 on the Division Thesis, and pp 165–169 on the *Fusion Thesis*). The *Fusion Thesis*, that every fusion of individuals is itself an individual, needs to be given up in any case as soon as we have a proper class of individual atoms, unrestricted composition, and global choice (Nolan 2002 p 169), so it would be very natural to restrict the *Fusion Thesis* in a setting like this in any case. Since my revisions require the use only of ontology found in possible worlds, the question of what to do with the proper-class-many mysterious atoms lying outside all the possible worlds evaporates, since the system no longer needs them.

Note that Lewis himself may have had some sympathies for this revision to his system. In Lewis 2002 Lewis says the case for postulating proper-class many possibilia such as electrons is “fairly persuasive” (p 8). If he endorsed that change, he would be able to accommodate all of his mathematical ontology within possible worlds after all. And given that he ended up endorsing a structuralist conception of the relationship between individuals and sets (Lewis 1993), he would have been able to have an ontology and ideology of mathematics that required no more than commitments he had already incurred in his theory of possible worlds.

Moving to a proper class of possible objects, and perhaps with it a proper class of possible worlds, would have some disadvantages as well, as Nolan 1996 pp 249-51 points out. Proper classes are not members of sets, so one has to be careful employing set-theoretic constructions out of possible objects or possible worlds for other purposes. Natural language semantics in the possible worlds tradition helps itself to functions from all sorts of classes that may well turn out to be proper classes on this proposal (see Partee 1988 for a classic introduction), and pressing classes of possible individuals into service in metaphysics (in the style e.g. of Montague 1968) will also face problems. Lewis 2002 pp 8-10 discusses some of the moves that might need to be made in the face of this challenge.

There are many options available to those tempted to operate with a proper class of possible objects. Some are canvassed by Nolan 1996 pp 249-153. Another option is to reconceive the task of possible worlds semantics as not providing the once-and-for-all semantic values of expressions, but just to be providing models of semantic values that have some perspicuous connections to the meanings of expressions. We can offer set-sized models with a set of "worlds" and a set of "possible objects" that can display e.g. systematic connections between the semantic values of simple and progressive tenses,

even if in reality there are more than set-many possible completed bakings of cakes and more than set-many possible bakings of cakes in progress. Operating as if semantic values can be modeled straightforwardly in set theory can be productive, even if there are foundational issues lurking about what these set-sized models have to do with modal space and the "real" semantic values of expressions, whatever those might be. The project of possible worlds semantics, as traditionally conceived, does not need to grind to a halt even if the models semanticists are working with are more limited than they might have realised.

Bringing out the tension in the ontologies of Lewis 1986 and Lewis 1991 is no mere pedantry. Resolving the tension between the two works provides us with another motivation to endorse a proper class of possible worlds and possible individuals, besides those suggested by Nolan 2002. (Nolan 2002 argues that moving to a proper class of worlds and individuals gives the modal realist a more satisfactory principle of recombination and an appealing alternative to the Parts of Classes machinery for class theory.) A modal realist who wishes to resist this resolution owes us an account of why it is an acceptable cost of her theory to deny that atomic possibilities of the sort described above are genuine possibilities, and why it is worth postulating “entirely impossible” ontology, i.e. objects that not only do not exist in worlds but which do not divide into parts which exist in worlds. Without motivating these bullet-bitings, a modal realist who resists a proper class of possible individuals would seem to be settling for second-best modal realism.[6]

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