SPACE, TIME AND PARSIMONY

Daniel Nolan
University of Notre Dame
The final version of this paper will be forthcoming in *Noûs.* Please cite that version when it is available.

Abstract

This paper argues that all of the standard theories about the divisions of space and time can benefit from, and may need to rely on, parsimony considerations. More specifically, whether spacetime is discrete, gunky or pointy, there are wildly unparsimonious rivals to standard accounts that need to be resisted by proponents of those accounts, and only parsimony considerations offer a natural way of doing that resisting. Furthermore, quantitative parsimony considerations appear to be needed in many of these cases.

When scientists evaluate scientific theories, they often have to rely on many criteria besides whether the relevant theories are consistent with the evidence so far. For one thing, it is harder than it looks to have evidence flatly contradict hypotheses, especially when we remember how easy it is to chalk recalcitrant results up to noise, or experimental error, or bias, or as-yet-undiscovered factors, and the fact that our theories often only have probabilistic connections to data. For another, it is relatively easy to come up with different theories that agree on any given body of data, though many will look like mere philosophers' tricks rather than serious scientific rivals. Finally, we need theories to help tell us about tricky cases that we have not observed so far: and when we are setting up experiments or observations we are not already sure about, we want to put extra effort into testing plausible theories of the new phenomena, rather than doing undirected data collection. But this suggests that we want criteria of plausibility that help choose between theories that have not already been ruled out by the evidence we already have. Scientists do seem to use criteria such as simplicity, coherence with theories in other areas, inductive considerations, valuing explanatory hypotheses, and so on. Call criteria of theory choice like these "theoretical virtues".[1]

On one conception of philosophical theorising, one which I share, evaluation of philosophical theories also involves appeal to theoretical virtues, at least implicitly. My view is that the basic theoretical virtues appealed to in philosophy are the same as the ones appealed to in the

[1] This usage of "theoretical virtues" may be a little wider than others, who may wish to e.g. exclude inductive methods. I doubt it will matter exactly where we draw the boundaries for the purposes of the current discussion.

sciences, and indeed the process in philosophy is often not too different in principle to the theoretical end of research in the natural and social sciences: and this includes the requirement that our philosophical theories cohere in the appropriate way with what the natural and social sciences tell us about the world around us and about ourselves. Even those who do not agree that there is a basic unity of method should agree, I think, that there are some important parallels and similarities between some criteria we should use in philosophy and criteria used in the sciences.

The question of exactly which criteria should be used in scientific and philosophical theorising remains, of course, as does the question of how important those criteria should be. In this paper, I want to focus on one often-mentioned criteria, that of theoretical parsimony, and to examine how it should play a role, and plausibly is already playing an implicit role, in the reasoning behind some prominent positions in the philosophy of space, time and spacetime. My focus will be on parsimony considerations about the *structure* of space and time, and in particular what divisions spacetime comes in. When we come to questions about how divisible space and time are, all of the standard views gain some support from parsimony considerations, at least against a range of rivals that have not received much attention. This does not yet show that parsimony considerations are in fact guiding people's theoretical preferences. However, I will argue that if we do not endorse parsimony considerations, the standard positions in the debate about the structure of space and time face unexpected challenges from unparsimonious variants. Furthermore, the cases I will look at suggest that supporting our favourite theories against unparsimonious rivals will often require an appeal to *quantitative* parsimony, that is, parsimony about how many entities of a given kind should be postulated, and not just parsimony about the kinds of entities in our ontology. Whether quantitative parsimony should play a significant role in theory choice is even more controversial than that some form of parsimony should be employed: so if we discover that quantitative parsimony has been an implicit part of our thinking, that is particularly methodologically interesting.

To keep things manageable, I will for the most part focus on theories of space and time that assume space and time *have* a structure. There are various theories that deny this. Some extreme theories deny that spatiotemporal phenomena are anything but illusions. Less extreme are theories according to which there are spatio-temporal phenomena, but there is no space or time. Perhaps the most discussed kind of theory under this heading is *relationalism*, the hypothesis that while there are entities that are spatiotemporally related, apparent talk of space and time is just indirect talk about those objects, and that there is no space or time. Another view that rejects the existence of space and time is *fictionalism* about space and time: there is only a convenient fiction that there is any space or time (See e.g. Hinckfuss 1986.) And there are others that, one way or another, reject the literal existence of space and time. Parsimony is obviously a potential motivation for such theories, and it is appealed to on behalf of relationalism by e.g. Huggett 2006 p 41 ftnt 1, and Vassallo et. al. 2017. However my topic will be appeals to parsimony, even granted there is such a thing as space or time.[2]

As it happens, I am among those convinced that space and time are not entirely distinct: instead, spatiotemporal goings-on are found in *spacetime*, a manifold of at least four dimensions, of roughly the same sort postulated by contemporary general relativity. But though I will often talk about spacetime, I do not think much of what I have to say here depends on whether one thinks that space and time are unified like this, or whether space and time are distinct manifolds.

After a discussion of some of the parsimony principles at play and how we might understand them, I will then turn to a central question where parsimony must play a role. This is the question of the fine structure of spacetime, and in particular what parts a region of spacetime has. Spacetime might be made up, ultimately, of minimal units of some positive magnitude; or might be resolvable into smaller and smaller regions ad infinitum; or might be ultimately made up of 0-sized points. Within each of these options, I will argue parsimony must play a crucial role if we are to support the positions with the most initial plausibility, and which are the positions in fact endorsed by mainstream proponents in the literature.[3]

Parsimony considerations have played an important role in two recent arguments about space and time, besides the motivation for relationalism mentioned above. Jonathan Tallant has argued that *presentism*, the doctrine that only present things exist (and, as Tallant develops it, that there are no concrete times), is supported by parsimony considerations. (Tallant 2013).

[2] Some relationalist theories in effect have ontologies very similar to substantivalists, for example relationalists who reject spacetime but postulate all-encompassing fields (electromagnetic, gravitational etc.) which have the structure that substantival space or time would have. For those theorists, questions about the fine structure of spacetime can be paraphrased into questions about the fine structure of the fields that should be posited.

[3] I expect creative philosophers and physicists will be able to come up with more options for the parts of spacetime than those listed here. There is also the option of claiming that subregions and points are not literally *parts* of the regions they are located within: those, if any, who prefer another account of this relationship can still appreciate the main point by translating my talk of parts into their preferred vocabulary.

Presentism does have a number of advantages on the score of parsimony, it seems to me. It also incurs some additional costs in simplicity, in requiring primitive tense operators. (That is, it may be less *ideologically parsimonious* than a theory committed to a block universe of temporal locations and their contents.) Parsimony considerations will only be part of the story in adjudicating between presentism, eternalism, and other rivals: and while that debate is fascinating, I will set it aside here.

Another interesting discussion of the interaction of parsimony considerations and the philosophy of space and time is found in Sorensen 2014. Sorensen raises the interesting question of whether parsimony would lead us to favour a finite past and future over infinite durations in one or both directions, and why, if so, philosophers and cosmologists have been relatively willing to postulate infinities in each direction. (Sorensen 2014) On the face of it, time and space are posits like any other theoretical posits, and postulating more years or light-years rather than less looks a lot like other postulations of more rather than less. Should we apply Occam's razor, and postulate no more time or space than is necessary? Even if we do not apply parsimony to time and space themselves, we may be under pressure to apply parsimony to the contents of time and space: if additional space comes with additional stars, or additional time comes with additional events, then principles of parsimony that tell us to minimise entities like stars or events will cut against such theories.

Sorensen argues that theorists have not been reluctant to posit enormous extents of space and time, or even infinite amounts (e.g. an eternal world or boundless space). This poses an apparent challenge to parsimony principles, at least on the assumption that these theorists where theorising well. I think this challenge is a very interesting one, and whether there is a problem here for the view that parsimony is a desideratum for theories turns both on difficult questions about the history of theories of space and time, and the other considerations at play in those discussions. (For example, if parsimony is a relatively weak desideratum and those who postulated infinite spaces or times had strong reasons for doing so, their failure to be held back or worried by parsimony might be easy to explain.) The questions Sorensen raises about the interaction of parsimony and the history of theories of space and time would repay close attention: however, since the aspects of theories of space and time that Sorensen focuses on are significantly different from my focus here, I will leave sustained discussion of them to another occasion.

While it is not a new idea that parsimony considerations might be used in theorising about space and time, it is not an uncontroversial one either. Huemer 2009 argues that we should never rely on parsimony considerations in philosophical theorising. Longino 1996 is suspicious of simplicity as a theoretical virtue in the sciences altogether, especially a form of simplicity that is in effect a parsimony principle: "the simpler theory is the one positing the fewest different kinds of fundamental entity (or causally effective entity)" (Longino 1996 p 53). And Parsons 1979 is among philosophers who reject parsimony considerations outright: "Theories should not be compared by counting entities, kinds of entities, or primitives" (Parsons 1979 p 660).

Presented with the arguments below, I hope readers will be convinced that we should prefer more parsimonious theories of the structure of space and time than less parsimonious rivals that do equally well in other respects. But those readers antecedently convinced that parsimony is not a guide to truth, or even a good guide to theory selection, this paper can be read as a challenge of a different sort. Each of the typically occupied positions on the structure of space and time have a wide range of cousins that resemble them except that the cousins are far less parsimonious. If we should accept one of the standard options, we need a reason to reject these cousins. So if parsimony should not be playing a role, we either need a *different* kind of reason that happens to reject the unparsimonious options in these cases, or we should retreat to taking no stand on which of the many cousins is correct. Those who do not accept parsimony as a criterion in theory selection have work to do here.

There are a variety of potential parsimony principles that might be employed: these can vary in terms of what we are parsimonious *about*; whether parsimony considerations should counsel *rejection* of less parsimonious options or just some form of agnosticism about them; and how parsimony considerations might interact with other theoretical considerations. It will be useful to briefly discuss these dimensions of variation before moving to look at how parsimony considerations may play a role in theories of the structure of spacetime.

1. A Variety of Parsimony Principles

There are a number of dimensions to parsimony: various ways in which our theory can postulate less rather than more. There is also a range of attitudes one can take to parsimony principles: whether parsimony, as variously described, is worth pursuing at all, and if so whether it is worth pursuing as a guide to the truth or merely for some more pragmatic end. It will be worthwhile to rehearse a partial menu of options, to make it easier to discuss later what sorts of parsimony are at stake and why it might matter.

An Occam's razor style principle, "do not multiply entities beyond necessity" can be implemented in various ways. It can be interpreted as an instruction about which theories to prefer, or as a rule taking some theories out of consideration altogether, but I think the best way to understand a parsimony principle is as a dimension of cost in an overall cost-benefit analysis of a theory: a theory is better, *in this respect*, if it has fewer postulates rather than more. Of course, it would be absurd to think this is the only dimension by which a theory could be better or worse, and it is sometimes worth paying the cost of new postulates for other theoretical benefits. (A theory of subatomic particles that postulates neutrons as well as protons and electrons did better, overall, than a theory that postulated protons and electrons alone, for example.) Thinking about parsimony in terms of costs, benefits and tradeoffs lets us capture several things: the *degree* of extravagance seems to matter, and whether a theorist is right to add postulates to a theory depends, in a range of ways, on the rest of the theorist's evidence and theoretical background in a way that would make a simple on/off rule for adding a postulate hard to formulate sensibly. When talking about costs and benefits I do not mean to automatically be talking about purely *pragmatic* costs and benefits: in scoring theories we may well be evaluating them as better or worse guides to the truth, or to what to believe based on the considerations before us.

There seem to be several dimensions of parsimony that matter. One is to limit the number of *kinds* of entities postulated by a theory. This "qualitative parsimony" seems to be the least controversial. Once a theory of light no longer required propagation in a luminiferous ether, orthodox physicists stopped postulating the ether. Once Kepler and Newton showed we could account for planetary orbits without epicycles, astronomers (eventually) ceased postulating epicycles. If Occam were right that there is no need to postulate universals in an adequate theory of generality, we would have good reason to not postulate universals. And so on. It would be ideal to have a clear criterion for determining when a new postulate is a new *kind* of entity, and when it is only more of the same. I do not know of an appealing general characterisation of this, though we seem to have a reasonable grip on the distinction in practice, at least in paradigm cases.

As well as quantitative parsimony, arguably we should also be parsimonious about how many postulates of each kind that we make. This further dimension of parsimony is known as *quantitative parsimony*. Quantitative parsimony is more controversial than qualitative parsimony, though I have offered an argument from case studies that quantitative parsimony does seem to be part of good scientific practice, on at least some occasions (Nolan 1997). Many of the applications of parsimony considerations to be discussed below are cases of being *quantitatively* parsimonious: those suspicious of quantitative parsimony will have reason to be suspicious that we have grounds for preferring the intuitively more appealing options over the less appealing ones; and on the other hand, those sympathetic to the application of the relevant parsimony considerations below will have additional reason to look kindly on the idea that there is something to a more general principle of quantitative parsimony.

When it comes to quantitative parsimony, to be useful we need an answer to the question of *which* counts matter. The most straightforward way to implement a quantitative parsimony constraint would be to look only at the total number of objects postulated by each rival theory, and to count a theory that postulates more overall as less quantitatively parsimonious overall. That would yield some implausible results: once a theory is committed to standard number theory, it would be committed to infinitely many objects, and so no finite difference in the number of entities like planets or parasites or conspiracies would make a difference to its degree of quantitative parsimony. Better, I think, to measure quantitative parsimony by looking at the number of entities *of each kind* postulated (see Nolan 1997 p 339-340): so the number of numbers a theory is committed to need have no bearing on whether it is unparsimonious in its postulation of planets or criminals or dark matter.

Another standard distinction that it will be worth keeping in mind is the difference between principles of parsimony that counsel *rejection* of theoretical postulates, versus those that merely counsel staying *neutral* on whether there are more kinds, individuals, or whatever than are needed. In Occam's razor terms, this is sometimes put as the distinction between an *atheistic* Occam's razor and *agnostic* Occam's razor. There is, of course, a distinction between a *theory* being neutral on an issue and someone who employs that theory being neutral on it. Fundamental physical theories are typically neutral on the question of whether koalas exist, but that does not mean that physicists in general have no opinion on the issue.

If we read theories of space and time as only saying there is *at least* as much structure as they explicitly postulate, then they will be consistent with most theories that are more ontologically extravagant than them. For purposes of this paper, I will take theories more at face value and read them as claims, not just that spacetime has *at least* a certain level of complexity and spatio-temporal features, but also that spacetime has *at most* that much structure. So, for example, a theory that says that a finite space is made up of finitely many regions of a fixed minimal size will turn out to be *false* if space is made up of infinitely many points.

Once we read theories in this committal way, using parsimony to motivate a theory with a smaller ontology is, in effect, to be using it to *reject* the larger ontology, not just to be neutral on it. (A finitist theory of space, then, is silent on the existence of koalas one way or another, but rejects the existence of space with a continuum of points.)

Another kind of disagreement among the friends of parsimony is worth mentioning, though it is not an issue I will attempt to resolve here. Even those who agree parsimony, of any form, is valuable may disagree about *why* it is valuable. One could take it to be a theoretical virtue that does not stand in need of further explanation, or perhaps better take it to be an aspect of a broader virtue of simplicity which is not explained in further terms. But one might also want to vindicate parsimony in further terms (or take it to have multiple sources of vindication). There could be a brutely inductivist one: it led to good results before! We could offer an account of the virtue of parsimony in terms of its contribution to valuable kinds of explanatoriness, as in Baker 2003. (Though see Jansson and Tallant 2017 for criticism of Baker's specific proposal.) No doubt there are many other further explanations of the desirability of parsimony that may be offered.

Note well that offering a further story about the value of parsimony is *not* denying that parsimony is a legitimate consideration in theory choice. If e.g. Baker should turn out to be right and parsimony brings explanatoriness of a sort we should prefer when choosing between theories, this does not mean that parsimony is not valuable: instead, it is valuable *because* of the desirable explanatoriness is ensures. If it is non-arbitrariness we prize, and there is some reason why the most parsimonious option is non-arbitrary but in general its rivals are not, then again we would have an explanation of parsimony's value rather than a rejection of it. If we were listing pros and cons of theories we may need to be careful to not double-count, of course, but that does not mean we should ignore whether theories are parsimonious when assessing them.

Once we are clear on what sort of parsimony we should pursue, there are further questions about what form of guidance parsimony considerations should offer. While we could interpret parsimony principles as absolute, forbidding any more postulates than absolutely required to save the phenomena, or at the other extreme we could treat parsimony principles as only tie-breakers when every other consideration is equipoised, I prefer to see parsimony as a consideration with some non-negligible weight, but also being the sort of consideration that can be outweighed. It does not seem not-negotiable: after all, scientists and other theorists do sometimes end up postulating more than they had initially been inclined to. On the other hand, I suspect it is a more powerful consideration than an "only when all else is equal" principle would have it. For example, sometimes when there is a recalcitrant piece of evidence that could only be accommodated by greatly complicating our theories, I think the right thing to do is to stick with the theory and assume the evidence is an anomaly that will eventually be resolved. The metaphor of weighing up different considerations and then picking one of the theories that does best on balance seems to me basically right as a story about what we should be doing. (Though there are plenty of places to cavil about details: perhaps the comparisons are not always clearcut, perhaps strength of considerations does not quite have the structure weight has, and adopting a theory may be less voluntary and less deliberate than choosing an object on a scale.)

Here is not the place to defend this general framework, or this particular take on how to understand parsimony considerations. Fortunately I think it will not matter much exactly whether we understand parsimony considerations as operating as weighted values, or rules, or something else in the following discussion. While I might talk in the language of a consideration to be weighed against others, I expect this can be translated easily enough into other idioms in the cases at hand. The conclusion I am concerned to argue for is that all the standard views on the parts of spacetime need to appeal to parsimony considerations if they are to be vindicated.

Many theories of space and time postulate one or more forms of *infinite* counts of their parts. Those familiar with the various sorts of infinite sizes that may be encountered in these theories may wish to skip to section 3: for other readers, I offer the following primer on some significant varieties of infinity in the next section.

2. Infinite Sizes

When counting things, there are larger and larger finite sizes of collections: 1, 2, 3... At the limit of all these sizes, there is the lowest infinite number $\aleph_0$ (aleph$_0$) often called a "countable infinity". It is the cardinality (i.e. the size) of the set of counting numbers. An interesting and important thing that Georg Cantor showed us is that it is not the only infinite size, and there are many infinite sizes above it. One interesting infinite size above it is $\beth_1$ (beth$_1$), the "cardinality of the continuum", sometimes represented as *c*. This is the size of the set of real numbers, and is also the size of the set of all the subsets of the counting numbers. (i.e. {{0}, {0, 1}, {1}, {0, 2} {0, 1, 2}, {2}, {1, 2}...}). Cantor showed that the "powerset" of any set is strictly greater in size than that set: that is, whenever you have a set S the set of all of S's subsets is strictly larger than it. $\beth_1$ is the size of the powerset of $\aleph_0$.[4] Sets of size $\beth_1$ also have powersets, and the powerset of such a set has cardinality $\beth_2$. $\beth_2$ in turn, is smaller than $\beth_3$, (defined as the size of the powerset of a set of size $\beth_3$), and so on. In the symbolism of transfinite arithmetic, we often represent the size of a powerset in terms of the size of the set with started with using exponentiation: where *n* is the size of an infinite set, the size of its powerset is 2^n.

Taking the limit of the sizes $\beth_1$, $\beth_2$, $\beth_3$... we get yet another size, $\beth_\omega$. (It is the size of the union of sets of each of the smaller beth sizes.) And we can keep going: $\beth_{\omega+1}$, $\beth_{\omega+2}$... With a combination of applications of the powerset operations and limit-taking, there is a healthy hierarchy of infinite sizes.

There is no highest infinite size in the above series of infinite sizes. Still, there are mathematical techniques to go further in defining even larger infinite sizes than the beths. We can, if we like, stipulate that there is an infinite set size greater than all of the infinite sizes we

[4] It is a pity that the terminology is inelegant in this way. Cantor introduced the alephs to name the infinities ranked in size, with $\aleph_0$ being the smallest, $\aleph_1$ being the next smallest, etc. Unfortunately, in ZFC the relationship between the alephs and the beths, as defined in the text, is not entirely settled: the system leaves open that there are other infinite sizes between $\aleph_0$ and $\beth_1$. $\beth_0$ is another name for $\aleph_0$, and using that would restore smoothness to the notation, but unfortunately aleph$_0$/$\aleph_0$ is so entrenched as the name for the first infinite cardinal that dropping it in favour of $\beth_0$ might confuse readers who have come across countable infinities before.

can reach through the methods of application of powerset and taking limits. This "first strongly inaccessible cardinal" is often labeled κ or κ_0. Once we have κ, we can apply powerset to get a cardinal that must be larger than it (2^κ), a cardinal larger than that, and in effect there will be a hierarchy above κ much like the one above $\aleph_0$. We can then hypothesise that there is another cardinality that stands to κ_0 as κ_0 stands to the beths: κ_1, or the "second strongly inaccessible cardinal". Indeed, we can postulate infinitely many strongly inaccessible cardinals, each larger than the one before in the way κ_0 is larger than the sequence of beths described above.

Postulating sizes such as κ_0, let alone infinitely many strongly inaccessible cardinals, takes us beyond the sizes that can be shown to exist just with ZFC set theory. These infinite sizes that are larger than sizes that can be established with ZFC alone are known as "large cardinals". Mathematicians wanting to work with structures of this size need additional axioms. Still, these axioms seem internally consistent and have been used to describe structures of mathematical interest, so there is little controversy that these sizes are available. (For example, Grothendieck universes, which are consistent if and only if ZFC plus an inaccessible cardinal axiom is consistent, were invoked in Andrew Wiles's proof of Fermat's Last Theorem. While many mathematicians are confident that proofs of Fermat's Last Theorem will be found that do not need to employ resources like this, this is a prominent case where postulating large cardinals, or the mathematical equivalent, is in practice useful even in pure number theory.[5])

The above series of inaccessible cardinals is only the beginning of the options for postulating larger and larger infinite sizes. So-called transfinite set theory typically postulates many kinds of large cardinals eye-wateringly larger than the kappas. One kind of large cardinal that goes beyond those discussed above are the so called measurable cardinals, in particular the real valued measurable cardinals. New kinds of large cardinals require new additions to ZFC, but many of these additions are well-investigated and show no signs of being subtly inconsistent.

Finally, beyond all the sizes of sets that can be consistently postulated within set theory, there is arguably at least one more size. Within set theory, we define cardinalities in terms of one-

[5] The exact relation between postulating Grothendieck universes and Wylie's proof is somewhat controversial. See McLarty 2010 for a useful discussion. I follow McLarty (among others) in thinking Wylie's proof, as presented, relied on theorems that had been proved using Grothendieck universes and had not been proved without them.

to-one mappings between sets. But there is no way within set theory to set up a function from *all* the sets to some other set (or even itself). So no matter how many sets of inaccessible size are posited, and of what kinds, there will be some things (e.g. the sets) that cannot be captured by any of the cardinalities in the system: as is sometimes said, they are "too many" to form a set, or to have a cardinality in the ordinary way. There are several ways to talk about the "size" corresponding to the size of sets, if it is discussed at all. One straightforward one is to introduce another kind of set-like object, the "proper class", that has members but is not itself a member of anything. Without contradiction we can discuss the *proper class* of all sets, and it can be put in one-one correspondence with itself, or with other sets, but employing a *proper class* of ordered pairs, the domain of which is the proper class of sets.[6]

Even if we do not treat proper classes as a new kind of object, we can allow ourselves other techniques to talk about there being as many objects of a certain sort as there are sets. One option is to postulate a new quantifier, "there are proper class many", or a new collective predicate "Xs are proper class many". (I am not saying that this postulation is philosophically uncontroversial, just that it is an available theoretical option.) Provided these expressions can somehow be meaningful even if the objects in question do not all belong to a single set or class, we will have a way to talk about how many of a certain kind of object there are without presupposing that they can be put into a one-one correspondence with any collection.

When these conditions are satisfied, we can say that there are "proper class many" of the objects in question, even if there are no proper classes. Even more loosely speaking, we can talk of "proper class many" as being *a size*, larger than all the sizes of sets. I will freely talk about the option of positing proper-class many objects of various sorts, and talking about this as a size larger than all sizes sets can have. But the reader should note that this talk may need some paraphrase unless we literally postulate proper classes. I will also, for convenience, talk as if there is a single "size" that is "proper class many". There are systems that allow proper classes to come in different sizes, but for present purposes there is little point leaving that option explicitly open.

[6] Can any proper class be put into one-one correspondence with any other? It depends on the system, or in particular whether a principle known as *Global Choice* is true of the system. (If and only if Global Choice is true, they are all of the same size.) How many proper classes are there? Since, in orthodox class theory, they do not belong to any sets or classes, we cannot find a class or set that they are in one-one correspondence to. If we use the analogue of one of the devices discussed in the text to let us talk in a sensible way about how many proper classes there are, there must be strictly more of them than there are sets.

What do all these infinite sizes have to do with space and time? According to one of the most standard contemporary conceptions of space and time, *regions* are infinitely divisible: indeed, infinitely *divided* in the sense that they have infinitely many smaller and smaller parts. The most common models of these regions postulate *at least* as many regions as there are real numbers: that is, continuum-many or $\beth_1$ many regions. In fact, the most standard model of space and time postulates $\beth_1$ many *points*, small zero-sized and zero-dimensioned pieces of space. With $\beth_1$ many points, if we allow that arbitrary sets of points correspond to (perhaps scattered) regions, as is also standard, there will be $\beth_2$ regions.

Some theories of space and time postulate many fewer regions. Some postulate only finitely many (e.g. by having a minimum positive size of regions and having larger regions made out of those smallest ones). Some postulate only countably infinitely many regions, though this is very rare. The standard approach to regions that takes them to be "gunky", with smaller and smaller regions all the way down, plus arbitrary fusions of all of those regions, postulates only $\beth_1$ regions rather than the $\beth_2$ regions standardly postulated by "pointy" theories. And of course some theorists resist postulating any space, time or spacetime at all. On the other hand, in principle theories of space and time could postulate many *more* parts of space and time than the mere $\beth_1$ regions of standard gunky space or the mere $\beth_2$ regions of orthodox pointy space. Some of those options will be discussed below.

When evaluating the postulates of different theories of space and time, then, we very quickly get into the business of comparing infinite sizes. Some of the options I will entertain below posit *vastly* more regions and points even than the relatively low levels of infinity such as $\beth_2$, including various inaccessible infinite sizes and postulating proper-class many regions or points. Many of these options have not received sustained attention, but once we do attend to them I think we should be very suspicious of these alternatives. If we reject these alternatives, as I think we probably should, we should make appeals to parsimony explicit to justify our preference for relatively small ontologies of regions and points.

3. Parsimony and Discreteness

Once we accept that there are parts of spacetime, we have a number of options for understanding the structure of those parts. Orthodoxy is that spacetime is divided into a continuum of points of zero-size, and regions of non-zero size are made up of those points. But a salient alternative, which has been defended explicitly at least since the time of the ancient atomists, is that space and time are made up of minima which themselves have non-zero size, and that there are only finitely many of these minima in a meter, or a second, or for that matter there are only finitely many in any finite distance or duration.

There are armchair philosophical arguments both for and against spatial, temporal, or spatio-temporal minima, and for and against infinitely divisible space and time. In my view both the picture of spacetime being made up of minima and the picture of it being continuous are coherent and not ruled out *a priori*: the decision between these rival pictures is to be made, if it can be made at all, on the basis of *a posteriori* considerations, and in particular what makes best sense of our evidence from physics. Showing that all the armchair arguments intended to demonstrate that space or time are discrete fail is a task that would have to be done piecemeal. It is also beyond the scope of this paper, though see Forrest 1995 for responses to a number of the traditional armchair arguments against discrete spacetime.

One armchair argument that may have some force, even it is not demonstrative, is that positing discrete spacetime results in postulating many fewer parts of space than positing space divided into infinitely many parts. (At least this is true if there is only a finite volume of spacetime.)[7] *If* we could do just as well with finitely divisible space and time, that might be preferable from a parsimony point of view. Forrest 2004 pp 355-356 is one philosopher who mentions this parsimony argument in favour of discrete space over infinitely divided space, though he does not take it to be decisive.

When we turn to contemporary physics, there are a number of lines of thought converging on the idea that we should treat spacetime itself as quantised. One of the best known comes from loop quantum gravity models of quantized spacetime, though there are a range of other theoretical approaches being tried.[8] This is an evolving area of physics, however, and recent astronomical data suggests that spacetime is "smoother" than a range of quantised approaches would have had it: see Perlman et al 2015. Quantised approaches postulate only finitely many smallest parts of spacetime in a given finite area, though the number is not a small finite

[7] Even a countable infinity of meters of discrete space, each divided finitely, will result in fewer parts of space that orthodox "pointy" continuous approaches postulate within a single meter.
[8] For an introduction to loop quantum gravity, see Rovelli and Vidotto 2014. One other promising approach to quantising space and time is the causal sets approach: see e.g. Bombelli et al 1987 or Dowker 2005.

number. One popular conjecture about the smallest size of space is that it will be a planck length long, or about 10^{-35}m. (If these minimum pieces are two-dimensional, the "planck areas" would be something in the range of 10^{-69}-10^{-70} in a square meter, if three-dimensional the minimum volumes might be around 10^{-103}-10^{-105} to a cubic meter, and so on.) The planck time is approximately 10^{-42} seconds, and is a plausible estimate for the minimum unit of time. With the minimum sizes being this tiny, treating space and time as approximately continuous at lengths and durations we normally come across does no harm for almost any practical purpose.

Apart from the appeal of specific approaches to quantising space and time, there is a general theoretical reason to want to do so. Traditional quantum mechanics, or even traditional quantum field theories, and the general theory of relativity have been spectacularly empirically successful in their usual domains of application. However, at very small scales and at very high energies it becomes very challenging to apply both theories at once (some physicists talk of an "inconsistency" between the theories, though whether there is a formal inconsistency depends on some subtle issues about formulation). One approach to take to this problem is to modify the theories so that gravity is in effect quantised as well, and it turns out a natural way to do that in this context is to introduce minimum finite spatio-temporal distances. While not compulsory, introducing a minimum quantum by which distance can change or time can advance offers an appealing approach to this crucial puzzle area in theoretical physics. The plausibility that we will somehow need to quantise distance is thus higher than the plausibility of any of the particular, highly controversial, implementations of this idea.

As I have argued in earlier work (Nolan 2008), many of these arguments take a step that should require an explicit defence. Suppose it has been established that physical phenomena are spatio-temporally quantised, in the sense that there is a minimum size that variation can happen across. (And let us suppose also, as is standard, that these minimum sizes "line up", so it is not that e.g. one event can start or end half a minima after another.) One way that could be is if the minimum parts of space and time themselves corresponded to the minimum quantum of variation among objects, or fields, or other phenomena. But another way it could be, at least in principle, is if spacetime was continuous, made up of 0-sized points, but the phenomena located in spacetime never "took advantage" of the extra degrees of variation allowed to them. An analogy might be a traditional pixelated computer screen: while the screen can in fact be divided into regions much smaller than the pixels, there is only variation in what the computer displays between pixels on the monitor.

Those who want to draw on the evidence from physics to support discrete spacetime, and not just discrete variation of entities located in spacetime, need to do more than show that the entities in spacetime come in spatio-temporally quantised sizes. After all, we can embed these discrete events in an underlying continuous spacetime, and the appropriate embedding will be observationally equivalent to a discrete model of spacetime where the smallest distinctions in spacetime itself correspond to the smallest distinctions in behaviour the phenomena display. (The picture is more complicated when spacetime is a dynamic part of the theory, e.g. when we treat gravitational attraction as being a matter of differences in spacetime curvature. One brute force way to have quantised gravity and a continuous spacetime would be to have a distinct "curved" gravity field with discrete values embedded in a continuous spacetime: this would be ugly, but the ugliness is due to the lack of parsimony such an option would display.)

As well as ruling out discrete-phenomena-in-continuous-spacetime, it would also be good to rule out models where spacetime is discrete but the smallest units are much smaller than the smallest units posited by our theory. Suppose our theory posits the minimum distance of spacetime is the planck length. A rival theory can be constructed where the minimum length is one millionth or one billionth of a planck length, but spatiotemporal phenomena only display planck-length differences. (For instance, particles quantum jump from being one plank length apart to being two planck length apart, but are never found one half or one millionth of a planck length apart.) This can be done in such a way that the resulting theory is observationally equivalent.

I argued in Nolan 2008 that this extra step is a parsimony step. If we are not to postulate any more structure to spacetime than is needed for our best theory, then the fact that we can treat physical objects and processes as being located in a discrete spacetime (if this is a fact) would require us to postulate only discrete spatiotemporal structure. I suspect an argument like this is implicit in the claims physicists make that there is evidence for quantised spacetime, though I have not seen it made explicitly. Instead, the form of the argument seems to be that the smallest quantities of spacetime needed, or that the laws operate on, or which are observable in principle... (etc.) are discrete and quantised, therefore space and time and discrete and quantised. The premise is controversial and a focus of scientific debate: but we also need to support the inference. Parsimony would do so, but I cannot see any other plausible way of moving from the scientific evidence to the desired conclusion.[9] So those who adopt discrete space or time had best be willing to endorse a parsimony principle, and a principle of quantitative parsimony at that.

4. Parsimony and Gunk

If we reject spatial or temporal minima of non-zero size, there is another salient alternative to orthodox continuous space and time. This is to treat space and time as *gunk*. In mereology, a gunky object is such that all of its proper parts themselves have proper parts: there are infinitely descending chains of whole-to-part, but there are no "least parts". The classic model of gunky space and time (Tarski 1929) is a Euclidean space with smaller and smaller regions, with smaller regions being parts of larger regions, but no points underneath the regions: there are no smallest parts of space. This model would need to be extended to spacetime: to model Minkowski spacetime, nested hypercones, rather than spheres, would seem to be the natural way to set up such a model, though I do not know of a presentation of the details in print.

There are a number of arguments for gunky space and time versus continuous, "pointy" space and time, as well as a number of arguments against. An old argument for gunky space is the idea that putting together zero-magnitude parts of space could never yield a more-than-zero magnitude region: this, allegedly, is as intuitively absurd as hoping to add zero to itself enough times to get to one. (See Arntzenius and Hawthorne 2005 pp 443-445.) Relatedly, you may have thought it was just definitional of parts of space or time that they have positive spatial or temporal magnitude, something points would lack. Another argument concerns dimensionality: that smaller and smaller parts of a region must have the same number of dimensions as it (line segments made up by line segments, areas by areas, volumes by volumes, and so on). If that is correct, space could not be ultimately be made up of zero-dimensional points, but a volume must only be made up by smaller volumes.

[9] Those who hold that parsimony can be explained further, e.g. in terms of preferring more explanatory theories (Baker 2003), could argue against these rivals on the grounds of doing less well on explanatory grounds. But this is not a rival to a parsimony argument, but, if Baker is right, it is spelling out a parsimony argument in further detail.

Put each of these premises together with the thought that space or time must be infinitely divisible (e.g. because it always makes sense to take half of any given magnitude), and we are naturally led to a view where space is made up of smaller and smaller regions of non-zero size, but has no smallest size. The analogous arguments suggest intervals of time have the same kind of structure. While this gunky option was repopularised in the twentieth century by writers such as Alfred Whitehead and Alfred Tarski, it has a much longer history: see Holden 2004 for evidence of this view in the early modern period, and Nolan 2006 for arguments that it was the approach taken to space and time by Chrysippus.[10]

A more recent argument for gunky space and time has its origins as a response to the "Banach-Tarski paradox". In a standard Euclidean three-dimensional space made up of points, if you start with a standard ball-shaped region of space, you will be able to specify an exclusive and exhaustive division of that ball into sub-regions, together with a mapping of those regions to a second set of regions, which preserves the size and shape of those sub-regions. (That is, each sub-region of the ball is mapped to another region of the same shape and size.) However, such a mapping can be defined so that the second set of regions make up a region of *twice* the size of the ball we started with (Banach and Tarski 1924). There are a variety of possible responses to this result, but one that has sometimes been suggested is that this shows there is something inadequate or defective in understanding regions as corresponding to arbitrary sets of points. A natural alternative for those who wish to preserve the infinite divisibility of space, but to employ a geometry that does not allow the Banach-Tarski result to be derived, is to allow smaller and smaller regions of non-zero size ad infinitum. See Forrest 2004 for discussion.

Beyond the arguments mentioned, there are at least three parsimony arguments for preferring gunky models of space and time to pointy ones, everything else being equal. One is that a gunky picture appears to commit us to a proper subset of the commitments of the standard pointy view of space and time. We are in any case committed to regions, and subregions of those, etc.: why, unless we have additional reason, should we accept the existence of point-like parts of space and time as well? Points seem further from our ordinary experience of spatial regions and temporal intervals than regions do: even the aspects of our experience that

[10] The arguments discussed are only two of a range of philosophical arguments offered in favour of gunky space: for another, see Zimmerman 1996.

involve edges of regions or boundaries seem adequately captured with very small boundary regions rather than literally magnitude-less points.

Note that in rejecting the existence of points, we reject a class of entities that seem qualitatively different from regions. As mentioned above, points are zero-dimensional parts of space/time, while typical gunky theories accept parts of space, time or spacetime that have positive magnitude and the same number of dimensions of the entire manifold. Points that are supposed to make up larger spaces or times have another surprising feature, in that even though each of them is zero sized, wholes made up of them can have positive non-zero sizes. Points, at least in continuous space or time as normally conceived, are never adjacent to each other: so not only would spacetime have to be made up of a lot of "zeroes", it would have to be made up of entities that are never even adjacent to each other! Since points are so unlike typical paradigm regions, positing regions without points, instead of both points and regions, seems to be an improvement in qualitative parsimony.

Finally, a standard gunky spacetime has vastly fewer entities in it than the corresponding pointy spacetime. Given a finite total volume, if we consider the regions we get by halving it, halving all of them, halving all of them, etc., we have finitely many regions at each stage of this process. However, we do not escape commitment to infinitely many objects: there are $\aleph_0$ stages of that process, and taking arbitrary fusions of the regions generated ends up yielding us continuum-many ($\beth_1$) regions. (The number is the same whether we consider one or three or four dimensions.) In the standard pointy conception of spacetime, however, there are continuum-many points ($\beth_1$): and since arbitrary fusions of points correspond to regions, we end up with 2^$\beth_1$ ($\beth_2$) regions. $\beth_2$ is vastly larger than the continuum: not only is the difference larger than the difference between $\aleph_0$ and any finite number, it is vastly greater than the difference between $\aleph_0$ and the continuum itself. It is striking that theorists are so willing to accept this extra ontology lightly, when gunky space and time seems to offer infinite divisibility and a kind of continuity as well, without the vast array of additional regions pointy space and time provide.[11] Preferring fewer regions to more, where fewer will do, is a matter of quantitative parsimony.

[11] Arntzenius 2012 pp 151-2 points out that while parsimony might favour gunky spacetimes, other respects of simplicity may favour the pointy view, including simplicity of statements of physical laws. So one line of response to these parsimony arguments would be to stress those other aspects of simplicity.

Of course, when gunky theories "recover" points by treating them as certain infinite sets of regions, they end up with as many "points" as a pointy theory, and as many sets of their constructed "points" as their rivals. But here it should matter that the gunk-theorist's points are only set-theoretic constructions from the genuine parts of spacetime. Everyone who adopts standard set theory is already committed to a staggering hierarchy of infinite sizes of sets. Given a set theory like ZFC, for almost any disagreement about which entities can be found in the physical world, we can find sets constructed out of those elements of mind-numbing sizes. The fact that a gunk theorist postulates fewer parts of space and time seems to be what matters here, rather than the fact that cardinality differences of total numbers of objects postulated wash out when we add the full power of set theory. After all, the cardinality of mathematical objects dwarfs the cardinality of non-mathematical entities postulated in standard e.g. physical theories, but that does not mean parsimony lacks a role in choosing between theories in the natural sciences. (See Nolan 1997 p 340 for this point.)

There is another reason partisans of gunky spacetime should want to employ parsimony considerations in defence of their theories: it is a concern about how to rule out gunky spacetimes with much higher cardinalities of gunky parts than the usual $\beth_1$ gunky regions. This final reason is analogous to the main reason I see that continuous, "pointy" theories of spacetime need to employ parsimony considerations: so for that reason I will delay discussion of it until the end of the next section, after I have outlined the issue for that view.

5. Parsimony and Points

Suppose, as many hold, that the case for discrete spacetime time is not successful, and that spacetime should have at least the structure of the continuum. Suppose further that the case for gunk should be rejected, and as well as smaller and smaller regions *ad infinitum*, spacetime should resolve into points, and those points should be dense (between any two is another) and continuous (the limit of a series of positions of points closer and closer together will itself correspond to a point). This picture of space and time is not just that of classical physics, but also of standard theories of special and general relativity. It is also presupposed in much of the mathematics of contemporary physics: trying to solve even relatively simple problems involving motion without using calculus or integration would be a nightmare, and both assume continuous variation of spatiotemporal quantities such as distance, acceleration, and so on. And defining qualities at points, such as instantaneous velocity or field strengths, is also ubiquitous in physics.

The standard "pointy" models of spacetime postulate continuum-many (*c*) points. This means there are $\beth_1$ points in such a system. It is standard to treat any non-empty set of these points as corresponding to a region (albeit often a scattered and gerrymandered region): this means there are 2^$\beth_1$ regions of spacetime in the standard models, i.e. $\beth_2$.

However, those who postulate only continuum-many points of space or time or spacetime must answer the question of why they postulate *only* continuum many points. There are many levels of infinity above the continuum: infinitely many, in fact. $\beth_1$ is the lowest cardinality that allows for points to be dense and continuous, but *any* infinity of points higher than $\beth_1$ might do the job as well, as I will argue below. In fact, as discussed in Section 3, the hierarchy of available infinite sizes is enormous: we have infinitely many sizes to choose from (and depending on which large cardinal axioms we accept, a very large infinity at that).

In each case a measure will need to be put on some sets of the points so that distances, areas, volumes and so on behave in familiar ways, and so that field strengths and other values can be defined across spatio-temporal regions in a familiar way. Several slightly different things get called "measures". The most orthodox thing to call a measure is a function from sets of objects to the real numbers that obeys certain further constraints: for example, that when a measure is defined for two disjoint sets, the measure of their union is the sum of their measures. (For illustration, if I have two line segments that do not overlap and are each 1cm long, the measure of the set of points from the first line plus the set of points from the second line will be 2cm.) Grünbaum 1970 is an accessible introduction to measures of (Euclidean) space in pointy settings.

The most standard way of doing measure theory maps certain sets of points to real numbers where the real numbers are thought of as the lengths, areas, volumes or whatever of the regions that are made up by those points. (Each measure will implicitly have a unit of measurement specified: e.g. if the measure is volume-in-cubic-meters, it will map volumes of $1m^3$ to the number 1, volumes of $2.5m^3$ to 2.5, and so on.) We can also define "measures", in a slightly different sense, as a function from aggregates of points to real numbers: this is plausibly a function directly from the lines, planes, volumes (or whatever) to real numbers that give their magnitude. Whether we say the domain of the measure is sets of points, or the regions that those points make up, will make little difference at least in standard settings.[12]

I do not think it would be useful to give all the conditions a function must satisfy to count as a measure here, but I do want to draw attention to one unusual feature of typical measure theory when applied to continuum-many ($\beth_1$) points. It turns out (given the Axiom of Choice) that there is no way to define such measures so that they meet the various conditions needed *and give a defined real-valued measure to every set of points*. Instead, only privileged sets of points receive values from a measure function, and most infinite subsets of the points receive no measure at all. This is often thought to be a drawback of our understanding of measure: why shouldn't every region get a volume, even if that volume is 0 when the points are too scattered? Still, this is a result that those who see spacetime as made up from points have largely made their peace with. Some partisans of discrete spacetime or gunk have pointed to this feature as one more reason to reject a pointy conception of space or time, however.

Measuring space and time with real numbers is traditional, and it is plausible that if space is continuous we will need at least that many values for length, area or volume to take. (Comparing volumes of cubes and spheres cannot be done in general with only rational numbers, for example, since fractions of π are often important!) But we can also define measures with systems *richer* than the real numbers: measures employing imaginary numbers are straightforward, and some theories with the right additional resources may employ functions from regions to *surreal* numbers: see below. Perhaps the easiest modification to grasp that uses more than real numbers is a measure on a space of infinite volume: while various sub-spaces of the volume may have arbitrarily high finite volumes, the measure will map the volume as a whole, as well as some of its sub-volumes, to something besides a real number: perhaps an infinite number such as $\aleph_0$.

Consider one minimal departure from orthodox continuous physics which is available if we have more than continuum points, and which clearly allows us to employ the formalism of continuous physics unchanged. Suppose, for specificity, we have $\beth_2$ (i.e. 2^c) many points which we will treat as the ultimate components of spacetime. Sort them into continuum-many

[12] It may make a difference in gunky settings, where we may wish to define a measure directly on regions: though most commonly the mathematical equivalent of points are constructed from infinite sets of regions through a method due to Tarski, and the measure is defined on them.

($\beth_1$) equivalence classes, each containing $\beth_2$ many points. Let us call these equivalence classes "bunches". Now apply a measure such that, within a bunch, all the members of it are 0 distance from each other. Start with a standard continuous distance measure, m_1 on the bunches as if each bunch were a point in ordinary continuum-spacetime, and adapt that measure to yield a measure, m_2, on the points within bunches as follows: when points x and y belong to distinct bunches b1 and b2, so that the "distance" between b1 and b2 is *r*, then the distance between x and y according to m_2 is also *r*. (i.e. the length of a line connecting the two is *r*: and extend this in the obvious way to areas, volume, etc.) That is, every member of a bunch agrees with its bunch-mates on its distances to all points. (If you wish, extend all the other measures from your favourite physical theory in the same way: define a measure on bunches that matches your original measure, and then apply it to the members of the bunches *mutatis mutandis*.)

If the measure on $\beth_2$ points is constructed in this way, it is easy to see that we can carry over our physics with little change. Even though the ultimate measure on points is our m_2, for almost any theoretical purpose we can ignore distinctions between distinct members of bunches and treat our spacetime as if it were composed of $\beth_1$ points. (Equivalently, we could use our measure m_1 on bunches to do our calculations and predictions, even though our model tells us bunches are not atomic but are sets with $\beth_1$ members.)

Without further stipulations, this model is susceptible to an unusual form of indeterminism. If a point particle moves one meter in one direction and one meter back, it will end up 0 distance from where it was. But there is no guarantee it will end up at the same point it started from: all that we will be able to show, given the laws of motion, is that it is at one of the points in the bunch that its starting point belongs to. In general, laws of motion expressed in terms of distance travelled will not distinguish between points within the same bunch. We could of course search for additional constraints that do distinguish between co-bunched points, but it is hard to see how any of these could make an empirical difference. Perhaps this indeterminism is a mark against such bunch theories, over and above the additional structure they posit: that will depend on whether there is anything per se preferable about deterministic theories over indeterministic ones. Even if it is a factor to be considered in theory evaluation over and above simplicity considerations, it seems to me that *this* form of indeterminism, at least, would be little to be concerned about.

I anticipate one potential objection to any version of this bunching strategy which may be a weightier concern. Why describe the strategy as one according to which our measure is defined over bunches, rather than this just being a theory according to which *the points themselves* are identical to the bunches, and it is just that the points of spacetime have some surprising internal structure? This objection is not obviously fatal: the $\beth_2$ objects are the partless parts of spacetime, and are plausibly its fundamental constituents in such a theory, so they have some claim to be considered the true points, even if the bunches play some of the roles orthodoxy assigns to points. Nevertheless, it would be good to be able to avoid this concern by constructing models where the putative points are non-zero distances from each other. Fortunately this is also straightforward.

To construct a slightly more complex model that handles this concern, we can tweak the above construction so that the members of the bunches are a little less differentiated. Consider a measure that agrees with the one just constructed about points from distinct bunches: as above, when *x* and *y* are any two points such that *x* belongs to one bunch b1 and *y* belongs to a distinct bunch b2, *x* and *y* stand in a distance, according to m_2, equal to the distance between the bunches specified by m_1. (And make the obvious associated stipulations for lengths, areas, volumes etc.) However, we add to this a further specification for points within a bunch. Let every point be 0 distance from itself, and let distinct points within a bunch stand in an "infinitesimal" distance *d*, defined so as to be greater than 0 but less than any distance represented with a greater-than-zero real number by our metric.

There are some more mathematically sophisticated approaches than either of the bunching strategy I have just described. Real-valued measures can be applied directly to sets of sets of points where there are more than $\beth_1$ points. This is rarely done for sizes of points such as $\beth_3$ or κ_0, but there are no very serious obstacles here. There does not seem to be anything gained either for the purposes of physical theory: the additional complexity would be entirely needless. Still, without some form of parsimony consideration, why not indulge in needless complexity? At the very least, these variations on a standard continuum-points account of continuous spacetime seem just as good as the orthodoxy, parsimony aside.

Some high levels of infinity might even be particularly appealing for applying the measures associated with space and time. As mentioned above, the standard mathematical continuum, with continuum many points, has an awkward feature that it is mathematically impossible to assign all of its subsets a measure between zero and one, while keeping other desirable features of a measure. This is why the orthodoxy about metrics on space, time or spacetime is that such a metric maps a distinguished set of subsets of points to real numbers. This is sometimes recognised as an undesirable feature of these measures (e.g. Forrest 2004 p 361-363), but it is often accepted as a mathematical fact of life. However, if we are willing to postulate enough points, we can ensure every set of points receives a measure as well.

Some larger cardinal numbers, it turns out, can support a function which has as its domain *all* of the subsets of a set of points to the range of real numbers 0-1, which also satisfies the usual conditions to be a genuine measure. These cardinalities are known as *measurable* cardinals (and if we want to impose the extra condition that we can have a measure from the subsets of the set to arbitrary real numbers, *real-value measurable* cardinals are needed, which is slightly more demanding). However, measurable cardinals must be very large, at least in ZFC set theory: they must be strongly inaccessible cardinals, but it must also be true that a measurable cardinal M must be larger than the first M inaccessible cardinals.[13] These sizes go well beyond the sizes required for the smallest models of ZFC. Nevertheless, it is widely supposed they are consistent postulates.

Even though real-value measurable cardinals have this attractive feature, I have not been able to find any physicist or philosopher proposing that we should therefore take the cardinality of spacetime points to be a real-value measureable cardinal. This may merely be because relatively few people working on the structure of spacetime have thought very much about various large-cardinal axioms. But I suspect another contributing reason is that theorists are reluctant to postulate such a mind-boggling cardinality of spacetime points even for the advantage of having a theory that fits at little better with some of our starting intuition about the sizes of regions: namely, that every set of points comes with a volume, zero or non-zero. If this is right (for some or all such theorists) this would be evidence that parsimony is functioning as more than a mere tie-breaker: theorists seem willing to avoid quantitative extravagance even when something of theoretical value is purchased by the postulation of additional ontology.

[13] I have not been able to find an accessible proof of this result, though see Fremlin 2009 p 24 for some relevant results.

Even if we postulate strongly-inaccessibly many spacetime points, or even that the spacetime points are as many as a real-valued measurable cardinal, there are plenty of models of infinite cardinals of much greater size still to be exploited by potential theories of spacetime. Indeed, we can push beyond all of these and postulate a *proper-class* of spacetime points. If there are a proper-class of spacetime points, then there will be more spacetime points than there are members of any set, at least given Choice. (In that sense that for any set, its members can be put into one-one correspondence with a set of points, but there is no set which all the points can be put in one-one correspondence with.) If we postulate that there is a proper class of spacetime points in ordinary regions of space or time with finite measures (for example in a centimeter of length or a second of time), then spacetime will be divided to an immeasurably greater degree than the pedestrian spacetime of ordinary continuum models. Incidentally, a model of space containing proper-class many points in finite volumes is one natural way to understand Charles Peirce's remarks about space: see Putnam 1995 for details.

Note that if we reject the existence of spacetime points, and instead endorse only the existence of spacetime regions with greater than zero magnitude, we face a similar range of options. The most usual models of so-called "gunky" spacetime only have continuum-many regions of gunk in a finite volume. But again, here, there are a dizzying range of theoretical options of postulating higher and higher cardinalities of gunky regions in a finite volume. There is nothing stopping us theorising about gunky spacetime with more than 2^c regions, or more than $\beth_\omega$ regions, or more than inaccessibly many regions, or a real-value-measurable-cardinal of regions, or even proper-class many regions. In many cases, we can provide models of gunky space with a Tarski-like construction, starting with a point space with a cardinality C, and constructing a space of regions corresponding to certain privileged sets of those points. (Extending this to gunky *spacetime* will require resources analogous to those mentioned at the start of section 4.) One limitation of this particular method of construction is that we must ensure that the cardinality of points we start with has a square root that is also a cardinal (in other words, that there is an *n* such that the cardinality C we start with is 2^n): but while not every cardinal meets this condition, there are arbitrarily large cardinals that do. This is not the only method of constructing models of gunk with large cardinalities: I mention it only as one relatively simple way to translate from certain "pointy" models to gunky ones.

The final option, to postulate a proper class of regions of spacetime, is the postulation of a form of "hypergunk". See Nolan 2004 for a defence of the possibility of hypergunk, and Reeder 2020 for a recent development of this strategy for space employing a measure theory based on surreal numbers. Reeder's interesting construction gives us extra reason, it seems to me, to see spacetime with proper-class many regions in any finite volume as a coherent option, and since a physical theory based on it can yield predictions that are observationally equivalent to our best continuous physics, it is reasonable to ask for a principled theoretical reason to reject it, especially if we allow that ordinary gunky theories are live theoretical options.

While an infinite range of possibilities opens up here for theories with higher and higher cardinalities of points of space, or of time, or of spacetime, I imagine many philosophers and physicists will not find these new rivals to standard theories of continuous spacetime appealing. I predict (hope?) that they will find this additional unnecessary structure unappealing and at least faintly absurd in the absence of any particular reason to invoke it. (I do not expect this repugnance will be universal: some have a taste for making things as complex as possible.)[14] Even the fact that measurable cardinals have a pleasing property that might be intuitive for space and time (that every sub-region of points can receive a size, even if it is zero) does not seem to be enough reason to overcome the pull of parsimony: mathematicians have been aware of real-valued measurable cardinals since 1930, but they are not standard in theories of continuous space. Indeed, as far as I know this is the first discussion of the proposal that physics might replace continuum-sized space and time with spacetime with real-value-measurably-many points.[15]

Note that if we do invoke parsimony in order to justify a preference for space with continuum-many points rather than one of the higher cardinalities, the kind of parsimony we

[14] Rudy Rucker has recently expressed a preference for theories of space that take the parts of space to be absolutely infinite (roughly, that there are a proper class of points): see the preface of Rucker 2019. I do not see an argument for this theory beyond an appeal to "the traditional philosophical Principle of Plenitude", but it seems that he does not feel the attraction of parsimony in this domain. And the Peirceian doctrine of the absolute infinite of points of space, discussed in Putnam 1995, is another example of a view that goes for the most points, rather than the least needed.

[15] The closest I have found is Sładkowski 1996. When discussing a slightly different structural issue about spacetime, and whether we can assume the structure is less rich than one that would require measurable cardinals to represent, he says "The existence of measurable cardinals cannot be proven in the standard axioms of set theory. Even if they do exist they must be so huge that it is unlikely that the spacetime is so 'potent'" (p 10). As soon as Sładkowski brings up an option for spacetime requiring it to have this rich a structure, he sets it aside on what looks like parsimony grounds.

will need to invoke will be quantitative parsimony and not merely qualitative. The additional points only need to be more of the same kind of thing as the points postulated by traditional theories of continuous space and time. The only difference is that there are many more of them, and (in some models) they stand in slightly more elaborate spatiotemporal relations to other points. So if we are to justify standard continuum conceptions of space and time, it seems we will need to employ a parsimony principle strong enough to take cognisance of quantitative parsimony as well, presumably, as qualitative.

6. Conclusion

Parsimony considerations arise if we are to try to justify any of the typical positions about the structure of space and time. That is not to say that every theory among the currently popular options is equally well supported by parsimony considerations: plausibly, discrete spacetime is more parsimonious than standard gunky spacetime (with continuum-many regions), which is in turn more parsimonious than standard pointy spacetime (with continuum-many points and 2^c-many regions). But crucially, each of these standard options have cousins which share the same general features but differ primarily in their lack of parsimony. Discrete theories have cousins with much smaller ultimate pieces of spacetime, as well as variants with continuous spacetime but only spatiotemporal phenomena which have their spatiotemporal features quantised. Gunky theories are arguably more parsimonious than their corresponding pointy ones, but also can be formulated in a vast infinity of rivals, each differing primarily in what order of cardinality the gunky regions come in. Finally, continuous pointy theories can be formulated in a huge range of infinite sizes, at the size of the continuum or higher, with nothing much to choose between many of the alternatives than parsimony considerations. (Indeed, even when some alternatives have intuitive advantages, such as those that postulate a real-measurable cardinal of points, those alternatives still seem inferior on parsimony grounds.)

We might employ parsimony considerations to add something to the scales in favour of discrete spacetime over gunky or (continuous) pointy spacetimes, or to prefer gunky spacetime to pointy spacetime. But even if we do not, or do not think in the end those considerations can bear much weight, we should think that parsimony gives us a reason to prefer the simpler discrete, gunky or pointy spacetimes to their gratuitously unparsimonious cousins. Or so it seems to me: but even if you do not think that is the right verdict, there are still some potential lessons in considering these cases.

There are a range of potential reactions to the presence, or apparent presence, of parsimony considerations in our theorising about space, time and spacetime. If you think parsimony is methodologically worthless, in general or in metaphysics in particular, then you should make sure parsimony is not smuggled in somewhere in the case for your preferred theory: that the appeal of discrete spacetime can be articulated in a way that does not, at some point, invoke parsimony-like considerations; that gunky theories are not being motivated by the thought that points seem otiose once we have the regions; and that continuous theories have some reason for supposing there are only continuum-many points in space, or time, or spacetime, rather than vastly many more. I think this third task will be the hardest: it is hard to see how we could get direct empirical evidence for their being exactly continuum many ($\beth_1$) points rather than, say $\beth_2$, in a meter of space or a second of time; and it would strain even the ingenuity of philosophers to come up with an a priori proof that space and time were one way rather than the other.

On the other hand, if you are inclined to trust that the more parsimonious options are preferable ones in each of these cases, that should lead you to have more sympathy for parsimony principles in general. In particular, given that the cases for discrete spacetime, ordinary gunky spacetime, and ordinary continuous spacetime all seem to be a matter of *quantitative* parsimony, you should likely be sympathetic to quantitative parsimony playing a role in at least some cases. Considering those options against their alternatives does not just give us insight into what reasons are at work here, but gives us methodological lesson to be applied more broadly.

The main focus on this paper has been to argue that whether a theorist inclines to discrete spacetime, gunky spacetime or continuous spacetime, parsimony seems to be helpful, or even needed, to justify their theory over some less appealing rivals. But there is a challenge lurking here for those who would impose more structure on spacetime than discrete theorists. Once you accept that parsimony is a legitimate consideration, and a powerful enough methodological principle to rule out some rivals, the question arises why your preferred view is better than its more parsimonious rivals: and indeed why its other features make it *sufficiently* better to overcome your parsimony disadvantage. No doubt those who adopt gunky or continuous frameworks already think their frameworks superior to discrete ones. But explaining why the virtues of their preferred views outweigh the ontological costs is something those theorists have rarely done explicitly to date, so I will end my discussion by urging them to do so.[16]

References

Arntzenius, F. (2012). *Space, Time and Stuff*. Oxford: Oxford University Press.

Arntzenius, F. & Hawthorne, J. (2005). Gunk and continuous variation. *The Monist*, 88, 441-465. doi: doi.org/10.5840/monist200588432

Baker, A. (2003). Quantitative parsimony and explanatory power. *British Journal for the Philosophy of Science,* 54, 245-259. doi: 10.1093/bjps/54.2.245

Banach, S. & Tarski, A. (1924). "Sur la décomposition des ensembles de points en parites respectivement congruentes". *Fundamenta Mathematicae*, 6, 244-277.

Bombelli, L., Lee, J., Meyer, D. & Sorkin, R. D. (1987). Space-time as a causal set. *Physical Review Letters*, 59, 521-524. doi: doi.org/10.1103/PhysRevLett.59.521

Dowker, F. (2005). Causal sets and the deep structure of spacetime. In Ashtekar, A. (Ed), *100 Years of Relativity: Space-time Structure: Einstein and Beyond* (pp 445-464) Singapore: World Scientific Publishing Company.

Forrest, P. (1995). Is spacetime discrete or continuous? – an empirical question. *Synthese*, 103, 327-354. doi: 10/1007/bf01089732

Forrest, P. (2004). Grit or gunk: implications of the Banach-Tarski paradox. *The Monist*, 87, 351-370. doi: 10/5840/monist200487317

Fremlin, D.H. (2009). *Real-valued Measurable Cardinals*, Version of 19.9.09.

[16] Thanks to Sara Bernstein, Aaron Cotnoir, Dan López de Sa, Roy Sorensen, Scott Shalkowski, Giuliano Torrengo, Robbie Williams, anonymous referees, and audiences at LOGOS in Barcelona, the Centre for Philosophy of Time at the University of Milan, the Centre for Metaphysics and Mind at the University of Leeds, and the Metaphysics Reading Group at the University of St Andrews for discussion and feedback.

<https://www1.essex.ac.uk/maths/people/fremlin/rvmc.pdf>. Accessed 29 March 2020.

Grünbaum, A. (1970). Zeno's metrical paradox of extension. In Salmon, W. *Zeno's Paradoxes* (pp 176-199). Indianapolis: Bobbs-Merrill.

Hinckfuss, I. (1986). Instrumental theories: possibilities and space and time in Riggs, P. (Ed) *Natural Kinds, Laws of Nature and Scientific Methodology* (pp 187-209). Dordrecht: Kluwer.

Holden, T. (2004). *The Architecture of Matter: Galileo to Kant*. Oxford: Oxford University Press.

Huemer, M. (2009). When is parsimony a virtue? *Philosophical Quarterly*, 59, 216-236. doi: 10.1111/j.1467-9213.2008.569.x

Huggett, N. (2006). The regularity account of relational spacetime. *Mind*, 115, 41-73. doi: 10.1093/mind/fz1041

Jansson, L. & Tallant, J. (2017). Quantitative parsimony: probably for the better. *British Journal for the Philosophy of Science*, 68, 781-803. doi: 10.1093/bjps/axv064

Longino, H. (1996). Cognitive and non-cognitive values in science: rethinking the dichotomy. In Nelson, L.H. & Nelson, J. (Eds). (1996). *Feminism, Science and the Philosophy of Science* (pp 39-58). Dordrecht: Kluwer.

McLarty, C. (2010). What does it take to prove Fermat's last theorem? Grothendieck and the logic of number theory. *Bulletin of Symbolic Logic*, 16, 359-377. doi: 10.2178/bsl/1286284558

Nolan, D. (1997). Quantitative parsimony. *British Journal for the Philosophy of Science*, 48, 329-343. doi: 10.1093/bjps/48.3.329

Nolan, D. (2004). Classes, worlds and hypergunk. *The Monist*, 87, 303-321. doi: 10.5840/monist200487320

Nolan D. (2006). Stoic gunk. *Phronesis*, 51, 162-183. doi: 10.1163.156852806777006796

Nolan, D. (2008). Finite quantities. *Proceedings of the Aristotelian Society*, 108, 23-42. doi: 10.1111/j.1467-9264.2008.00234.x

Parsons, T. (1979). The methodology of nonexistence. *Journal of Philosophy*, 76, 649-662. doi: 10.2307/2025698

Perlman, E.S., Rappaport, S.A., Christiansen, W.A., Ng, Y.J., DeVore, J. & Pooley, D. (2015). New constraints on quantum gravity from x-ray and gamma observations. *The Astrophysical Journal*, 805, 10-20. doi: 10.1088/0004-637X/805/1/10

Putnam, H. (1995). Peirce's continuum. In Ketner, K.L. (ed) *Peirce and Contemporary Thought* (pp 1-22). New York: Fordham University Press.

Reeder, P. (2020). Quick and easy recipes for hypergunk. *Australasian Journal of Philosophy,* 98, 178-191. doi: 10.1080.00048402.2019.1602151

Rovelli, C. and Vidotto, F. (2014). *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*. Cambridge: Cambridge University Press.

Rucker, R. (2019). *Infinity and the Mind: The Science and the Philosophy of the Infinite*. Princeton: Princeton University Press.

Sładkowski, J. (1996). Spacetime models, fundamental interactions and noncommutative geometry. arXiv preprint hep-th/9610093

Sorensen, R. (2014). Parsimony for empty space. *Australasian Journal of Philosophy*, 92, 215-230. doi: 10.1080/00048402.2013.827732

Tallant, J. (2013). Quantitative parsimony and the metaphysics of time: motivating presentism. *Philosophy and Phenomenological Research*, 87, 688-705. doi: 10.1111/j.1933-1592.2012.00617.x

Tarski, A. (1929). Foundations of the geometry of solids. A modified translation of his 1929 Les fondemonts de la gèomètrie des corps. In Tarski, A. (1956). *Logic, Semantics, Meta-Mathematics* (pp 24-29). Oxford: Oxford University Press.

Vassallo, A., Deckert, D., & Esfeld, M. (2017). Relationalism about mechanics based on a minimalist ontology of matter. *European Journal for Philosophy of Science*, 7, 299-318. doi: 10.1007/s13194-016-0160-2

Zimmerman, D. (1996). Could extended objects be made out of simple parts? An argument for 'atomless gunk'. *Philosophy and Phenomenological Research*, 56, 1-29. doi: doi.org/10.2307/2108463