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Approximation and Idealization: Why the Difference Matters

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It is proposed that we use the term "approximation" for inexact description of a target system and "idealization" for another system whose properties also provide an inexact description of the target system. Since systems generated by a limiting process can often have quite unexpected, even inconsistent properties, familiar limit systems used in statistical physics can fail to provide idealizations, but are merely approximations. A dominance argument suggests that the limiting idealizations of statistical physics should be demoted to approximations.

1. Introduction

The terms approximation and idealization are used extensively in the philosophy of science literature for a range of practices, descriptions and structures and with more or less care to distinguish them. My concern here is not to unravel the tangled use of the terms. It is to note an important difference among the practices and structures to which the terms are applied. While it is ultimately a matter of indifference to me how these differences are reflected in our naming, it is a practical necessity for what follows that we fix terminology. I will propose a use of the terms that roughly reflects common usage: approximations merely describe a target system inexactly. Idealizations refer to new systems whose properties approximate those of the target

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system. The key difference is referential: idealizations, in the way I shall henceforth use the term, carry a novel semantic import not carried by approximations.

Attending to this difference, it will be argued, is essential to understanding how approximations and idealizations are used. The extended example will be the use of large component number limits in statistical physics. The properties of thermal systems are a function of the number of components. The limit functions are recovered by the purely mathematical operation of letting this number go to infinity. They provide inexact descriptions of systems with very large numbers of components; these "limit properties," as I shall call them, are approximations. We may also posit an infinity of components and examine the resulting system's properties. These "limit systems" are idealizations.

There are many traps in these limits—more, I assert here, than the literature has acknowledged. My concern is not the widely recognized fact that the limit functions may be singular, a fact that is connected with the analysis of phase transitions. I am concerned with far more serious oddities. The limiting system may prove to have properties radically different from the finite systems, violating both determinism and energy conservation. Or the limit may be set up in such a way that there can be no limiting state, so idealization is impossible. Approximations may also be mistaken for idealizations. Such is the case with renormalization group methods, whose celebrated results on phase transition are recovered, I will argue, from approximations and not idealization. Far from being ineliminable, there are no infinite idealizations employed and a dominance consideration argues against their introduction.

In the following, Section 2 will provide a more extensive characterization of the difference proposed between approximation and idealization. Section 2.4 will sketch how the difference separates a realist from an antirealist response to the pessimistic meta-induction. Section 3 will illustrate how infinite limits can be well- or badly-behaved. Section 4 will show each of these behaviors is implemented in analyses in statistical physics that employ the thermodynamic and related limits. Section 5 will develop the dominance argument against introduction of infinite idealizations. An Appendix recounts the emergence of indeterminism in simple, infinite systems.

2. Approximation and Idealization Distinguished

2.1 Characterizations

The term "approximation" and "idealization" are applied to a wide range of activities and structures in science. Sometimes the two terms are carefully distinguished, as in Frigg and Hartmann (2009, §1.1). Other accounts carefully dissect one term, typically "idealization," into types. McMullin (1985) distinguishes six types of idealization; and Weisberg (forthcoming) finds three. More commonly, however, the terms are used fluidly, without much discipline, and even interchangeably.[2]

My concern here is not the lexicographic task of discerning precisely how the terms are presently used. That would lead to an unproductive profusion of competing meanings. Rather my concern is to identify an important division in the range covered by the terms. Do they involve novel reference? Whether they do involve novel reference will turn out to matter a great deal to their roles in explanation, reduction and emergence in the applications below. Hence, as a notational convenience for the remainder of this paper, I will stipulate characterizations of the two terms that indicate this division and, I hope, reflect more or less common usages[3]:

> An *approximation* is an inexact description of a target system.
>
> An *idealization* is a real or fictitious, idealizing system, distinct from the target system, whose properties provide an inexact description of the target system.

These are not definitions; they merely specify important properties. They neglect to identify, for example, how inexact a description may become before we cease to admit it as an approximation of some target system. They neglect pragmatic considerations often deemed essential, such as the simplicity of the description or the intelligibility of the idealizing system. However they assert the distinction that will drive the remainder of the discussion: only idealizations introduce reference to a novel system.

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[2] For another account of the terms in statistical mechanics, see Liu (2004).

[3] The referential element of idealization recalls its origins in Plato's theory of ideal forms, whose properties inexactly describe the imperfect things of ordinary experience

The characterization of idealizations is quite permissive when it comes to the nature of the idealizing system. They may be other real systems,[4] or fictitious systems, or mathematical objects, or even parts of the target system itself. For present purposes, this level of permissiveness is quite acceptable. The analysis to follow will not depend on the nature of the idealizing systems. It will depend on whether they can exist at all; or, if they do, whether they have the properties intended.

2.2 Examples

The differences between the two can be illustrated with the familiar example of a body of unit mass, a stone, falling in a weakly resisting medium, air. Its speed v at time t is given by

$$dv/dt = g - kv$$

where g is the acceleration due to gravity and k is a friction coefficient. Its speed when falling from rest at v=t=0, is given by

$$v(t) = (g/k)(1 - \exp(-kt)) = gt - gkt^2/2 + gk^2t^3/6 - \ldots$$

When the friction coefficient k is small, the speed of fall in the early stages is described nearly exactly by the first term in the power series:

$$v(t) = gt$$

This inexact description is an approximation of the fall. We can promote this approximation to an idealization by introducing reference to a fictional system, a mass of the same size falling under the same gravity in a vacuum. This idealizing system's fall is described exactly by $v(t)=gt$ and this property provides an inexact description of the target system.

A second example foreshadows the problems to come. A colony of bacteria numbers n(t) at time t. Since the population will keep doubling in the same time unit under favorable conditions, its growth is often described by an exponential law

$$n(t) = n(0) \exp(kt)$$

for some constant k. Of course the law is an inexact description since n(t) must always be a whole number and n(0)exp(kt) will almost never be so. However the fractional error becomes smaller as the number of bacteria increases and the law becomes a better approximation.

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[4] In this case I would call the idealization a model; more generally, an idealization is more akin to a model the more the idealization has properties disanalogous to those of the target system.

One might image in this last example that one can promote the approximation to an idealization by just "taking the limit to infinity" and imagining a system of infinitely many bacteria as the idealizing system. The attempt fails. If n(t) is actually infinite, it can no longer enter into the exponential law, which would now merely assert "infinity = infinity." So we have an approximation that can be made more exact without restriction by taking larger n, but taking the limit system of an actual infinity of bacteria does not yield the intended idealization.

2.3 Promotion and Demotion

Approximations and idealizations are, to some extent, interconvertible. These last two examples indicate an important relation between approximations and idealization:

> An idealization can be demoted to an approximation by discarding the idealizing system and merely extracting the inexact description; however the inverse promotion by the introduction of an idealizing system will not always be possible.

2.4 Other Applications

The distinction drawn here is useful for approaching other debates in philosophy of science, beyond the issues to be raised below. Consider, for example, realist and antirealist responses to what has come to be known as the pessimistic meta-induction. According to it (Laudan, 1981), history of science is replete with theories that are proven to be referential failures when an antecedent theory is replaced by a contradicting successor. In spite of its theory's successes in the late 18th and early 19th centuries, there is no caloric. So we should not expect referential success of our present theories.

Antirealists affirm this conclusion. They regard the antecedent theories (and probably the successors also) as inexact descriptions without referential success; they are all mere approximations. Realists, however, regard the antecedent theory as an idealization of the successor theory. It is referentially successful in that the idealizing system is a part of the same system the successor theory describes. The "caloric" of caloric theory refers to the same thing as the "heat" of thermodynamics, but in the confines of situations in which there is no interchange of heat and work.[5]

3. The Problem of Limits

Limits can be badly behaved and this bad behavior will create problems when we try to use limit systems as idealizations. This section reviews three cases, one of good behavior and two of bad behavior, and illustrates them with simple examples. Butterfield (2010, Section 3) has also noted the importance of the diverging of limit properties and limit systems.

3.1 Limit Property and Limit System Agree

Consider a sphere of unit radius. It is elongated into a capsule, a cylinder with spherical end caps, as shown in Figure 1. Its total length grows through the sequence of cylinder lengths a = 1, 2, 3, 4, …

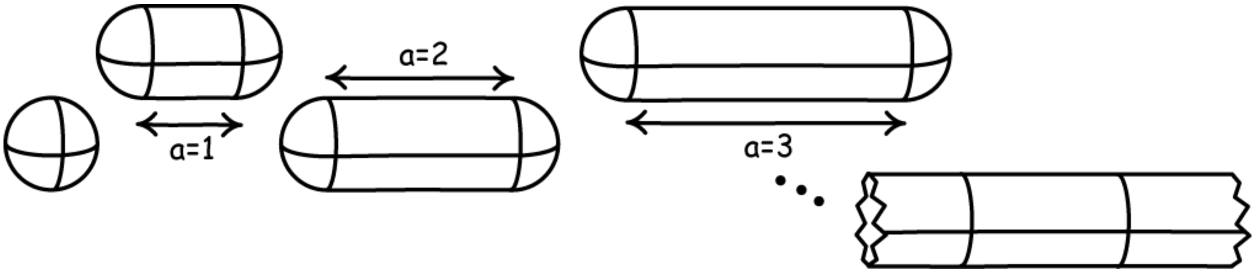

Figure 1. Sphere Elongated as Capsule

In the infinite limit, the capsule becomes an infinite cylinder of unit radius. The surface area of a capsule of cylinder length a is $2\pi a + 4\pi$ and its volume is $\pi a + 4\pi/3$. Hence the ratio of surface area to volume is $(2\pi a + 4\pi)/(\pi a + 4\pi/3)$ and the ratio approaches a limiting value of 2 as a goes

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[5] What of Feyerabend and others' claim that the referent of a term is fixed by the theoretical context, so the terms "caloric" and "heat" in different theories cannot have the same referent? In Norton (manuscript), I argue that the space of meanings in these cases is sparse, so that there are few candidates to which the terms can attach. The slight differences in properties ascribed by the two theories is insufficient to disrupt the successful reference.

to infinity. This limit of the properties of the sequence of capsules agrees with the corresponding properties of the limiting system, the infinite cylinder, whose ratio of area to volume is also 2.

The example implements the general scheme in which we have the sequences:

System$_1$, System$_2$, System$_3$, …, Limit System
agrees with
Property$_1$, Property$_2$, Property$_3$, …, Limit property

and the two cohere in that the limit property is the corresponding property of the limit system. Hence the infinite cylinder is an idealization of the larger capsules.

3.2 There is no Limit System

Consider a unit sphere whose radius r grows as r = 1, 2, 3, … as shown in Figure 2.

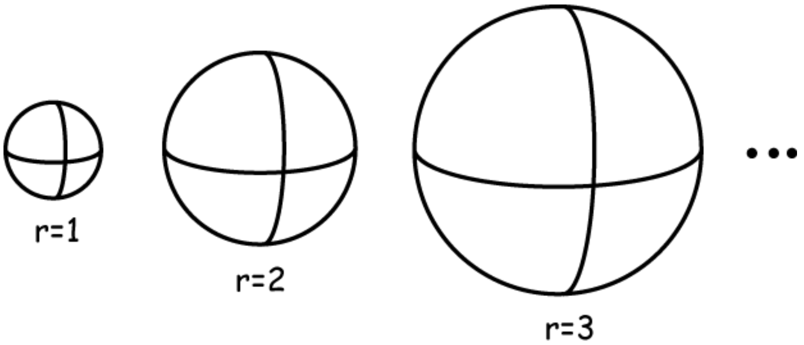

Figure 2. Sphere Expands Uniformly

The area of the sphere is $4\pi r^2$; and its volume is $4\pi r^3/3$. The ratio of surface area to volume is $(4\pi r^2)/(4\pi r^3/3) = 3/r$ and this ratio goes to zero as the radius r goes to infinity. Hence the sequence of properties has a limiting value. The sequence of systems, that is, of spheres, however, has no limit system. One might casually speak of “an infinitely large sphere” as the limit system. But that talk is literally nonsense. A sphere is the set of points equally far away from some center. An infinitely large sphere would consist of points infinitely far away from the center. But there are no such points. All points in the space are some finite distance from the center.

This example implements the scheme

$\text{System}_1, \text{System}_2, \text{System}_3, \ldots$ (No limit)

$\text{Property}_1, \text{Property}_2, \text{Property}_3, \ldots$ Limit property

There is a limit property, but it is not a property of a limit system, since there is none. The zero area to volume ratio is not a property of an impossible infinite sphere. It is a property of the set of all finite spheres; specifically, it is the greatest lower bound of the ratios of the set's members.

In this case, the limit property is an approximation, an inexact description, of the properties of the later members of the sequence of systems. However the limit provides no idealization because there is no limit system to bear the limit property.

3.3 Limit Property and Limit System Disagree

Consider once again a sphere of unit radius. Uniformly expand it in one direction only so it becomes an ellipsoid with semi-major axis a. Continue the expansion until a goes to infinity. The limit system is a cylinder of unit radius,[6] as shown in Figure 3:

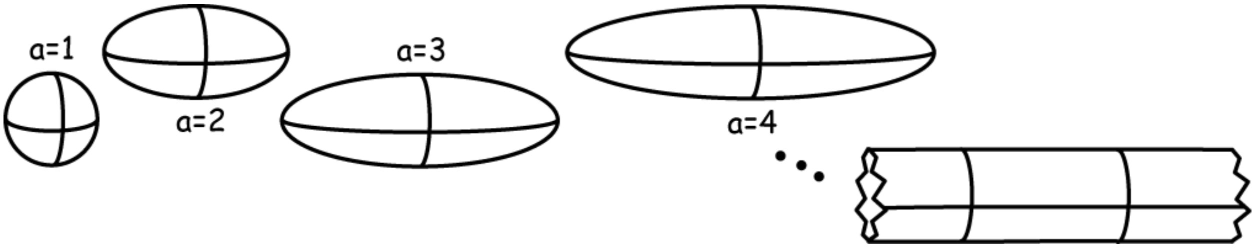

Figure 3. Sphere Elongated as Ellipsoid

The volume of the ellipsoid is $4\pi a/3$. The surface area of the ellipsoid nears a value of $\pi^2 a$ arbitrarily closely for large a. Hence the ratio of surface area to volume approaches $(\pi^2 a)/(4\pi a/3) = 3\pi/4$ as a goes to infinity. This limit ratio is *not* the same as the corresponding ratio of 2 for the limit system, an infinite cylinder.

The example implements the general scheme in which we have the sequences:

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[6] In Cartesian coordinates, the ellipsoid is $x^2/a^2 + y^2 + z^2 = 1$, which becomes an infinite cylinder, $y^2 + z^2 = 1$, aligned with the x axis, as a becomes infinite.

System$_1$, System$_2$, System$_3$, … Limit System
*dis*agrees with
Property$_1$, Property$_2$, Property$_3$, … Limit property

where the limit property is not a property of the limit system. This limit can be used to provide an approximation of the systems leading up to the limit. The limiting ratio of 3π/4 is a close approximation of the area to volume ratio of the very large ellipsoids. However the limit does not provide an idealization in that the properties of the limit system disagree with those in the sequence. That is, an infinite cylinder has an area to volume ratio of 2, which is a poor approximation of the ratio for large ellipsoids.

4. The Limits of Statistical Mechanics

Thermal systems consist of very many components: atoms, molecules, radiation modes and so on. The statistical mechanics of these systems is only able to return thermodynamic behavior when there are very many components, whose behavior is analyzed statistically. As a result, taking a limit with infinitely large numbers of components is a standard device.

When the limit is construed quite literally, we analyze the properties of systems of infinitely many components. This fictitious limit system is the idealizing system of an idealization and its properties provide inexact descriptions of the real thermal system.

In a more cautious approach, we note that the properties of real thermal systems are functions of many parameters. The probability that an oxygen molecule in a chamber of air lies within some specified energy range is a complicated function of many parameters, including the number of oxygen molecules. However, when there are very many oxygen molecules, as is the case in real air chambers, the function is almost completely independent of the number of molecules. Whatever the actual number, the function can be approximated extremely well by the probability density exp(-E/kT)/Z, for energy E and Z the normalizing partition function. This probability density is produced by the taking of the limit as a purely mathematical operation. This limit property provides the (very slightly) inexact description of an approximation.

When an analysis takes this limit, is it forming an approximation or an idealization? We can find both cases and more that are less clear.

4.1 Idealizations

The most commonly discussed limit is the "thermodynamic limit" in which both the number of components n and the volume V they occupy are taken to infinity in such a way that the ratio n/V remains constant. At least some texts are clear that the system to be investigated is the limit system of an actual infinity of components. Ruelle (2004, p.2; emphasis in original) writes:

> The physical systems to which the thermodynamic formalism applies are idealized to be actually infinite, i.e. to fill $\mathbf{R}^\nu$ (where $\nu=3$ in the usual world). This idealization is necessary because only infinite systems exhibit sharp phase transitions. Much of the thermodynamic formalism is concerned with the study of *states* of infinite systems.

Another motivation for examining an infinite system directly is that one no longer has to accommodate effects from the boundary that contains every finite system. What makes the transition to an infinity of components admissible is the assumption that the infinite systems will provide a good description of large, but finite systems. Ruelle (1999, p.11) remarks "…if a system exhibits thermodynamic behavior the states defined by the ensemble averages for large […] finite systems approach in some sense states of the corresponding infinite system…"

The hope expressed by Ruelle is that the limit property and limit system will agree as in the well-behaved case of Section 3.1. Then the limit provides an idealization in the sense defined here. The long-recognized difficulty with this idealization is that the infinite systems often have properties very different from those of the finite system. That is, they exhibit the discord of limit property and limit system of Section 3.3, so that the limit does not provide an idealization.

The difficulty is well-known. Lanford (1975, §4) describes an infinite system of hard spheres all of which are at rest until some moment of excitation, after which a disturbance propagates in "from infinity," setting all but finitely many of the spheres into motion. The system manifests a violation of determinism and also a violation of the conservation of energy and momentum. This is an early contribution to the now flourishing literature in supertask systems. It describes how infinitely many particles in classical and, sometimes, relativistic physics can interact to produce analogous violations of determinism and the conservation of energy and momentum. For recent contributions to this literature, see Lee (2011) and Atkins (2007); and for a survey, see Laraudogoitia (2011).

A version of this supertask that is not driven by carefully tuned collisions is a chain of masses connected by Hooke's law springs as shown in Figure 4. It is a simple model of a one dimensional crystal.

Figure 4. Masses and Springs

The Appendix shows that an infinite chain of these masses can spontaneously excite, violating determinism and energy conservation, and sketches how similar pathologies may arise for other infinite systems governed by dynamics that is well-behaved when applied to systems of finitely many components.

In these cases, the infinite limit system fails to provide an idealization and we have a more elaborate case of the limit property and limit system disagreeing, as in Section 3.3. All the finite systems have the properties of determinism and energy conservation; hence the limit properties are determinism and conservation. The infinite limit system, however, is indeterministic and non-conservative.

The remedy is to add further conditions. To exclude the indeterministic behavior of his system of hard spheres, Lanford (1975, p. 54) imposes a boundary condition on solutions that limits the magnitude of changes in position of the spheres. Lanford and Lebowitz (1975) consider the time evolution of harmonic systems such as crystals and one of their examples is the one-dimensional chain of masses and springs above (pp. 148-149). They do not discuss the indeterministic time evolution of this system and proceed to theorems that assert the uniqueness of time evolution. This uniqueness seems to depend upon a condition that bounds the maximum magnitude of displacements and momenta of the masses. This condition is more readily apparent in simpler versions of the uniqueness theorems, such as in Lanford (1968, p. 180, Theorem 2.1), where the component positions are directly required to be bounded functions of time.

In sum, there is a real difficulty facing the use of the thermodynamic limit as an idealization. One cannot assume that the limit of well-behaved finite systems will be a well-behaved infinite system. The remedy involves a kind of reverse engineering. We know the properties of infinite systems that are pathological, so we seek to restrict the systems for which the thermodynamic limit is taken in such a way that the pathological properties are not manifested. This is a result of some importance and we will return to it below in Section 5.4. The finite systems control the infinite systems in the sense that, if there is a conflict, we modify the infinite systems to match the finite ones.

4.2 Approximations

Another type of limit used in thermodynamics cannot be used for idealizations. Its limiting processes are beset with pathologies so that it either yields no limit system or yields one with properties unsuited for an idealization.

4.2.1 The Continuum Limit

In the continuum limit described by Campagner (1989), the number of components n goes to infinity in such a way that the extensive magnitudes of the system, such as volume and energy, remain constant. If d is a parameter that measures the size of the individual components, this condition entails that nd^3 remains a non-zero constant; for nd^3 is proportional to the volume of the system occupied by matter. Similarly, Boltzmann's constant k goes to zero since nk remains a non-zero constant. For the mean energy of a mole of a monatomic gas, n = N, is given by (3/2)NkT and this remains constant in the limit.[7]

While many properties will approach well-behaved limits, the system itself has no well-defined limit state. One might imagine that the infinitely many, infinitely small components spread over a finite volume have become a uniform matter distribution. However such a uniform distribution is not approached by the system in the limit. To see the problem, imagine that the system consists of massive components that half fill the volume and that the matter density within the components is uniformly unity.[8] The occupied portions of the volume consist of many

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[7] Since k sets the scale of thermal fluctuations, there are no fluctuations in this continuum limit. Compagner (1989, p. 106) suggests that "[t]he continuum limit is to be preferred above the thermodynamic limit when macroscopic dependencies on space and time are present…" and illustrates the claim with the example of capillary phenomena.

[8] Compagner's (1989) components are interacting points with a length parameter ρ in the interaction that goes to zero in the limit. We can conventionally fix the extent of each component as ρ or some function of it; and set the matter density at a point of space to unity just in case the point lies within ρ of a component's center.

islands of matter, where the matter density is unity; and they float in a sea of emptiness, where the matter density is zero. As we approach the limit, the matter islands are divided into smaller islands. However at no stage does the system consist of anything other than regions with matter density unity and regions with matter density zero. Hence, the density of matter at an arbitrary point in space will oscillate between 0 and 1. It will not approach the limiting value 1/2 of the uniform matter distribution.

We can see the difficulty clearly in a simplified example. Consider a unit square that is divided into half, quarter, eighth squares, … in stages 1, 2, 3, … of a process as shown in Figure 5. At each stage, half the squares are occupied—represented by shading—and half are not.

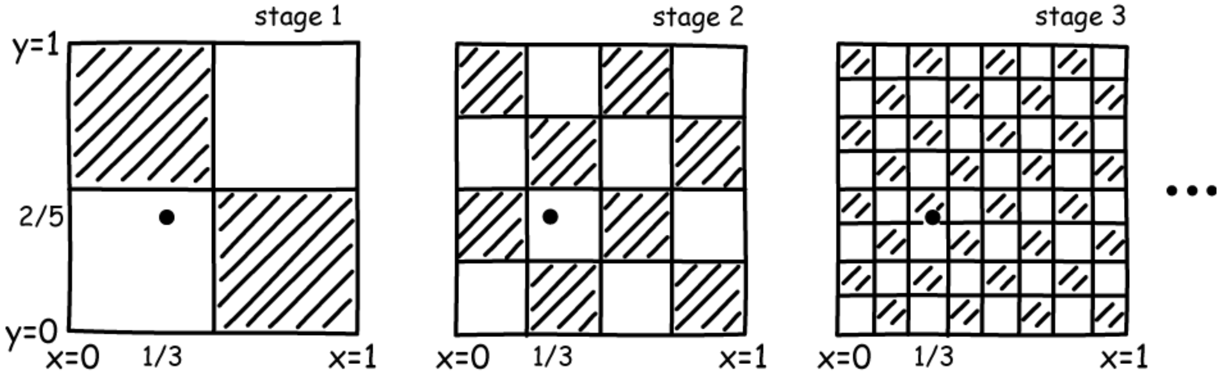

Figure 5. Sequences in Halftone Printing

The sequence mimics halftone printing, which simulates grey scales even though the printer can only assign either black or white to each point. Some points will approach limiting values. For example, points on the diagonal, such as (1/3,1/3) will have the state "white" at all stages and thus carry "white" as a limiting value. Others, however, will oscillate indefinitely. The states at the point (1/3, 2/5) shown in the figure will oscillate indefinitely as

white, white, black, black, white, white, black, black, …

as we pass through the stages and thus admit no limiting value.[9] Hence the limit will return points whose limiting color is black, whose limiting color is white and points with no limiting color. It does not yield a unit square uniformly covered in a 50% grey tone.

The situation with the continuum limit of a system of extended masses is the same. It will not produce a limit system in which the matter is somehow uniformly spread over the volume of space. It will produce points that either carry matter fully, not at all or have no limit state. We come close to the halftone printing example if we consider the system of masses to be cooled to the absolute zero of temperature, so that the masses are at rest in an equilibrium configuration. If the resulting crystal is a cubic lattice, then the halftone printing analysis can be applied directly.

4.2.2 The Boltzmann-Grad Limit

A second example in which idealization fails is the Boltzmann-Grad limit taken in generating the Boltzmann equation. (Lanford, 1975, p.70-89; 1981.) The system consists of n hard spheres of diameter d. In the limit, n goes to infinity and d to zero in such a way that nd^2 remains a non-zero constant. Since the volume of space occupied by matter varies with nd^3 = constant x d, the fraction of the volume of space occupied by matter drops to zero. Hence the limit state consists of a countable infinity of extensionless points occupied by matter and the rest of the volume of space is empty. Therefore the limit state is not beset by a lack of convergence of the halftone printing problem; or at least the problem arises at worst in a measure zero set.

In the analysis leading up to the Boltzmann equation, the non-zero size of the bodies and their spherical shape determines whether two nominated bodies will collide and which will be the resulting out-going motions. This resolution of collisions leads to the computation of the changes with time of the distribution of the spheres over the possible positions and velocities.

However the limit state is too impoverished to support this computation. It now consists of an infinity of points of zero size in motion. If two points of the limit state collide because their positions coincide (a measure zero event), we can no longer determine the collision outcome. We need to determine six quantities: three velocity components for each of the two outgoing points.

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[9] The rule for computing this series requires that the coordinates be expanded as binary numbers. $1/3 = 0.010101010\ldots_2$ and $2/5 = 0.011001100\ldots_2$. The point is white at the nth stage if the nth digits of the two numbers agree; and black if they disagree.

We have only four equations: three for momentum conservation and one for energy conservation. Hence any collision that may happen has become indeterministic. Until we reach this limit state, collision outcomes can be determined uniquely since we have the added condition that, when spheres collide, the momentum transfer is perpendicular to the plane of contact of the two sphere's surfaces. (Lanford, 1975, p.8). Two colliding points no longer have a definite plane of contact.[10]

Hence neither continuum limit nor Boltzmann-Grad limit support idealization. The first has no limit system; and the second has a limit system too impoverished to supply an inexact description of the finite systems.

4.3 Renormalization Group Methods are Approximations not Idealizations

The examples of the last two sections are less common. More commonly, when a limit to infinitely many components is considered, it is left unclear whether the limit is taken only for properties (approximation); or whether the limit system of an actual infinity of components is intended (idealization).

While some authors, such as Lanford and Ruelle are clearly investigating the properties of infinite systems, others give definitions of the thermodynamic limit that mention only the existence of limit properties. Le Bellac et al. (2004, pp. 112) consider an extensive magnitude A(T, V, N), where T is the temperature, V the volume and N the number of components. They consider the limit

$$\mathrm{Lim}_{N,V \rightarrow \infty} A(T,V,N)/V = a(n, T)$$

where n = N/V is kept finite and, presumably, non-zero in the limit. The thermodynamic limit of A(T,V,N) is said to exist if a(n,T) is finite. Conspicuously absent is any condition on an actual infinity of components and the corresponding behavior of that infinite system.

It is routine for accounts of renormalization group methods to remark that the thermodynamic limit is essential for recovery of the discontinuities in thermodynamic quantities

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[10] We could, of course, declare that the limiting distribution of the finite system to be the distribution to be applied. However that is to add a property not resulting from the dynamics of the indeterministic collision dynamics and without an assurance that the added distribution is compatible with the indeterministic dynamics.

at critical points. Kadanoff (2000, p. 239) reviews the governing fact that a partition function of a system of finitely many components is analytic. It becomes non-analytic only in the limit of infinitely many components, whereupon the thermodynamic quantities derived from it can harbor discontinuities that characterize critical points. He continues:

> We reach the important conclusion:
>
> The existence of a phase transition requires an infinite system. No phase transitions occur in systems with a finite number of degrees of freedom.

But which limit is actually used in the methods? In so far as they yield results, we shall see the limit taken by renormalization group methods is of the properties only, such as in the thermodynamic limit of Le Bellac et al.

The methods are applied in a space of reduced Hamiltonians and used to create transformations between different Hamiltonians in it. We start with a thermal system with Hamiltonian H and suppress explicit dependence on some of the thermodynamic degrees of freedom to arrive at a new Hamiltonian H'. In "real space" renormalization, the components in space are collected into clusters, hiding the degrees of freedom in the clusters. Each is a component for a new Hamiltonian H' of the same mathematical form as H, but with different parameters. The clustering reduces the number of components from N to N'. If the dimension of the space is d, the two are related by

$$N/N' = b^d$$

for some constant b. This transformation is only well defined if both N and N' are finite. In momentum space renormalization, we Fourier transform our descriptions, replacing position variables by momentum variables. We "trace out," that is, sum over and thus hide, the high frequency (=high momentum) modes of the Hamiltonian H to arrive at the new H'.

The transformation of thermodynamic quantities is derived by recalling that the two Hamiltonians H and H' are just different descriptions of the same system; so they must have the same total thermodynamic properties, such an energy, entropy and free energy. Since total free energy F = -kT ln Z(H), where Z(H) is the partition function derived from H, it follows that equality of total free energy of the two systems, F=F', entails equality of the partition functions

$$Z(H) = Z'(H')$$

Fisher (1982, p. 68)[11] calls this essential equality "unitarity." The transformation of all thermodynamic quantities is derived from it. For example, the free energy f per component is given as $f = -(kT/N) \ln Z$ and the energy u per component is given as $u = (kT^2/N)\, \partial \ln Z/\partial T$. Hence they transform as $f = (N'/N)\, f'$ and as $u = (N'/N)\, u'$.

These derivations, in both real space and momentum space renormalization, depend essentially on the finitude of the system. The partition function of homogeneous systems of components is a product of many equal terms, one for each component. If the system consists of infinitely many components, then its partition function is zero or infinity and unitarity will no longer induce non-trivial transformations.

The finite systems used to generate the transformations may be large subsystems of still larger finite systems, or even subsystems of an infinite system, if one knows the infinite system's behavior is not pathological. But the Hamiltonians related by the transformations must describe finite systems, so that the Hamiltonians yield the finite, non-zero partition functions of a non-degenerate unitarity condition.

The renormalization group transformations induce a flow over the space of reduced Hamiltonians and, for the reasons just given, this portion of the space must correspond to systems of finitely many components. Systems of infinitely many components at best enter as limit points of the flows, since the transformations cannot map a finite system to an infinite system. They might correspond, for example, to the limit of the reversed sequence of transformations that undoes the suppression of component number degrees of freedom:

$$N^{(1)} = b^d N \qquad N^{(2)} = b^{2d} N \qquad \ldots \qquad N^{(n)} = b^{nd} N \qquad \ldots$$

Critical points connected with infinitely many components appear in the diagrams of the space of reduced Hamiltonians (e.g. Fisher, 1982, p. 85). They are introduced as limit points that topologically close an open region of space filled with the renormalization group flow.

The methods do not directly compute the properties of systems of infinitely many components associated with the critical points. Rather properties attributed to the critical points are just the limit properties of finite systems. Thus we cannot take these critical points to represent an actually infinite system and their properties. For such systems may have properties very different from finite systems. They may, as we have seen, violate determinism and energy

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[11] See also Yeomans (1992, p. 107).

conservation, so that the whole framework of statistical physics would collapse. Without an analysis that precludes these anomalies, the points are best seen as mathematical artifices completing a diagram.

As an explanatory analogy, consider the space of all spheres. It is one-dimensional and coordinatized by the curvature (=1/radius^2). A map on the space expands and contracts the spheres; typical transformations would map spheres to others twice, thrice, … their curvature. The resulting flow fills the portion of that space where curvature is greater than zero as shown in Figure 6. We close the space by adding the point at zero curvature, where a natural extension of the map has a fixed point. We might imagine that the added point represents a "sphere of zero curvature" and attribute limiting properties of finite spheres to it, such as the possession of a unique center and an inside and outside. However there is no sphere of zero curvature. If the point represents anything at all, it represents an infinite Euclidean plane, which does not carry the sphere's limiting properties.

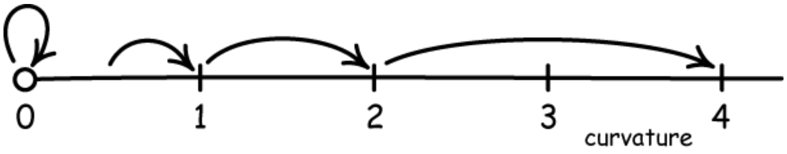

Figure 6. Space of sphere curvature.

Critical points enter real space renormalization group methods as points in a diagram: mathematical pegs on which to hang limit properties. They do not arise from an investigation of the properties of infinite limit systems. They are not idealizations.

The acclaimed results of the methods pertain to critical exponents. For many systems, thermodynamic quantities at temperatures near the critical temperature turn out to be powers of the reduced temperature $t = (T-T_c)/T_c$, commonly written as $|t|^{-\alpha}$, $(-t)^{\beta}$, $|t|^{-\gamma}$, etc. Renormalization group methods have enjoyed great success in explaining universality: that very few numerical values of the critical exponents, α, β γ, … suffice for very many substances. The results are recovered by examining the renormalization group flow in the vicinity of the critical point. In this vicinity, the systems are finite, so that the results recovered apply to finite systems, albeit of arbitrarily large size. That is, in so far as limits enter, the results are recovered for limit properties, not limit systems, so that the acclaimed results concerning critical exponents are recovered by approximations.

5. The Elimination of Idealizations

There is a spirited debate in the present literature over whether phase transitions are emergent phenomena that cannot be recovered reductively from statistical mechanics.[12] The debate is wide-ranging and subtle. The most perspicacious of many noteworthy contributions is Butterfield's (2010). He argues that emergence, properly understood as novelty and robustness, is compatible with reduction, so that one may have both. My concern here is just one argument used to support the anti-reduction view. It asserts, contrary to the analysis of the last section, that renormalization group methods do employ infinite idealizations, that these infinite idealizations somehow outstrip the reductive powers of statistical mechanics and, moreover, that they are ineliminable.[13]

My purpose in this section is to show that careful attention to the difference between approximation and idealization leads one to a different conclusion. If infinite idealizations are employed, far from being ineliminable, the infinite idealizations of statistical mechanics can be and should be eliminated.

5.1 Explanations from Approximations

Before proceeding to the main argument in the next section, we should note that use of limits to provide explanations of the behavior of finite target systems is delicate, for infinite limits may behave in ways we do not expect or intend. A simple example is the use of the continuum limit in which Boltzmann's constant k vanishes. That means that the limit eradicates fluctuations and cannot be used to explain fluctuation phenomena like Brownian motion and

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[12] For an entry into this literature see Batterman (2002, 2005, 2010, 2010a), Belot (2005), Butterfield (2010, 2010a), Butterfield and Bouatta (2011), Callender (2001), Jones (2006), Liu (1999) and Menon and Callender (manuscript).

[13] For a survey and defense, see Jones (2006).

critical opalescence. As noted in Section 4.1, the finite systems control the limits and the latter cede whenever there is a conflict.

Given this fragility, explanations derived from approximations are the more secure for they assume less. One only considers the limit properties, that is, the properties of finite systems in the limit in which their number of components grow indefinitely large. Their use in a successful explanation requires that two conditions are met:

(i) As we consider systems with larger numbers of components, the properties will eventually settle down to stable, limiting values. This stability must be achieved by the stage in which the limit process has arrived at the number of components possessed by the target system.

(ii) The limiting values of the properties do match those of the target system. This seemingly innocuous condition can fail, as it did in the case of fluctuations and the continuum limit.

If both conditions are satisfied, the limit properties will match closely with those of the target system and good explanations will be supported. Note that this conclusion is independent of the account of explanation one may favor.

5.2 The Dominance Argument

Explanations that employ idealizations have an extra complication: they are at least in part analogical: they depend on the limiting system and the target system agreeing sufficiently in their properties for the explanation to proceed. Thus, in addition to (i) and (ii), successful explanation with idealizations requires a third condition:

(iii) The limit properties of the finite systems match those of the limit system. [14]

For a given limit, we may choose to base our explanations on the limit properties of the approximation or on the limit system of the idealization. We now conclude that dominance[15] considerations direct that we choose the approximation. There are two cases:

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[14] There is one exception that I discount as unlikely. Condition (ii) on limit properties may fail and the infinite system may disagree with the limit properties in a way that fortuitously cancels so that the limit system ends up matching the target system.

Case 1. The limit properties of the finite systems do not agree with the properties of the limit system.

In this case, we definitely should employ only the approximation lest the explanation fail.

Case 2. The limit properties of the finite systems agree with the properties of the limit system.

It was noted in Section 2.3 above that an idealization of this type can always be demoted to an approximation by discarding the limiting system and extracting the inexact description. In this case, there is no gain in retaining the infinite idealization; all its results are already available from the approximation. In Case 1, we gain choosing the approximation; in Case 2 we are indifferent. Dominance directs selection of the approximation, especially in the common occurrence that we are unsure which case is at hand.

Perhaps the only escape from this dominance is a parochial one. It may turn out that determining the properties of an infinite limit system is computationally feasible in a way that taking the limit properties of finite systems is not.[16] The difficulty in such a case is to know that the limit properties do agree with those of the limit system; and if we have sufficient understanding of the limit properties to know that, why not just base the explanation on those limit properties directly?

5.3 Illustration of Non-Analyticity with Finite Systems

What of the argument that infinite systems are needed if thermodynamic functions are to be non-analytic and thus support the discontinuities of phase transition? The argument overstates what is needed. One does not need the limit system with an infinity of components. Consider the functional dependence of some property on the number of components n in a set of systems all of whom have a finite n. Taking the limit as n goes to infinity in this function, as a purely mathematical operation, can yield a non-analytic function that is a very good approximation of

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[15] The term dominance is used in the decision-theoretic sense. That is, a strategy weakly dominates another if the first yield outcomes that are sometimes better and never worse than those yielded by the second.

[16] Lanford (1975, p. 17) seems to suggest this when he justifies the analysis of infinite systems as "the only precise way of removing inessential complications due to boundary effects, etc."

the analytic functions of system of large, finite n. Butterfield (2010) has described this effect with the slogan of "emergence before the limit."

The mathematics of the partition functions is complicated even in simple cases. See Le Bellac et al. (2004, p. 183) and Jones (2006, Section 3.1.3) for the simpler case of an Ising chain. The essential point, however, can be illustrated in a toy model. Imagine that, in some theory, a particle of type n generates a potential well of the form

$$\phi_n(x) = (x/L)^{2n}$$

where n = 1, 2, 3, … is always some finite, whole number. For each n, this potential well is an analytic function of the position coordinate x. When n grows large, however, $\phi_n(x)$ will approach the limit of an infinitely high square well, as shown in Figure 7:

$$\begin{aligned} \phi_{lim}(x) &= 0 \quad |x| < L \\ &= 1 \quad |x| = L \\ &= \infty \quad |x| > L \end{aligned}$$

This limit function is not analytic (and not even continuous), but has been recovered from considerations of particles with finite n only. Moreover, the square well $\phi_{lim}(x)$ will approximate $\phi_n(x)$ extremely well, especially when n is large, say 10^{23}.

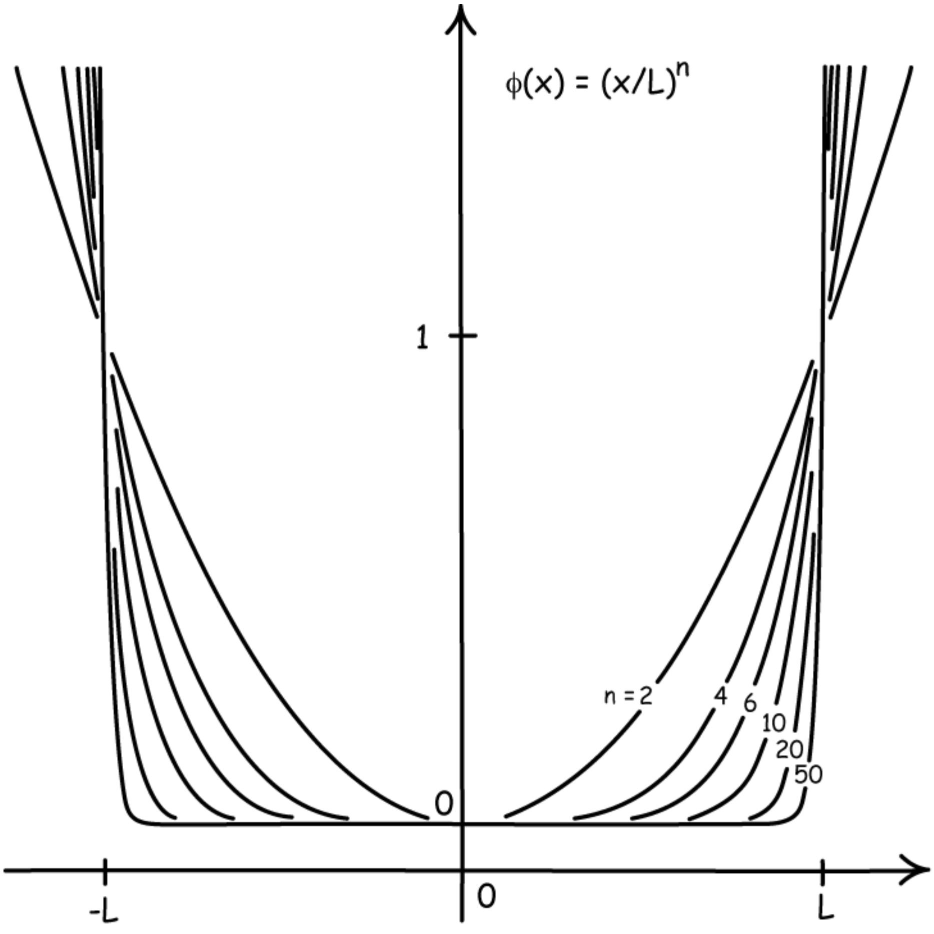

Figure 7. Analytic potential functions approach non-analytic limit.

5.4 Finite Systems Control Infinite Systems

While we can recover non-analytic functions from operations on the functions governing arbitrarily large systems of finitely many components, we should not lose sight of the fact that these non-analytic functions and their discontinuities have a pragmatic value only. If the atomic theory of matter is true, then ordinary thermal systems cannot display discontinuous changes in their thermodynamic properties. The changes they manifest are merely so rapid as to be observationally indistinguishable from discontinuous behavior.[17] Indeed, if we could establish that the phase transitions of real substances exhibit these discontinuities, we would have refuted

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[17] This has also been emphasized by Callender (2001) and Butterfield (2010). For a survey and response, see Batterman (2005) and Jones (2006, Ch. 5).

the atomic theory of matter, which holds that ordinary thermal systems are composed of finitely many atoms, molecules or components. It must be feared that a similar refutation is at hand if the positing of infinitely many components is necessary to recover the observed behavior of phase transitions.

The moral is that the properties of finite systems control and infinite systems cede to them when there is a conflict. One finds this view expressed in the physics literature. Lanford (1975, p. 17) writes:

> We emphasize that we are not considering the theory of infinite systems for its own sake so much as for the fact that this is the only precise way of removing inessential complications due to boundary effects, etc., i.e. we regard infinite systems as approximations to large finite systems rather than the reverse.

Fisher (1982, p.14) portrays phase transitions with true discontinuities as unrealized in the laboratory and even experimentally refutable:

> …in the laboratory one would always be dealing with a finite system, with a finite number of atoms confined in a bounded region of space. A perfectly sharp phase transition can take place only in a truly infinite system, i.e., in the *thermodynamic limit* where the system is infinitely large in extent but its density, pressure, and all other intensive quantities are fixed and finite. However large a system is in practice, it will still be finite and, ultimately then one will reach the point where the specific heat singularity is seen to be rounded off. Experiments deliberately done on small samples certainly show these rounding effects. So in talking about a phase transition one really should always have in mind the thermodynamic limit. (Fisher's emphasis)

Le Bellac et al. (2004, p. 184) are similarly concerned to qualify their statement that the non-analytic behavior of phase transitions occur only in the thermodynamic limit:

> The reader, who has undoubtedly observed an ice cube floating in a glass of water, may find this statement a bit surprising. What is meant by this statement is that the *mathematical* signature of a phase transition can only be seen in the infinite volume limit. (Le Bellac et al.'s emphasis)

The physics of phase transitions is complicated and difficult and its philosophical analysis rewarding. However once one sees past the traps of the infinite limits, it is hard to find philosophical discontinuities.

In the venerable deductive-nomological ("DN") or "covering law" account of explanation of Hempel and Oppenheim (1948), one explains some phenomenon by deducing it from physical laws with the assistance of particular conditions. The model has been widely and justly criticized and there seems to be every reason to expect that the practice of explanation in science is so irregular as to admit no clean account. However there are a few pedestrian cases in which the DN model works. The use of limits in statistical mechanics as approximations provides such a case. The phenomenon to be explained is, for example, universality: that many substances manifest the same critical exponents. Renormalization group methods take the theoretical framework of statistical mechanics as the covering law. They select as the particular conditions a broad class of Hamiltonians pertinent to the materials. They then derive universality under conditions close to criticality. The renormalization group analysis simply is a covering law explanation.

While the ontological reduction of ordinary matter to atoms, molecules and like is as secure as any result of science, we cannot have the same confidence for explanatory reduction. The traffic in Los Angeles may ontologically be nothing but atoms, molecules and heat radiation. Yet we surely cannot expect their statistical mechanics to provide an explanation of traffic jams. Nagel's (1961, Ch. 11) is the venerable account of reduction in which the less fundamental theory is derived from the more fundamental. It too has been much criticized and justly so. However there are a few pedestrian cases in which it still seems to apply.[18] In providing a covering law explanation of critical exponents, renormalization group methods are also providing a Nagel-style reduction, or at least something like it, such as the more sophisticated version of Schaffner (1967).

Finally, the idea that discontinuous changes of phase transitions are emergent phenomena is difficult to penetrate. For the phenomena at issue are not possible objects of experience. Real phase transitions cannot exhibit the discontinuities on pain of contradicting the atomic theory of matter and, were the discontinuities established factually, the atomic theory would fall.

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[18] For a recent defense of this form of reduction with similar applications intended, see Butterfield (2010a).

6. Conclusion

This paper has sought to distinguish two sorts of analytic activity. One employs only inexact descriptions of some target system and is here labeled "approximation." Another introduces a new system whose properties provide inexact descriptions; it is here labeled "idealization." It is important to attend to the difference between the two. The extended example was of the use of limits in statistical mechanics. They may merely provide approximations as the limiting properties of finite systems, as their number of components grown large. Or they may provide idealizations if we posit a system of infinitely many components and examine the new system's properties. Since an infinite system can carry unexpected and even contradictory properties, the latter practice carries considerably more risk. Renormalization group methods are sometimes described as employing ineliminable, infinite idealizations. I have argued that their methods only employ approximations in the form of the limiting properties of large systems that always have finitely many components. If idealizations are present, a dominance argument favors their replacement by approximations.

Appendix: Violation of Determinism and Energy Conservation for Systems of Infinitely Many Components

Consider a system of n components interacting under some dynamics that is well-behaved in the sense that it is deterministic and conserves energy and momentum. This good behavior can persist when the number of components, n, grows arbitrarily large, but is still finite. However, if we allow the number of components to become infinite, we can lose both determinism and conservation.

A General Sketch

The simplest way to see this possibility is to construct a pathological solution in which the infinite system spontaneously excites from a quiescent state, even though the dynamics for all finite systems is well-behaved. The following sketch shows how one can construct such a pathology.

Consider a subset of n components of the infinite system. We could write down a pathological solution for this subsystem in which the system spontaneously excites from a quiescent state. If the totality of the system consisted of just these n components, that pathological solution would be inadmissible. By supposition, the dynamics applied to finite systems is well-behaved.

However these n components are a subsystem of the larger system. If the pathological solution for the n components is carefully chosen, there will be some motion for the next m components that will drive the pathological solution for the n components. We now have arrived as a pathological solution for (n+m) components.

The analysis now repeats. This pathological solution for (n+m) components can be driven if the next p components have suitable behavior. By repeating the analysis further, the pathological solution is propagated over all components to produce a pathological solution of the dynamics of the infinite system.

The infinity of the system plays an essential role. If there were just finitely many components—N, say, then the analysis would fail. For once the pathological solution was propagated to all N components, there would no longer be any further components to drive the pathological solution.

Masses and Springs

The example of the masses and springs of Section 4.1 illustrates this mechanism for generating pathological solutions.[19] Infinitely many unit masses are connected in a chain, infinite in both directions, with the masses numbered, …, -2, -1, 0, 1, 2, … The springs that connect neighboring masses are governed by Hooke’s law and are assumed to have a unit spring constant. Hence, if the displacement of the n-th mass from its equilibrium position is x_n, its equation of motion is

$$d^2x_n/dt^2 = (x_{n+1} - x_n) - (x_n - x_{n-1}) \tag{A1}$$

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[19] The analysis follows Norton (1999). The resulting indeterminism manifests as a failure of an infinite system of differential equations to admit a unique solutions. Most of the literature on such systems is devoted to determining conditions under which the system has unique solutions. There is a small literature that investigates when uniqueness fails. See, for example, Hille (1961).

This same equation (A1) holds if we consider the displacements of the masses to be restricted to the one-dimension of the chain; or if they are constrained only to move orthogonal to the chain. If we set initial conditions

$$dx_n(0)/dt = x_n(0) = 0 \qquad \text{for all n} \tag{A2}$$

we see immediately that a future time development is the quiescent

$$x_n(t) = 0 \qquad \text{for all n, all t} \tag{A3}$$

We construct a pathological solution by stipulating motions for the masses 1 and 2 that conform with the initial conditions (A2), but deviate from the quiescent solution (A3) for some t>0. It turns out that we will need to stipulate in addition that the functions $x_1(t)$ and $x_2(t)$ satisfy

$$d^m x_1(0)/dt^m = d^m x_2(0)/dt^m = 0 \qquad \text{all m} \tag{A4}$$

Hence $x_1(t)$ and $x_2(t)$ cannot be analytic functions of time, excepting the uninteresting case of the constant function. A suitable choice is

$$x_1(t) = x_2(t) = (1/t) \exp(-1/t) \tag{A5}$$

These two functions form the heart of a pathological solution of the infinite chain in which the chain is quiescent at t=0 and then spontaneously excites into motion after t=0. The remaining motions are computed iteratively using (A1). That is,

$$x_3 = d^2x_2/dt^2 + 2x_2 - x_1 \tag{A6}$$

and, by differentiation,

$$dx_3/dt = d^3x_2/dt^3 + 2dx_2/dt - dx_1/dt \tag{A7}$$

The resulting function $x_3(t)$ will satisfy the initial condition (A2) since $x_3(t)$ and $dx_3(t)/dt$ are linear functions of $x_1(t)$ and $x_2(t)$ and their derivatives at t=0, all of which vanish at t=0.

The motion x_4 is computed as

$$x_4 = d^2x_3/dt^2 + 2x_3 - x_2$$

$$dx_4/dt = d^3x_3/dt^3 + 2dx_3/dt - dx_2/dt$$

so that, by (A6) and (A7), $x_4(t)$ and $dx_4(t)/dt$ are also linear functions of $x_1(t)$ and $x_2(t)$, whose derivatives vanish at t=0. Hence this $x_4(t)$ will satisfy the initial condition (A2).

This iterative computation is repeated for all remaining masses. In general, $x_n(t)$ and $dx_n(t)/dt$ are linear functions of of $x_1(t)$ and $x_2(t)$ and their derivatives, all of which vanish at t=0. Hence they satisfy initial condition (A2), but they differ from the quiescent (A3) for some t>0.

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