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The Dome: An Unexpectedly Simple Failure of Determinism

John D. Norton[1]
Department of History and Philosophy of Science
and Center for Philosophy of Science
University of Pittsburgh
www.pitt.edu/~jdnorton

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Because of the specific shape of the dome at its apex, Newton's equations of motion tell us that a mass at rest at the apex can spontaneously be set into motion. It has been suggested that this indeterminism should be discounted since it draws on an incomplete rendering of Newtonian physics; or it is "unphysical"; or it employs illicit idealizations. I analyze and reject each of these reasons.

1. Introduction

It has been widely recognized for over two decades that, contrary to the long-standing lore, Newtonian mechanics is not a deterministic theory. The clarion call came in John Earman's (1986, Ch. III), which recounted the failure of determinism, including the then recent discovery by Mather and McGehee of "space invader" systems of interacting particles that spontaneously

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[1] My thanks to Stephan Hartmann, Alexandre Korolev and Dmitri Tymoczko for their interest in engaging in discussion of the dome; to Bernie Goldstein for help in figuring out "unphysical"; to David Malament for his efforts in organizing the symposium; and to my co-symposiasts, David Malament and John Earman, whose work and personal example have long been energizing and inspiring.

rush into an empty space from spatial infinity. Further, simpler violations of determinism emerged. Pérez Laraudogoitia (1996) described an especially simple example of "supertask" indeterminism in which a countable infinity of masses confined to a unit interval are spontaneously energized; and Norton (1999) described a correspondingly simple example of a countable infinity of masses connected by springs that are spontaneously energized.

In this developing tradition, the simplest example so far of indeterminism in Newtonian physics is what has come to be known as "the dome," described in Norton (2003, §3). The indeterminism involves none of the complications of infinitely many systems interacting or masses appearing with unbounded speed from spatial infinity. A mass sits on a dome in a gravitational field. After remaining unchanged for an arbitrary time, it spontaneously moves in an arbitrary direction, with these indeterministic motions compatible with Newtonian mechanics.

In this note I will review briefly the physics of the dome in Section 2, paying attentions to aspects of it that seem to have attracted attention or caused confusion. I will then address a concern I have heard repeatedly voiced. Is it possible that the very great simplicity of the indeterminism of the dome has come through some improper maneuver in Newtonian theory? I will outline the candidates for this improper maneuver in Section 3, in so far as I can discern them, and describe there why I find none of them improper. We shall see that resolving this issue proves to be of unexpected philosophical interest. It requires a careful appraisal of three questions: Just what is Newtonian theory? What do we mean by the notion "unphysical"? Are some idealizations improper? In Section 4, I will review how I see the dome fitting into the distribution of indeterminism in Newtonian theory. Using an informal argument, I will suggest that indeterminism is generic in Newtonian systems with infinitely many degrees of freedom; whereas it is exceptional in Newtonian systems with finitely many degrees of freedom. The dome is one of the latter exceptions.

2. The Dome in Brief

2.1 Description

The dome is a radially symmetric surface shown in Figure 1. Its shape is defined by the relation

$$h = (2/3g)\, r^{3/2} \qquad (1)$$

where r is the radial distance coordinate in the surface of the dome, h is the vertical distance of each point below the apex at r=0 and g is the constant acceleration of free unit mass in the vertical gravitational field surrounding the surface. A point like unit mass slides frictionlessly over the surface of the dome. Initially, it is at rest exactly at the apex.

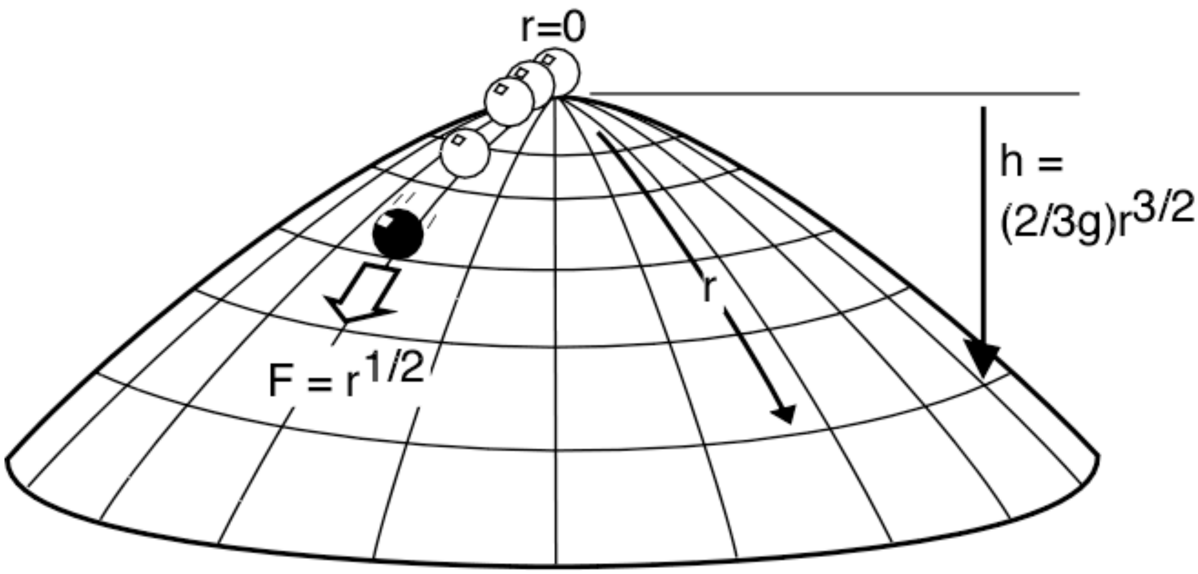

Figure 1. The Dome

Newton's laws of motion are applied in the form:

If the net applied force F vanishes, then the acceleration a vanishes. (N1)

and, for a body of mass m,

$$F = ma \quad \text{(N2)}$$

For the unit mass on the dome, the net force F acting on it is directed radially outward. Its magnitude is that of the component of the gravitational force tangent to the surface, which is g sin θ, where θ is the angle between the tangent to the surface in the radial direction and the horizontal. Since sin θ =dh/dr, we have that F = g dh/dr = $r^{1/2}$. Using Newton's second law (N2) and that a = d^2r/dt^2 we recover the equation of motion of the mass as

$$d^2r/dt^2 = r^{1/2} \quad (2)$$

The expected solution is

$$r(t) = 0 \quad (3)$$

in which the mass simply remains at rest for all times t. Another family of solutions represents spontaneous motion at an arbitrary time T in an arbitrary radial direction:

$$r(t) = (1/144)\,(t - T)^4 \quad \text{for } t \geq T \qquad (4)$$
$$= 0 \quad \text{for } t \leq T$$

One sees immediately that (4) satisfies (2) if we compute the radial acceleration $a(t) = d^2r(t)/dt^2$, which is

$$a(t) = (1/12)\,(t - T)^2 \quad \text{for } t \geq T \qquad (5)$$
$$= 0 \quad \text{for } t \leq T$$

and note that a(t) as given in (5) is the square root of r(t) as given in (4).

2.2 Individual versus Collective Indeterminism

The family of solutions (4) are indeterministic[2] in the standard sense that a single past can be followed by many futures. The mass may be at rest, for example, for all times up to t=0, an example of one fixed past. It may then spontaneously move at any time after that in an arbitrary direction.[3]

This manifestation of indeterminism differs from those already in the literature in an interesting way. In the case of supertask indeterminism, each individual component is well-behaved. If a component is set into motion, it is because is has been, struck, pushed or pulled by another component. The space invader form of indeterminism is odder in the sense that it involves components that pop into being "from spatial infinity" with unbounded speeds and unbounded energies. However it remains true that each individual component is well-behaved locally. That is, if we look at the motion of any component in some part of spacetime of finite

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[2] The mathematically minded may note that (2) with initial conditions r(0) = dr(0)/dt = 0 fails to satisfy a familiar condition sufficient for existence and uniqueness of a solution, a Lipshitz condition. I originally concocted the dome example by starting with a text-book example of a system that violates a Lipshitz condition and has multiple solutions; and then worked backwards to a plausible physical instantiation.

[3] I have used the fact that Newtonian theory assigns no probabilities to these different outcomes as a way of illustrating my claim elsewhere that inductive inference need not be probabilistic. See Norton (forthcoming).

spatial and temporal extension, the component will only change its motion if it is struck, pushed or pulled by another component. In both supertask and space invader cases, the indeterminism only arises when we assemble these many individually well-behaved parts into the pathological totality.

In this regard, we can develop an intuitive sense of how the indeterminism comes about. We can metaphorically "pop the hood" and check the inner workings of the engine that brings us the unexpected behavior, noting that it comes from the combinations of many parts, each of which function normally.

The dome is unlike this. There is only one component, the mass, and no new interaction brings about its spontaneous motion. The only relevant force, the external gravitational field, acts on the mass in exactly the same way at the moment of spontaneous excitation as it had in all the moments prior. No ordinary, intuitive story in terms of smaller, individually well-behaved parts is possible for why the mass changed its state of motion at just the moment of excitation and not at any other. The best we can say is that it did because it could and it could because Newton's laws allow it.

2.3 Newton's Laws

That there is no new interaction bringing about the spontaneous motion of the mass on the dome may make one wonder whether the motion properly conforms to Newton's first law of motion. We can affirm that it conforms to the law, stated as (N1), by inspecting the expression (5) for acceleration. For all times $t \leq T$, up to an including the moment of spontaneous excitation $t=T$, the mass has zero acceleration. Since for all these times, the mass is placed at $r=0$, the net force acting on it also vanishes, as (N1) requires. Once the motion is underway, that is for times $t > T$, the mass is accelerating. But since it is no longer at $r=0$, there is a non-vanishing net force acting on it, so (N1) no longer applies.

If we care to graft causal language onto the spontaneous motion, we can express quite concisely what makes it puzzling. We expect a change of motion to have an initiating cause, a first cause. Since the motion starts at $t=T$, we expect that first cause to be active at that moment $t=T$. In Newtonian physics, the only admissible candidate is a net force. Yet at $t=T$, there is no net force acting on the body. So there is no first cause for the motion. In trying to accommodate how the motion can fail to have a first cause, it is helpful in reflecting on the moment of spontaneous motion, t=T, *not* to think of it as the first moment of the motion, but to think of it as the last moment of rest. Thus the time interval in which the mass moves, t>T, has no first moment at which a first cause could first act.

Or perhaps there is a simpler solution. The form (N1) of Newton's first law is not as he originally stated it. His versions, in their time-honored, archaic translations are (Newton, 1729, Vol. 1, p. 13)

> Law 1. Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.
>
> Law 2. The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Does the motion (4) really conform with Newton's version of the first law? It seems to. Newton's version of the law applies to the motion (4) only in the time interval $t \leq T$, for that is the time interval during which no net force acts on the mass. During that time interval, the mass remains at rest, as Newton's law required.

The complication is that the phrasing "compelled to change" suggests that changes of motion must be brought about by forces acting at the same time as the change, if not even earlier. Yet we have just seen that no force acts on the mass at the time t=T at which the motion changes, while something seems to be changing at t=T. Although the acceleration a(0)=0, the fourth time derivative of r has a discontinuity at t=0. It is

$$
\begin{aligned}
d^4r(t)/dt^4 &= 1/6 \quad \text{for } t > T \\
&= 0 \quad \text{for } t < T \\
&= ? \quad \text{for } t=T
\end{aligned}
\qquad (6)
$$

where the quantity proves to be not well-defined at t=T.

It is by no means clear to me that this amounts to a violation of Newton's form of the first law. Newton's wording suggests, but does not clearly assert, that forces must be first causes. More importantly, the position, velocity and acceleration of the mass are all vanishing at t=T and that seems sufficient to meet Newton's requirement of "state of rest" or "uniform motion." If it is not, then we will be creating difficulties with other canonical examples. Take the motion of a simple harmonic oscillator, a mass on a spring. In suitable units, its displacement x may be given as a function of time t by

$$x(t) = \sin t \qquad dx(t)/dt = \cos t \qquad dx^2(t)/dt^2 = -\sin t \qquad dx^3(t)/dt^3 = -\cos t \tag{7}$$

The mass will pass through a position of vanishing net force at t=0, when x(t) = 0 and the acceleration $dx^2(t)/dt^2 = 0$. We normally think of the mass at just that one moment as moving inertially—there is no net impressed force, so the velocity is constant, in the sense that the acceleration vanishes. Yet at that same moment, the third derivative does not vanish $dx^3(t)/dt^3 = 1$.

The strongest reason for not seeking to conform the motions (4) to Newton's expressions of his laws of motion is that Newton's expression of his laws are not applicable to cases of continuously varying motion. As readers of *Principia* know, Newton used an indirect, geometrical method to investigate continuously varying motions, such as planetary orbits. They were first approximated by polygonal trajectories, such as shown in Figure 2.

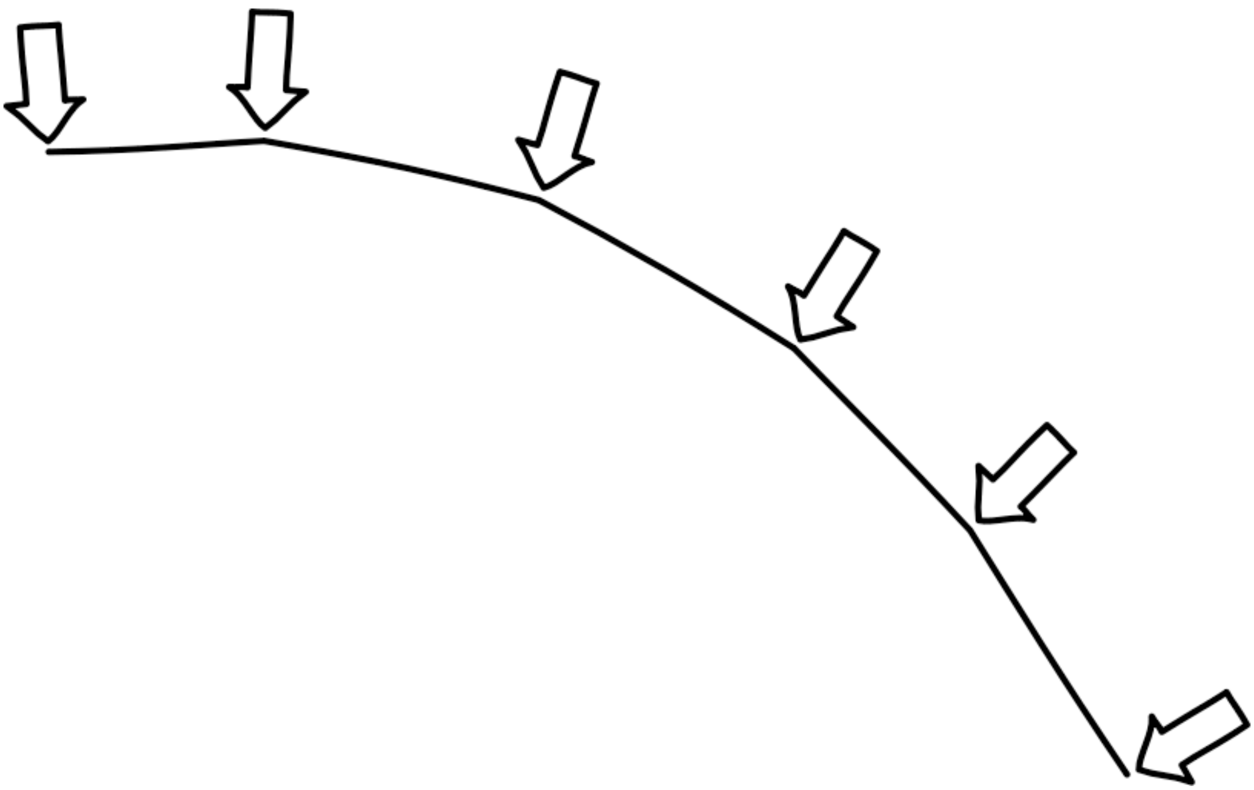

Figure 2. Newton's Polygonal Trajectory

The effect of a continuously acting force is approximated as a series of discrete forces, each acting at just one moment, with the trajectory consisting of force-free, inertial segments in between. The continuously varying trajectories of Figure 3 were then recovered by allowing the size of these segments to become vanishingly small.

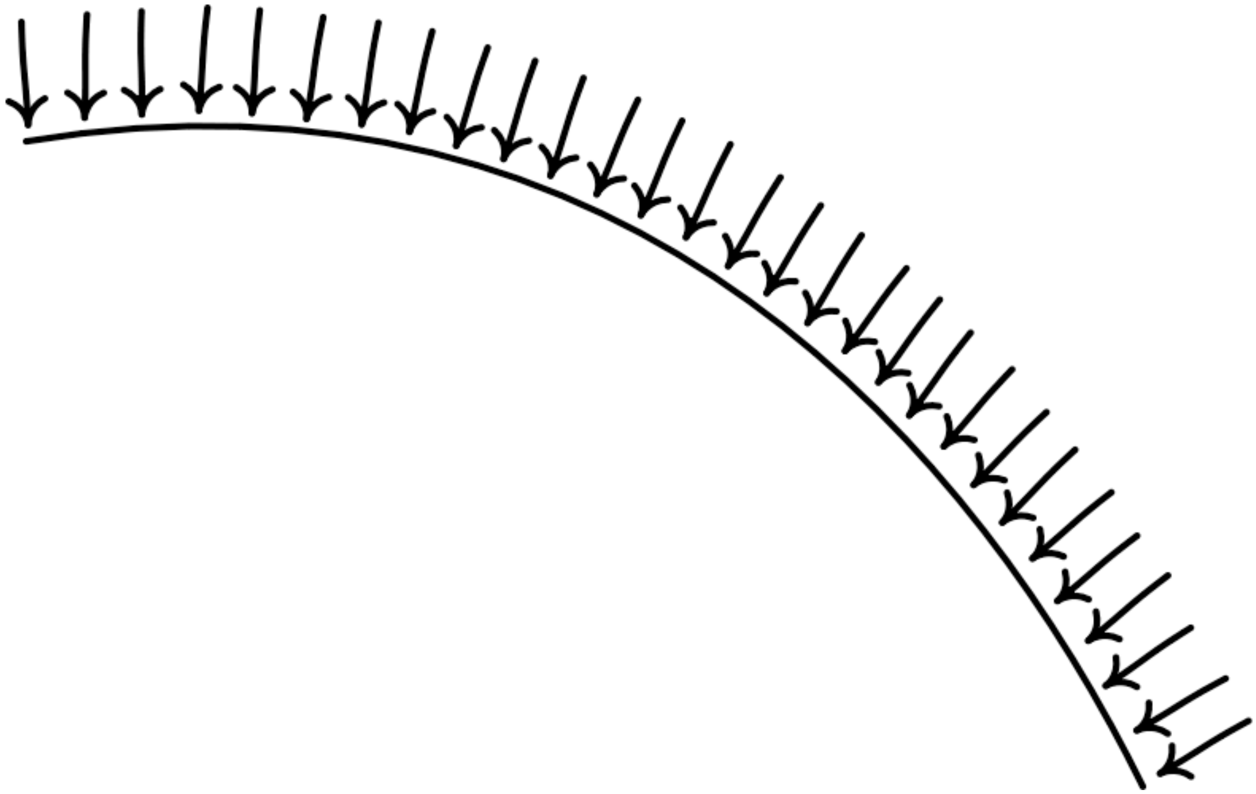

Figure 3. A Continuously Varying Trajectory

Newton drafted his laws of motion to apply to the polygonal trajectories and not to the continuously varying trajectories. The clearest indication of this is that his second law talks of the "change of motion," referring to the change wrought by the applied force between two polygonal segments. Modern presentations commonly edit Newton's language to "*rate* of change of motion" (e.g. Leech, 1978, p. 1) to render the law applicable to continuously varying trajectories.

In terms of the polygonal trajectories, every inertial segment that changes motion with respect to the one before must have an applied force at its beginning. In that sense, applied forces are first causes of the change. However that need no longer be the case when the limit of the continuously varying trajectory is taken. Consider the harmonic oscillator (7). The moment t=0 marks the beginning of a period during which the mass' acceleration is directed away from the origin x=0. However there is no force acting at the event x=0, t=0, even though the velocity is different at every moment in the immediately ensuing time interval $0 < t < 1$.

Since above I investigate continuously varying trajectories directly, I follow the later literature that defines Newton's laws in terms of forces and accelerations. Indeed in it, the first law (N1) is logically superfluous. As has been noted frequently, it is entailed by the familiar

F=ma of (N2). Thus it is entirely possible for a canonical text in classical mechanics, Goldstein (1980, Ch. 1) to formulate Newtonian mechanics using the second law only, without even mentioning the first law and to have no entry for it in its index.

2.4 Losing Touch

David Malament (manuscript) has explored an aspect of the dome I did not think to look at. If the mass has any initial velocity other than 0 at r=0, it does not remain on the dome's surface, but flies off. I emphasize that this does not compromise the dome as an example of indeterminism in Newtonian theory; the dome is the case of initial velocity 0, and that remains indeterministic. However I will review some aspects of these extra cases.

First, the fastest way to see the that non-zero velocity leads to losing touch, is to consider a radial motion h(t), r(t) that lies on the dome's surface, with h and r both strictly increasing with t. It is assumed to start at r=0, t=0 with a non-zero initial velocity v(0) = dr(0)/dt > 0. We have that the vertical velocity is dh/dt = dh/dr.dr/dt, where we use the fact that for this motion h can also be expressed as a function of r since r is strictly increasing with t. It now follows that the vertical acceleration is

$$d^2h/dt^2 = dh/dr. \, d^2r/dt^2 + d^2h/dr^2.(dr/dt)^2 \qquad (8)$$

Recalling (1) $h = (2/3g)\, r^{3/2}$, we have $dh/dr = (1/g).r^{1/2}$, so that

$$d^2h/dr^2 = 1/(2gr^{1/2}) \qquad (9)$$

Thus d^2h/dr^2 is infinite at the apex h=r=0. It now follows immediately from the second term of (8), that the vertical acceleration d^2h/dt^2 at t=0, when the mass is at h=r=0, must also be infinite, as long as the initial velocity v(0) = dr(0)/dt > 0. So if the mass is to remain on the surface, it must have an infinitely large vertical acceleration[4] at its first instant at t=0. That is impossible since the greatest acceleration gravity can provide is g; and at the apex it is g.sin 0 = 0. So the mass cannot remain on the surface.

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[4] In the case of the spontaneous motion (4), the expression (8) becomes indeterminate at t=0, since its second term becomes 0/0. Other arguments show that the vertical acceleration d^2h/dt^2 at t=T is zero. To see this, note that combining (1) with (4) we find that h varies with $(t\text{-}T)^6$ in the interval t ≥ T, so that d^2h/dt^2 varies with $(t\text{-}T)^4$, which vanishes at t=T.

We could however modify the example slightly. Instead of sliding a mass over the surface of the dome, we replicate the essential properties of the dome with a rather extravagantly idealized bead on a wire. The wire of no thickness is formed into the profile of the dome and threads an infinitely small bead of unit mass that slides frictionlessly along it. The diverging of d^2h/dt^2 at t=0 above shows that the wire must exert an infinite reaction force on the bead just at the moment t=0, if the bead is to slide along the wire from the apex with an initial velocity greater than zero. We should not, however, be too quick to dismiss momentarily infinite reaction forces. They arise in common examples in Newtonian mechanics. Consider a ball that bounces off a hard surface in which we idealize the bounce to be momentary. A momentary, infinite reaction force is needed to reverse the direction of the ball instantly.

Of course all these difficulties disappear if we posit *ab initio* the existence of at outward, radially directed force field in some two dimension space that exerts a force of magnitude $r^{1/2}$ in the radial coordinate r on a unit mass. It then follows automatically that the equation of motion of the unit mass is (2) and the calculations above show that its time development is indeterministic if it is initially as rest at the origin.

If the idea of a force field everywhere directed away from an empty point in space is bothersome, a slight modification can put sources at the centers of force. Imagine a mass point that exerts a short-range, attractive force of magnitude

$$f(x) = (L-x)^{1/2} \text{ for } x \leq L$$
$$= 0 \qquad \text{for } r > L$$

on a unit mass at distance x from the mass. We create an outward directed field along a straight line in the interval $-L<r<L$ simply by placing one of these masses at the position r=L and another at r=-L. A unit mass in that interval is then governed by the equation of motion (2).

Finally, if we just consider motions within a two-dimensional surface governed by the equation of motion (2), then it can be shown that, for the case of non-zero initial velocity v(0) = dr(0)/dt > 0, there is a unique solution of the equations of motion (2). See Appendix.

3. What's Wrong with the Dome?

Since 2003, there have been many reactions to the dome. Some are amused to see that indeterminism arises in so simple an example in Newtonian physics. Others are indifferent. The response that surprised me, however, came from those who had a full grasp of the technical issues, but nonetheless experienced a powerful intuition that the dome somehow lies outside what is proper in Newtonian theory. It is not always easy to discern the grounding of such instinctive reactions. My efforts to do so have identified three bases for the judgment. They are described below, along with my reasons for finding them unconvincing.

3.1. Does the Dome Employ an Incomplete Formulation of Newtonian Physics?

This response to the dome is that Newton's laws, as formulated in (N1), (N2) and a corresponding form of the third law, is an incomplete formulation of Newtonian theory. Perhaps, we might say that the intent of Newtonian theorists was that their theory would be deterministic and all the traditional Newtonian systems they dealt with are deterministic. Therefore any formulation of Newtonian theory that admits indeterministic motions is incomplete and must be strengthened to preclude them. Or we might insist that forces must be a first cause of motion after all (see Section 2.3 above), so that the present formulation of Newton's theory must be strengthened to block systems like the dome in which motions can be initiated without a force as first cause.

My first and final reaction to this proposal is that we must distinguish what the canonical formulation of Newtonian theory says and entails from what its proponents may mistakenly think their theory entails. The dome conforms to Newton's laws in their standard forms; so the dome is a Newtonian system. The old lore was mistaken to think that simple systems like the mass on the dome must be deterministic in Newtonian theory.

While that really is the essential point, there are two additional complications. First, strengthening Newton's theory is not so straightforward. In effect we need to find some additional postulate to supplement the three laws, a "fourth law." But what can in be? Merely asserting as the fourth law that determinism holds is inadequate. The three laws allow competing, indeterministic motions. In merely asserting that determinism holds, the fourth law is not telling us which of the competing motions, if any, are to be admitted into the theory. As we strengthen the fourth law to address this problem, we must then worry that we are restricting the theory excessively and perhaps precluding systems that are unobjectionable; or that we have such a narrow formulation of the fourth law that it is tailored to resolving indeterminism only in special cases. While the proposals are preliminary, I have seen none yet that can resolve these problems.[5]

The second complication is that it is hard to determine what Newtonian theory properly is. So it is hard to know if a proposed fourth law properly belongs in it. What Newtonian theory is cannot be established by an experiment, since Newtonian theory is a false theory; and it cannot be determined by mathematical demonstration, since Newtonian theory is not defined as a theorem of some grander system.

Perhaps the best we can say is rather weak: that Newtonian theory is delineated historically by the community of physicists. That community has the right to add a fourth law to Newtonian theory. As long as sufficient continuity with earlier communal decisions is maintained, I can see no objection to the addition. It would mean, however, that we have multiple versions of Newtonian theory—an older three-law and a newer four-law version. This would just be part of the evolution of the theory, whose content is defined by communal agreement.

What we should object to, however, is decreeing that the original three-law version is no version at all, so that the only real version is the four-law version. For that obscures the fact that through most of its history, the community of Newtonians mistakenly thought that the three laws by themselves were enough to guarantee determinism and that, in effect, the three and four law

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[5] For example, Korolev (manuscript) suggests that it is a tacit presumption of Newtonian physics that the differential equations of motion derived from it satisfy a Lipschitz condition, which guarantees the existence of a unique solution. A Lipschitz condition is a sufficient condition mathematically for differential equations to have a unique solution, which in turn expresses the physical idea of determinism. Since it is only sufficient, it may be too strong. Tymoczko (manuscript) has proposed that the informal import of Newton's first law are not fully expressed by the condition of vanishing acceleration at an instant if the net force vanishes at that instant. Rather the law asserts in addition that this inertial motion continues over a non-vanishing time interval provided that inertial motion does not in turn trigger further forces. That leaves open the question of whether determinism is preserved if the continuing motion does trigger further forces.

versions are the same. The dome shows that this equivalence fails even in the simple case of a mass sliding over a dome.[6]

3.2 Is the Dome "Unphysical"

Another common reaction to the indeterminism of the dome is that it is "unphysical" and hence can be safely dismissed. The grounding for this reaction may come from several places. Perhaps most important is a sense that Newtonian theory is true, well enough, of our ordinary world and that ordinary world just doesn't admit masses that spontaneously set themselves into motion. As we saw in Section 2.2, the motion seems to be initiated without a force as a first, initiating cause. Ordinary things like expensive crystal vases do not just launch themselves off the table. They have to pushed, or at least nudged first. And, unlike supertask and space invader indeterminism, there is no way to understand the indeterminism as resulting from the combination of behaviors of components, each of which are individually well-behaved, and thus at least individually "physical."

In short my response is that a careful examination of just what "unphysical" may mean fails to license the idea that the dome is unphysical in a way that would allow us to excise it from Newtonian physics. The principal difficulty is that the dome is intended to explore the properties of Newtonian theory, not the actual world. As a false theory, Newtonian theory can certainly have consequences that do not agree with ordinary expectations, especially if the background conditions are artfully contrived.

The sense of what is "unphysical" or "physical" is quite fundamental to the intuitions of physicists. At the same time, it is a kind of primitive notion that does not attract further explication. Indeed "unphysical" in this sense even escapes definition in standard dictionaries, including the authoritative *Oxford English Dictionary*. Yet if we are to know if the dome ought properly to be dismissed as "unphysical," then we must analyze the notion.

The term "unphysical" means, as far as I can tell, "cannot obtain in the real world," so the assertion that something "unphysical" happened must be a falsehood. There are different ways

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6 In a related response, Alper and Bridger (in Alper and Bridger, 2000) in effect assert that energy and momentum conservation has been a tacit fourth law of Newtonian physics that would preclude indeterministic supertask systems from counting as properly Newtonian.

that unphysical processes can emerge from a theory and the type of remedy called for, if any, depends upon the way the unphysical process arose. I have been able to distinguish three ways, two of which are relevant to the dome:

(a) *Unphysical as gauge (overdescription)*. In these cases, a theory admits more structures than are in the world for descriptive convenience. The simplest example is the origin of a coordinate system in a Euclidean space. It is plainly unphysical to think of this origin as the preferred center of the space. There's nothing physically different about the origin compared to all other points of the space. It has just been selected for descriptive convenience. The remedy is just to keep that arbitrariness in mind when using the coordinate system. I do not think this is the sense of unphysical at issue with the dome, since the motions described do not arise from arbitrary choices made as descriptive conveniences.

(b) *Unphysical as false*. In this usage, a theory makes a prediction that turns out to be false and quite far from approximations to the actual. For example, a classical electrodynamical analysis of heat radiation predicts the "ultraviolet catastrophe," that heat radiation at equilibrium has an infinite energy density. This prediction in unphysical in that it directly contradicts the finite energy densities found in experiment. The remedy is to renounce the offending theory as a false theory; or at least false in the domain in which the failed prediction was made.

If this is the sense of unphysical that is at issue with the dome, then it gives us no license to absolve Newtonian theory of indeterminism. At best it would tell us that Newtonian theory is just a false theory to be renounced. That it would predict indeterminism is a little surprising if we are used to thinking of Newtonian theory as giving a very good account of ordinary systems in which the sort of indeterminism of the dome does not arise. However the fact that the prediction of the theory is surprising is no basis for concluding that the theory does not entail it. It is a false theory, so it will have failings we must learn to accommodate.

As it turns out, however, I do not think we need even to do this. The dome is not intended to represent a real physical system. The dome is purely an idealization within Newtonian theory. Indeed, on our best understanding of the world, there can be no such system. For an essential part of the set up is to locate the mass *exactly* at the apex of the dome and *exactly* at rest. Quantum mechanics, which gives us our best description of the worlds in the small, assures us that cannot be done. It would violate the uncertainty relations. Similarly, the dome depends upon a very particular shape for the two-dimensional surface of the dome in an arbitrary small neighborhood of the apex of the dome. As we pursue smaller neighborhoods of the apex, we will enter the domain of atomic and then quantum theory and the granularity of the atoms would preclude us realizing exactly the shape required.

The central question at issue with the dome is not " How is the world factually?" The question is "What are the properties of Newtonian theory?" We explore those properties by looking at what the theory says of various hypothetical systems. We learn from the dome that it is, at least in some cases, a theory that supports indeterminism. That fact is unaltered by the objection that no actual system is like the dome. Analogously, classical electrodynamics predicts the factually false "ultraviolet catastrophe." The result is unphysical, but we accept that it is a consequence of the theory.[7]

(c) *Unphysical through under description*. In this third case, a theory may under describe or under constrain a system's properties, with the outcome that the theory admits solutions that do not apply to the system. It is standard practice to dismiss these superfluous solutions as "unphysical." For example, consider a ball that bounces in a perfectly elastic collision off a hard floor. If its speed just before the bounce is u and v just after, we have from the conservation of kinetic energy that $u^2=v^2$. Taking square roots, we have that u=v or u=–v. We discard the first of the two solutions as "unphysical" since it corresponds to the ball penetrating the floor without impediment.

This discarding of superfluous solutions as unphysical seems to be closest to what is intended in the declaration that the dome is unphysical. For its import is that the dome can somehow be excised from Newtonian theory, without discarding the full theory. However this third category cannot be applied to the dome, since it requires that there is a target system we seek to describe of which we have independent knowledge. That independent knowledge is used to determine that some solutions are superfluous and may be discarded. In the case of the

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[7] Or should we escape the difficulty by saying that classical theory should not be applied to these cases? The difficulty returns when we give our reason: for if we did apply it, we would have to concede, it would give a patently unphysical result.

bouncing ball, those further conditions come from our knowledge that the floor is impenetrable, an aspect of the target system not captured in the law of conservation of energy, $u^2=v^2$. To get physically meaningful solutions, we must implement that impenetrability by noting that u and v have opposite signs.

In the case of the dome, there is no target system of which we can have independent knowledge. As we have seen, the dome cannot exist in the world. It is a creation that resides entirely within Newtonian theory. So we can have no means of judging that some solutions delivered by the theory are superfluous and should be discarded.

3.3 Does the Dome Use Inadmissible Idealizations?

This reaction allows that some idealizations are admissible in theorizing. The concern however is that the idealizations required by the dome are so extreme as to be inadmissible, so that the dome displays no inherent indeterminism in Newtonian theory, but only the pathology of an inadmissible idealization. Under this reaction, there are many idealizations that we could identify as suspect. The mass is a point, so it has zero extension in space and infinite density. It slides frictionlessly over a perfectly even surface. The curvature of the surface diverges at the apex.[8] The mass must be placed at perfect rest exactly at the apex.[9]The surface is assumed perfectly rigid so that it does not deform under the weight of the mass.[10]

In short my response will be that, while some idealizations may be inadmissible in certain circumstances, those used in the dome are not. Every one of them individually can be found in one form or another in standard textbooks in Newtonian theory. To pursue one example, while the surface admits a curvature singularity at its apex, the tangent to the surface is everywhere defined. In this regard it is more differentiable than another surface routinely appearing in

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[8] This is examined by Malament (manuscript), although he finds this idealization no more troublesome that others routinely employed in Newtonian theory.

[9] Stephan Hartmann has emphasized to me in correspondence that fluctuations in initial velocity are unavoidable.

[10] Korolev (manuscript) points out that allowing any elastic deformation of the surface would destroy the shape needed at the apex for indeterminism to obtain.

Newtonian texts, a tabletop with a sharp edge formed by the intersection of two flat surfaces. At the edge, both curvature and tangent are undefined.

The complaints leveled against the idealizations of the dome seem to arise through a process of reasoning backward. The real goal is to disallow indeterminism. So when the dome manifests indeterminism, the strategy is to see if dismissal of its idealizations will defeat the indeterminism. When it is found that they will, the idealizations are declared inadmissible. In other examples these same idealizations and others even more extreme (such as a sharp table top edge) would be allowed to stand, as long as they are not complicit in the emergence of indeterminism. For those proceeding this way, there can be no objection to the idealizations grounded in their intrinsic properties. The dismissal is simply an oblique way of imposing determinism as an extra law of Newtonian physics.

This short response masks complications that can be developed by distinguishing two types of failure of idealization, external and internal. I will argue that neither failure arises in the case of the dome.

The external failures of idealization arise when a theory is attempting to describe a system that is specified independently from the theory and the idealization compromises that description. The most obvious way this happens is when we are using the theory to describe a real system. Because of the atomic and quantum restrictions mentioned earlier, no real dome can be exactly as the one described in the idealization. As a result, these idealizations are inadmissible, in so far as we are seeking to provide a true description of an actual or possible domed surface with a mass sliding over it. However, since we are seeking to discover the properties of Newtonian theory and in a domain in which it cannot describe the world, this form of failure is not the relevant form.

Internal failures arise when there is difficulty intrinsic to the relationship of the idealization to the theory. One way this can arise is if we idealize to the extent that we contradict the theory. For example, in a spacetime theory that prohibits propagations outside the light cone, we cannot idealize bodies as perfectly rigid, in so far as perfect rigidity allows infinitely fast propagation of disturbanced through the bodies. There will be borderline cases, in which standard methods of a theory fail, but we may wonder if some non-standard method may allow the theory still to be applied. Consider, for example, a surface that is continuous but nowhere differentiable. If that surface is located in a gravitational field and we place a point-like unit mass on it, can we say what the ensuing motions are? The usual rule is that they are the motions driven by the net gravitational force, which is the component of the vertical gravitational force tangent to the surface. Yet a nowhere differentiable surface has no tangent anywhere, so this quantity is undefined. This form of failure is not present in the dome, even of the borderline type. The geometrical tangent to the dome and thus the net force acting on the mass is everywhere defined.

There is another internal effect that might tempt us to rule an idealization as inadmissible. Some idealizations can be recovered at the limit of more realistic structures. A point mass is the limit of successively smaller, perfectly spherical masses of correspondingly greater density. We may be tempted to say that if some property changes discontinuously when the limit is taken, we have an inadmissible idealization. Consider, for example, that a sharp table edge is the limit of a smoothly beveled table edge with successively smaller bevels. A mass sliding off the horizontal table will be projected in a parabolic arc. As we consider successively smaller bevels, so that the limit of a sharp edge is approached, the parabolic arcs of the mass projected will approach that of the mass projected over the sharp edge. That agreement assures us the idealization is benign. Now consider domes with finite curvatures at their apexes. For all, a point mass will remain on the apex indefinitely, according to Newtonian theory. Thus, as the curvature become arbitrarily large, determinism will prevail. Yet when we arrive at the limit of infinite curvature, only then does indeterminism enter. Does this mean that the idealization of infinite curvature is inadmissible?

It does not. First, we should not confuse this case with the case of external failures. If we were trying to predict the behavior of real surfaces that have nowhere diverging curvature, then this different between the limiting case of infinite curvature and nearby cases of finite curvature would be relevant. For it would tell us that we cannot use the dome to approximate such surfaces. But we are not trying to predict the behavior of real masses on real surfaces. Indeed all the surfaces in this example are unrealistic idealizations. A real surface is granular, consisting of atoms interacting with the mass above it by forces, which, although short range, only weaken asymptotically with distance. So there is no sharp, two dimensional surface in a real space that exactly demarcates the substance of a dome from the emptiness above.

Thus the failure we are seeing is of one idealization to approximate another. That there can be discontinuous changes in the properties of systems when limits are taken is surely a commonplace and no reason for discounting the system approached in the limit. To give one illustration of the many possible, it has long been recognized in statics that some systems are statically indeterminate. A simple example is shown in Figure 2.

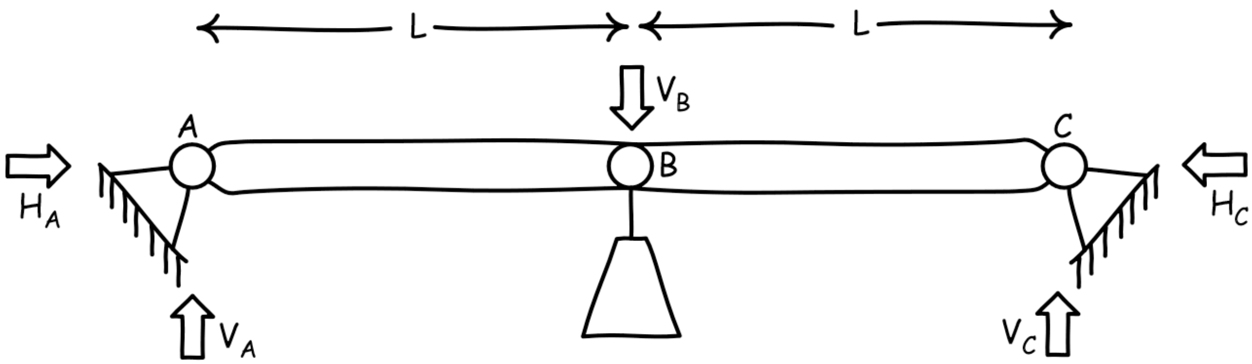

Figure 2. Statically Indeterminate Structure

A perfectly rigid beam AC of length 2L, supported at either end, is loaded at its midpoint B, with a known force V_B. We seek the four unknowns H_A, V_A, H_C and V_C, which are the horizontal and vertical components of the reaction forces at A and C. Balancing shear forces and bending moments,[11] it turns out that we have only three independent equations in these four unknowns, so we end up being able to solve for $V_A = V_C = V_B/2$. But H_A and H_C remain indeterminate, constrained only by $H_A = H_C$.

A standard resolution is to relax the idealization of perfect rigidity and to model the beam as elastic, but very stiff. Then the beam will deflect slightly, as shown in Figure 3.

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[11] The vanishing of the bending moments about A yields $2L\ V_C = L\ V_B$, so that $V_C = V_B/2$. The vanishing of the bending moments about C analogously yields $V_A = V_B/2$. These two equations now entail the vanishing of bending moments about B: $L\ V_C = L\ V_A$ and the vanishing of vertical shear forces everywhere on the beam: $V_A+V_C=V_B$. The balance of forces in the horizontal direction yields $H_A = H_C$, but no conditions constrain these horizontal components further. To see the origin of the indeterminateness, imagine the simpler case of a perfectly rigid beam stressed only by horizontal reaction forces at each end. Equilibrium of forces can only tell us that the two reaction forces are equal, but not their magnitude.

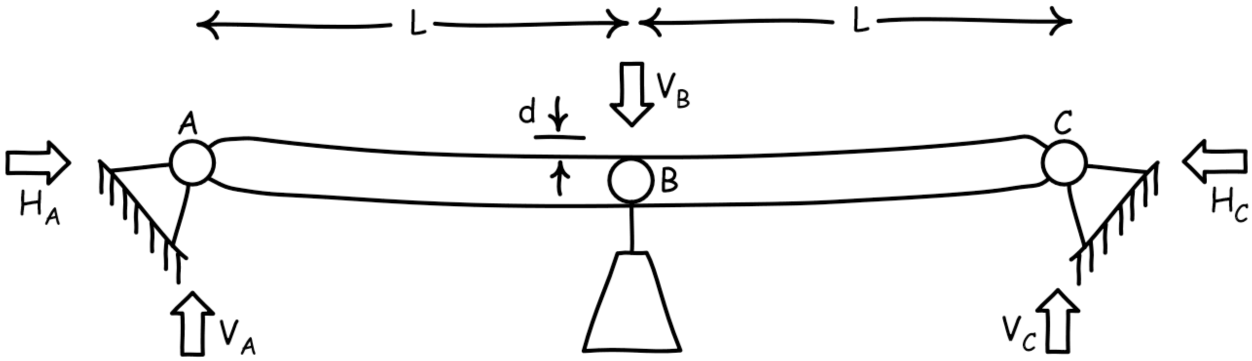

Figure 3. Statically Determinate Structure

The deflection of the center of the beam d is fixed by the known load V_B and the elastic properties posited for the beam. A balance of forces about A will now eventually give an additional relation[12] that will fix the value of H_A thus also H_C.

The structure shown has determinate forces as long as we model the beam as having some finite elasticity, no matter stiff. In the limiting case, however, when the elasticity becomes infinite and the beam perfectly rigid, the forces are indeterminate. We not infer from this that the limit case is no Newtonian structure at all. We merely allow that it is a different case with qualitatively different properties.[13] The conditions that are sufficient to determine forces in a system with elastic members are not sufficient to determine the forces in one with perfectly rigid members.

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[12] Informally, the beam is now stressed like an archer's bow. The reaction forces $H_A$and H_C correspond to the determinate tensile forces from the bow's string needed to sustain the precise bending of the bow associated with the load.

[13] There are many more cases like this. Elsewhere (Norton, 2004, §2.2.3) I have described an idealized helicopter rotor. As its speed is halved and its size doubled, the non-zero lift L it generates remains the same. In the limit of infinitely many doublings of size, we have an infinite rotor at rest that generates no lift. Are we to infer that the discontinuity in lift generated permits us to dismiss an infinite rotor at rest as an inadmissible idealization? Or just that it has properties different from finite, rotating rotors?

Analogously, the dome with its indeterminism may have different properties from other idealized systems that approximate it arbitrarily closely. Initial conditions that fix the future motion of one do not fix those of the other. That does not make the latter inadmissible; it is just different.

4. When Determinism is Exceptional or Generic

While for many the import of the dome is an examination of what counts as a Newtonian system and which idealizations it may employ, in my view the dome in useful in illustrating another aspect of how indeterminism arises in Newtonian theory. The broader moral is that determinism is generic for Newtonian systems with infinitely many degrees of freedom; whereas it is exceptional for Newtonian systems with finitely many degrees of freedom.[14] The dome is a case of finitely many degrees of freedom in which indeterminism arises. However it is exceptional in the sense that small changes in it, such as discussed in the last section, eradicate the indeterminism.

One can quickly arrive at this rather informal moral of the distribution of indeterminism across systems of finitely and infinitely many degrees of freedom, if we write the dynamical equations governing a Newtonian system with n degrees of freedom as a set of n coupled first order differential equations[15]

$$dx_1/dt = f_1(x_1, \ldots, x_n)$$

$$dx_2/dt = f_2(x_1, \ldots, x_n)$$

$$\ldots$$

$$dx_n/dt = f_n(x_1, \ldots, x_n)$$

---

[14] A Newtonian system, in this context, has n degrees of freedom if its dynamical evolution with respect to time t can be tracked by specifying n real valued variables $x_1(t)$, … , $x_n(t)$.

[15] This can be achieved without loss of generality. The equations governing the system may be set in a Hamiltonian form; or, more directly, a single equation in dx^2/dt^2 can be replaced with two equations, one of which defines a new variable $v = dx/dt$ and the second merely restates the original in terms of dv/dt.

We have n equations in n unknowns, which, generally speaking, yields a unique solution for the n unknowns. Generally speaking, determinism reigns here. Of course “generally speaking” is not a complete assurance. There are exceptions, such as when a Lipschitz condition fails. The dome and space invader indeterminism are illustrations of different types of failure. They are, however, unusual in the universe of Newtonian systems as is suggested by the uncomfortable reception they receive.

If we have a system with infinitely many degrees of freedom, its infinitely many variables are governed by the infinite set of equations

$dx_1/dt = f_1(x_1, \ldots , x_n)$

$dx_2/dt = f_2(x_1, \ldots , x_n)$

…

Whether the system admits indeterminism or not depends on the amount of coupling between the equations. (The equation in dx_i/dt is coupled to the equation in dx_k/dt just in case the function $f_i(x_1 , \ldots, x_k , \ldots , x_n)$ varies in value with changes in x_k.) In one extreme case, the coupling is minimal in the sense that the infinite set of equations degenerates into infinitely many subsets of finitely many equations, with coupling only within each subset. For example, the system represented may just be infinitely many non-interacting masses. Then the overall system will be deterministic if the individual systems are, which is generally the case for systems with finitely many degrees of freedom.

If, however, there is more coupling, then the situation changes radically. Consider the case in which each pair of equations in dx_i/dt and dx_{i+1}/dt are coupled, for all i.

$dx_1/dt = f_1(x_1, x_2)$

$dx_2/dt = f_2(x_1, x_2, x_3)$

$dx_3/dt = f_3(x_2, x_3, x_4)$

…

and each of the functions $f_n(x_{n-1}, x_n, x_{n+1})$ is invertible in x_{n+1} and the inverse function arbitrarily differentiable. A simple example of a minor variant of this type of coupling is the string of infinitely many masses and springs: mass-spring-mass-spring-… described in Norton (1999, §1.1). For such systems, indeterminism follows from a simple iterative calculation. We will stipulate two arbitrarily differentiable solutions for $x_1(t)$ that agree up to t=0 and then differ pretty much[16] as we please. We affirm that the two functions for $x_1(t)$ belong to solutions of the dynamical equations by generating the remaining compatible sets of variables for each. Selecting one of the two functions for $x_1(t)$, we use the equation in dx_1/dt to solve for $x_2(t)$. With the solutions for $x_1(t)$ and $x_2(t)$ in hand, we can now use the equation for dx_2/dt to solve for $x_3(t)$; and so on for all the remaining variables. Hence we are able to construct competing solutions whose behavior will agree up to t=0 and disagree thereafter; that is, we have found pairs of solutions implementing indeterminism. (See Norton, 1999, §1.1 for an illustration of this procedure.)

What is essential for this procedure is that the set of equations is infinite. Otherwise, if there are just n equations, the iterative construction will likely fail at the nth step. We would solve the equation in dx_{n-1}/dt for $x_n(t)$. However, since there is no $x_{n+1}(t)$, the final equation of the set

$$dx_n/dt = f_n(x_{n-1}, x_n)$$

would relate only quantities already computed in earlier steps. This additional constraint is likely to block the freedom of choice of the infinite case in stipulating the initial function $x_1(t)$.

It is clear from the construction that this form of indeterminism is robust in the sense that small changes in the function f_i will not compromise the indeterminism; and, moreover, that it depends principally only on the presence of infinitely many well-coupled variables in the dynamical equations.

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[16] The complication is that the time derivatives of $x_1(t)$ at t=0 in the different solutions must not alter the common initial values of $x_2(0)$, $x_3(0)$, … Since $x_n(0)$ will be a function of $dx_{n-1}(0)/dt$ for all n, it follows by iteration that $x_n(0)$ will be function of $d^{n-1}x_1(0)/dt^{n-1}$. Hence the derivatives $d^nx_1(0)/dt^n$ to all orders n will be fixed by the initial values of $x_2(0)$, $x_3(0)$, …; and similarly for all the derivatives of the remaining variables values of x_2, x_3, …. Since we seek competing solutions that agree in their values at t=0, it now follows that they must also agree in all their derivatives at t=0 as well. As a result, at least one of them must be non-analytic. For an analytic function is specified everywhere by its value and the values of all derivatives at t=0.

5. Conclusion

That Newtonian theory admits indeterministic solutions has been recognized for decades. What the example of the dome adds to this knowledge is that indeterminism can arise in cases that do not have the exotic properties of previously known examples. We do not need supertask systems, with their infinite degrees of freedom, which (as shown in Section 4) are generically indeterministic. Nor do we need space invader systems, in which masses materialize from spatial infinity with unbounded speed and energies. We need only a single particle on a dome, with a special shape.

In examining the responses of those who resist the inclusion of the dome in Newtonian theory, the one that called up the notion of "unphysical" is especially noteworthy. My sense is that this notion has attracted much less attention among philosophers of science than is commensurate with its importance for physicists. For them, the notion of "physical" or "unphysical" functions as a powerful guide and filter, with an authority that is almost as oracle-like as it is unanalyzed. In Section 3.2, I gave as careful an explication of the notion as I could. My analysis is deflationary, portraying the notion as the coalescence of three related ideas, each of which is unremarkable when examined in isolation. It will be interesting to see if further analysis of the notion bears out this deflationary view or if, perhaps, there is something of greater epistemic moment at hand.

Appendix: Unique Solution of the Equation of Motion $d^2r(t)/dt^2 = r(t)^{1/2}$ for Initial Conditions $r(0)=0$, $v(0) = dr(0)/dt > 0$

It is shown that

$$d^2r/dt^2 = r^{1/2} \tag{2}$$

has a unique solution for the boundary condition $r(0) = 0$ and $dr(0)/dt = v(0) > 0$. The strategy is to show that any solution $r(t)$ of (2) with these boundary conditions must satisfy the integral condition (A4) and then that this condition admits a unique solution.

If the solution satisfies $v(0) > 0$ and it is continuous and twice differentiable, then $v(t)$ must remain positive for some time interval following the instant $t=0$. Hence $r(t)$ is strictly increasing in t in this interval, so that we may also write v as a function of r, i.e. as $v(r)$. The demonstration of existence and uniqueness will be limited to this interval. (That is sufficient, since once the system has left r=0, it enters regions in which a Lipshitz condition is satisfied, so existence and uniqueness of the solution is guaranteed.)

This solution r(t) must satisfy (2) rewritten as

$$r^{1/2} = d^2r/dt^2 = dv/dt = dv/dr \,.\, dr/dt = v \,.\, dv/dr = (1/2).(dv^2/dt) \quad \text{(A1)}$$

where we have used the fact that v is a function of r. Its first integral is

$$v^2(r) - v^2(0) = (4/3)\, r^{3/2} \quad \text{(A2)}$$

which is equivalent to

$$v = dr/dt = (v^2(0) + (4/3)\, r^{3/2})^{1/2} \quad \text{(A3)}$$

This last equation can be integrated again to yield

$$t = \int_0^t dt' = \int_0^r \frac{dr'}{\sqrt{v^2(0) + (4/3)r'^{3/2}}} \quad \text{(A4)}$$

Any solution r(t) must satisfy (A4). We can now see that (A4) admits a unique solution. Since the integrand of (A4) is always positive, it follows that t is a unique, strictly increasing function of r. Hence the function t(r) is invertible to the unique function r(t), in which r is a strictly increasing function of t. Therefore also $dt/dr > 0$ and $v(t) = dr/dt > 0$ in the interval of the solution.

We can now undo the calculations of (A1) to (A4) to affirm that r(t) does indeed solve (2). Differentiation of (A4) gives us an expression for dt/dr, which may be inverted to yield (A3) and (A2), since dt/dr is not zero. A second differentiation returns (A1), which is equivalent to (2). We can see that the r(t) constructed this way satisfies r(0)=0, by inspecting (A4), and v(t=0) = v(0), by inspecting (A3). Finally, the solution cannot be more than twice differentiable at t=0. Differentiating (2), we have $d^3r(t)/dt^3 = (1/2r(t)^{1/2}).\, dr/dt$. At t=0, $dr/dt > 0$ and r=0, so this third derivative diverges.

Finally, even though (2) has a unique solution with r=0, v(0)>0, it does not satisfy a Lipshitz condition with this boundary condition. (A Lipshitz condition is sufficient but not necessary for existence of a unique solution.) To see this, rewrite (2) as two, coupled first order differential equations

$$dr(t)/dt = v(t) \qquad dv(t) = r(t)^{1/2}$$

A Lipshitz condition is satisfied in a region surrounding the points r=0, v=v(0) if there exists a K such that for every pair of values in the region (t, r_1, v_1) and (t, r_2, v_2)

$$(v_1 - v_2)^2 + (r_1^{1/2} - r_2^{1/2})^2 \leq K^2 [(v_1 - v_2)^2 + (r_1 - r_2)^2]$$

This condition can be rewritten as

$$(K^2 - 1)(v_1^2 - v_2^2) + \left[K^2 - \frac{1}{\left(\sqrt{r_1} + \sqrt{r_2}\right)^2}\right](r_1 - r_2)^2 \geq 0$$

No value of K can make this inequality hold. Since, for each K, its first term can be made to vanish by choosing $v_1=v_2$; and its second term can be made negative by choosing values of r_1 and r_2 close enough to 0. If, however, the region does not contain the point r=0, then the second term cannot assuredly be made negative and we can find a K for which the inequality holds.

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