Marek Nowak

On Some Generalizations of the Concept of Partition

Abstract. There are well-known isomorphisms between the complete lattice of all partitions of a given set A and the lattice of all equivalence relations on A. In the paper the notion of *partition* is generalized in order to work correctly for wider classes of binary relations than equivalence ones such as quasiorders or tolerance relations. Some others classes of binary relations and corresponding counterparts of partitions are considered.

Keywords: Partition, Quasiorder, Tolerance relation.

1. Introduction

A *partition* of a given set A is any family Π of subsets of A such that: (i) any member of the family is nonempty and (ii) each element a of the set A belongs to exactly one set X from Π. Given the partitions Π_1, Π_2 of A we say that Π_1 is a *refinement* of (or is *finer* than) Π_2 iff $\forall X \in \Pi_1 \exists Y \in \Pi_2,\ X \subseteq Y$. The relation of refinement, denoted as $\leq$, turns out to be a partial order on the set $Part(A)$ of all the partitions of A, with the poset $(Part(A), \leq)$ being a complete lattice. This lattice is isomorphic to the complete lattice $(E(A), \subseteq)$ of all equivalence relations defined on A. The mapping $f : E(A) \longrightarrow Part(A)$ such that for any $\rho \in E(A),\ f(\rho) = \{[a]_\rho : a \in A\}$, where $[a]_\rho = \{x \in A : x\rho a\}$, is the isomorphism. The inverse isomorphism $g : Part(A) \longrightarrow E(A)$ is of the form: $g(\Pi) = \bigcup\{X^2 : X \in \Pi\}$.

We are going to apply two methods to obtain new classes of families of subsets of a given set A (other than the one of all partitions of A). These new classes when equipped with refinement-like relations, form posets, which we shall show are isomorphic to the posets composed of appropriate classes of binary relations defined on A, ordered by inclusion. In special cases, when a class $\Theta \subseteq P(A \times A)$ (P is the operation of powerset) of binary relations on A and a class $\mathcal{F} \subseteq P(P(A))$ of families of subsets of A are such that $E(A) \subseteq \Theta$ and $Part(A) \subseteq \mathcal{F}$, the isomorphism $f' : (\Theta, \subseteq) \longrightarrow (\mathcal{F}, \leq)$ should fulfil the expected condition that the restriction f' to $E(A)$ is just

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the above mentioned isomorphism f. It turns out, as a result of the first method, that several classes of binary relations defined on a set A, such as quasiorders, so-called antiquasiorders and semiequivalence relations, can be associated with appropriate classes of families of subsets of A. The second method leads to two results: firstly, we obtain two different isomorphisms joining the class of so-called semitolerance relations defined on A with two classes of families of subsets of A; and secondly, we provide a similar result for tolerance relations. This result is important since the tolerance relations are used to form the "quotient" algebras, in particular lattices composed of the "blocks" of "partitions", leading to the important method of gluing of lattices, cf. for example [6,9]. Such quotient modulo tolerances algebras are considered without any justification, i.e. without providing a form of isomorphism joining the class of all tolerances with an appropriate class of families of "blocks", cf. [1,3–5,11]. According to our knowledge, [2] is the only paper which gives an appropriate family of blocks, called the τ-covering of a set, corresponding to a given tolerance relation. However, the conditions defining the notion of *τ-covering of a set* are rather non-intuitive and complicated, in comparison with our more natural approach to describe a "partition" corresponding to a tolerance.

Although some of the results presented here can be generalized also to the case $A = \emptyset$, it is assumed in the whole paper (sometimes explicitly) that the set A on which the considered relations are defined and whose subsets form "partitions" of different types, is nonempty.

2. Preliminaries Concerning the First Method

The following simple observation is basic for the first method of searching for other types of families of subsets of a given set A which are isomorphic to classes of binary relations defined on A. The isomorphism $f : E(A) \longrightarrow Part(A)$, is a composition of two mappings: the first, $\phi : E(A) \longrightarrow P(A)^A$, assigns to each equivalence relation ρ the canonical mapping $k_\rho : A \longrightarrow P(A)$ defined by $k_\rho(a) = [a]_\rho$. The second assigns to each $k \in P(A)^A$ its counterdomain $k[A]$. The first mapping, ϕ, can be generalized to be defined on the set of all binary relations on A, as follows. First, for any $\rho \subseteq A \times A$ and any $a \in A$, we define the *equivalential class* of a to be the set $(a]_\rho = \{x \in A : x\rho a\}$. Now let $\phi(\rho)$ be the function defined on A by $\phi(\rho)(a) = (a]_\rho$. This defines a function $\phi : P(A\times A) \longrightarrow P(A)^A$. We note that $(P(A\times A), \subseteq)$ is a complete lattice, under the usual set inclusion, and that $(P(A)^A, \leq)$ is also a complete lattice, under the ordering given by $k_1 \leq k_2$ iff for each $a \in A$ we have $k_1(a) \subseteq k_2(a)$.

PROPOSITION 1. *The mapping* $\phi : P(A \times A) \longrightarrow P(A)^A$ *is an isomorphism from the lattice* $(P(A \times A), \subseteq)$ *to the lattice* $(P(A)^A, \leq)$. *The inverse isomorphism is given by the mapping* $\psi : P(A)^A \longrightarrow P(A \times A)$ *defined by* $\psi(k) = \{(a, b) \in A \times A : a \in k(b)\}$, *for any* $k \in P(A)^A$.

PROOF. Straightforward. ∎

From now on we will consider the correspondence $A \times A \supseteq \rho \Longrightarrow k_\rho \in P(A)^A$, where for each $a \in A$, $k_\rho(a) = (a]_\rho$, and the inverse correspondence: $P(A)^A \ni k \Longrightarrow \rho_k \subseteq A \times A$ such that for any $a, b \in A :\ (a, b) \in \rho_k$ iff $a \in k(b)$. In view of Proposition 1, for each $\rho \subseteq A \times A$ we have $\rho = \rho_{k_\rho}$ and for each $k \in P(A)^A$ we have $k = k_{\rho_k}$.

Moreover, with each map $k : A \longrightarrow P(A)$ we can associate its counterdomain $k[A] = \{k(a) : a \in A\} = \{(a]_{\rho_k} : a \in A\}$. In general, the assignment: $P(A)^A \ni k \Longrightarrow k[A] \in P(P(A))$ is obviously neither 1-1 nor onto. However, for some special subsets of $P(A)^A$ and of $P(P(A))$ the correspondence is in fact 1-1 and onto. In such a case, when there is a 1-1 and onto correspondence between a set $\mathcal{K} \subseteq P(A)^A$ and $\mathcal{P} \subseteq P(P(A))$ one can establish immediately that the posets $(\{\rho_k : k \in \mathcal{K}\}, \subseteq), (\mathcal{P}, \leq')$ are isomorphic. Here, by definition, for any $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{P} :\ \mathcal{R}_1 \leq' \mathcal{R}_2$ iff $k_1 \leq k_2$, where k_1 corresponds to $\mathcal{R}_1$, and k_2 to $\mathcal{R}_2$. The isomorphism of these posets is a composition of two isomorphisms: $\rho_k \xrightarrow{\phi} k \longrightarrow k[A]$, where ϕ over the arrow denotes the restriction of the isomorphism ϕ from Proposition 1 to the set $\psi[\mathcal{K}]$ of binary relations on A. Obviously, the final form of the isomorphism is as follows: $\rho_k \longrightarrow \{(a]_{\rho_k} : a \in A\}$, for each $k \in \mathcal{K}$.

For example, consider the set $Part(A)$ in a role of the subset $\mathcal{P}$ of $P(P(A))$ and a subset $\mathcal{K}$ of $P(A)^A$ composed of all the mappings k fulfilling the following conditions:

(1) $\forall a \in A,\ a \in k(a)$,

(2) $\forall a, b \in A\ (k(a) \cap k(b) \neq \emptyset\ \Rightarrow\ k(a) = k(b))$

or, equivalently, the following one:

(3) $\forall a, b \in A\ (a \in k(b)$ iff $k(a) = k(b))$.

Then given $k \in \mathcal{K}$, the family $k[A]$ is a partition of A. Conversely, given any $\Pi \in Part(A)$ define the mapping $k_\Pi : A \longrightarrow P(A)$ by setting $k_\Pi(a)$ to be the single element of Π to which a belongs, for each $a \in A$. Then the mapping k_Π fulfils the conditions (1) and (2). Moreover, for each $k \in \mathcal{K}$ we have $k_{k[A]} = k$ and for any $\Pi \in Part(A)$ we have $k_\Pi[A] = \Pi$. Thence the correspondence: $\mathcal{K} \ni k \Longrightarrow k[A] \in Part(A)$, is 1-1 and onto. In this way we see that the posets $(\{\rho_k : k \in \mathcal{K}\}, \subseteq)$, $(Part(A), \leq')$ where for any

$\Pi_1, \Pi_2 \in Part(A): \ \Pi_1 \leq' \Pi_2$ iff $k_1 \leq k_2$, and $\Pi_i = k_i[A]$, $i = 1, 2$ are isomorphic. However, one can show that $\{\rho_k : k \in \mathcal{K}\} = E(A)$ (to this aim it is convenient to use the condition (3) rather than (1) and (2)); moreover, for each $k_1, k_2 \in \mathcal{K}: \ k_1 \leq k_2$ iff $\forall X \in k_1[A] \ \exists Y \in k_2[A], \ X \subseteq Y$, that is $k_1 \leq k_2$ iff the partition $k_1[A]$ is finer than $k_2[A]$. So finally, using this method, we rediscover the well-known theorem we started from.

Now we are going to apply the method to find the analogous connections between some other classes of binary relations defined on a given set A and the corresponding classes of families of subsets of A.

3. Quasiorders and Quasipartitions

Definition. A family $\mathcal{R}$ of subsets of a given set A will be said to be a *quasipartition of the set* A iff

(1) for each $a \in A, \ \mathcal{R}_a = \{X \in \mathcal{R} : a \in X\} \neq \emptyset$,

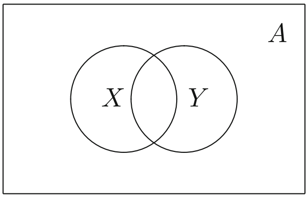
(2) $\forall a \in A, \ \bigcap \mathcal{R}_a \in \mathcal{R}$,

(3) $\forall X \in \mathcal{R} \exists a \in A, \ X = \bigcap \mathcal{R}_a$.

In other words, a family $\mathcal{R} \subseteq P(A)$ is a quasipartition of A iff $\mathcal{R} = \{\bigcap \mathcal{R}_a : a \in A\}$. (The name "quasi-partition" already occurs in the literature: in [5] it is used in an informal way as the name of the partition counterpart corresponding to a tolerance relation, which in [2] is referred to as the τ-covering of a set.)

As a simple example of a quasipartition, consider a nonempty set A and $X, Y \subseteq A, X \neq Y, X \cap Y \neq \emptyset, X \cup Y \neq A$. Then a family $\mathcal{R} = \{X \cap Y, X, Y, A\}$ is a quasipartition of A. Here

$$\begin{aligned}
\mathcal{R}_a &= \{A\}, \text{if}\, a \notin X \cup Y, \\
\mathcal{R}_a &= \{X, A\}, \text{if}\, a \in X - Y, \\
\mathcal{R}_a &= \{Y, A\}, \text{if}\, a \in Y - X, \\
\mathcal{R}_a &= \{X \cap Y, X, Y, A\}, \text{if}\, a \in X \cap Y.
\end{aligned}$$



Notice that any ordinary partition Π of a set A is a quasipartition of A. Here for each $a \in A, \ \Pi_a$ is a singleton.

Now consider the class of all the mappings $k : A \longrightarrow P(A)$ fulfilling the following conditions:

$$\forall a \in A,\ \{k(x) : x \in A \ \&\ a \in k(x)\} \neq \emptyset, \tag{$k\bigcap 1$}$$

$$k(a) = \bigcap\{k(x) : x \in A \ \&\ a \in k(x)\} \text{ for each } a \in A. \tag{$k\bigcap 2$}$$

It is easily seen that given such a function k from that class, its counterdomain $k[A]$ is a quasipartition of A (the condition $(k\bigcap 2)$ means: $\forall a \in A,\ k(a) = \bigcap k[A]_a$). Conversely, given any quasipartition $\mathcal{R}$, define a function $k_{\mathcal{R}} : A \longrightarrow P(A)$ by setting $k_{\mathcal{R}}(a) = \bigcap \mathcal{R}_a$, for each $a \in A$. Then the function $k_{\mathcal{R}}$ fulfils the conditions $(k\bigcap 1), (k\bigcap 2)$. The condition $(k\bigcap 1)$ holds due to the fact that for each $a \in A$ we have $a \in \bigcap \mathcal{R}_a$, i.e., $a \in k_{\mathcal{R}}(a)$. The inclusion $(\supseteq)$ of $(k\bigcap 2)$ follows due to the same fact. The inclusion $(\subseteq)$ is equivalent to the expression: $\forall a, x \in A\ (a \in k_{\mathcal{R}}(x) \ \Rightarrow\ k_{\mathcal{R}}(a) \subseteq k_{\mathcal{R}}(x))$. When $a \in k_{\mathcal{R}}(x)$, that is $a \in \bigcap \mathcal{R}_x$, then $\bigcap \mathcal{R}_x \in \mathcal{R}_a$, so $k_{\mathcal{R}}(a) = \bigcap \mathcal{R}_a \subseteq \bigcap \mathcal{R}_x = k_{\mathcal{R}}(x)$.

Furthermore, for any mapping k such that the conditions $(k\bigcap 1), (k\bigcap 2)$ are satisfied we have $k_{k[A]} = k$. On the other hand, for any quasipartition $\mathcal{R}$ of A we have $k_{\mathcal{R}}[A] = \mathcal{R}$. This means that there exists a 1-1 correspondence between the class of all the quasipartitions and the class of all mappings k fulfilling $(k\bigcap 1), (k\bigcap 2)$.

Now it is sufficient to read the conditions $(k\bigcap 1), (k\bigcap 2)$ in terms of corresponding binary relations in order to obtain an appropriate connection between the quasipartitions and a suitable class of relations. So the condition $(k\bigcap 2)$ is equivalent to the following one:

$$\forall a, b \in A\ (b\rho a \text{ iff } \forall x \in A\ (a\rho x \Rightarrow b\rho x)) \tag{qo}$$

which holds for a relation ρ if and only if ρ is a quasiorder on A (that is a reflexive and transitive binary relation on A). The condition $(k\bigcap 1)$ is equivalent to the condition: $\forall a \in A \exists x \in A\ a\rho x$ (which means by definition that ρ is serial), therefore yields nothing more to (qo).

Let $QOrd(A),\ QPart(A)$ are the sets of all quasiorders and of all quasipartitions of a set A, respectively. Then the following connection follows:

COROLLARY 2. *The complete lattices* $(QOrd(A), \subseteq),\ (QPart(A), \leq)$ *are isomorphic. Here the relation of refinement* $\leq$ *is of the form:* $\mathcal{R} \leq \mathcal{S}$ *iff* $\forall a \in A, \bigcap \mathcal{R}_a \subseteq \bigcap \mathcal{S}_a$.

Obviously we have just stated that the poset $(QPart(A), \leq)$ is a complete lattice as it is isomorphic to the complete lattice $(QOrd(A), \subseteq)$ in which $A \times A$ and the identity relation id_A on A are the greatest and the least elements respectively, and for any nonempty $\Theta \subseteq QOrd(A)$ we have $inf\Theta = \bigcap\Theta$ and $sup\Theta = \overline{\bigcup\Theta}$, where for any binary relation ρ defined on A the relation $\overline{\rho}$ is the transitive closure of ρ.

One may show that the isomorphism from Corollary 2 restricted to the class $E(A)$ of all the equivalence relations on A is just the isomorphism $f : (E(A), \subseteq) \longrightarrow (Part(A), \leq)$.

4. Antiquasiorders and Dual Quasipartitions

Definition. A family $\mathcal{R}$ of subsets of a given set A will be said to be a *dual quasipartition of the set A* iff

(1) $\forall a \in A,\ \bigcup\mathcal{R}^a \in \mathcal{R}$,

(2) $\forall X \in \mathcal{R}\exists a \in A,\ X = \bigcup\mathcal{R}^a$,

where for each $a \in A$, $\mathcal{R}^a = \{X \in \mathcal{R} : a \notin X\}$. In other words, a family $\mathcal{R} \subseteq P(A)$ is a dual quasipartition of A iff $\mathcal{R} = \{\bigcup\mathcal{R}^a : a \in A\}$.

The word "dual" is used here in the following sense:

Fact 3. *For any family $\mathcal{R} \subseteq P(A)$, $\mathcal{R}$ is a dual quasipartition of A iff there exists a quasipartition $\mathcal{S}$ of A such that $\mathcal{R} = \{-X : X \in \mathcal{S}\}$, where $-X$ is the complement of X in the Boolean algebra of all subsets of A.*

Proof. $(\Rightarrow)$: Assume that $\mathcal{R}$ is a dual quasipartition of A. We show that $\mathcal{S} = \{-X : X \in \mathcal{R}\}$ is a quasipartition of A. To this aim consider any $X \in \mathcal{R}$. Then $\exists a \in A,\ X = \bigcup\mathcal{R}^a$. So $-X = -\bigcup\{Y \in \mathcal{R} : a \notin Y\} = \bigcap\{-Y : Y \in \mathcal{R} \;\&\; a \in -Y\} = \bigcap\mathcal{S}_a$. Therefore, any element from $\mathcal{S}$ is of the form $\bigcap\mathcal{S}_a$ for some $a \in A$. On the other hand, for any $a \in A$, $\bigcap\mathcal{S}_a = \bigcap\{-Y : Y \in \mathcal{R} \;\&\; a \in -Y\} = -\bigcup\{Y \in \mathcal{R} : a \notin Y\} = -\bigcup\mathcal{R}^a$, while $\bigcup\mathcal{R}^a \in \mathcal{R}$, so $\bigcap\mathcal{S}_a \in \mathcal{S}$. Finally, $\mathcal{S}$ is a quasipartition of A. Obviously, $\mathcal{R} = \{-X : X \in \mathcal{S}\}$.

$(\Leftarrow)$: Now we show that the family $\mathcal{R} = \{-X : X \in \mathcal{S}\}$, where $\mathcal{S}$ is a quasipartition of A, is a dual quasipartition of A. So consider any $X \in \mathcal{S}$. Then $X = \bigcap\mathcal{S}_a$ for some $a \in A$, that is $X = \bigcap\{Y \in \mathcal{S} : a \in Y\}$. So $-X = \bigcup\{-Y : Y \in \mathcal{S} \;\&\; a \notin -Y\} = \bigcup\mathcal{R}^a$. Thus any element of $\mathcal{R}$ is of the form: $\bigcup\mathcal{R}^a$ for some $a \in A$. On the other hand, for any $a \in A$ we have $\bigcup\mathcal{R}^a = \bigcup\{-Y : Y \in \mathcal{S} \;\&\; a \notin -Y\} = -\bigcap\{Y \in \mathcal{S} : a \in Y\} = -\bigcap\mathcal{S}_a$. Therefore, $\bigcup\mathcal{R}^a \in \mathcal{R}$, since $\bigcap\mathcal{S}_a \in \mathcal{S}$. Finally, $\mathcal{R}$ is a dual quasipartition of A. ■

Now consider the class of all the mappings $k : A \longrightarrow P(A)$ fulfilling the following condition:

$$k(a) = \bigcup\{k(x) : x \in A \,\&\, a \notin k(x)\} \text{ for each } a \in A. \qquad (k\bigcup)$$

It is easily seen that given a function k satisfying condition $(k\bigcup)$, its counterdomain $k[A]$ is a dual quasipartition of A. Conversely, a function $k_{\mathcal{R}} : A \longrightarrow P(A)$ defined by a dual quasipartition $\mathcal{R}$ in the following way: $\forall a \in A,\ k_{\mathcal{R}}(a) = \bigcup \mathcal{R}^a$, fulfils the condition $(k\bigcup)$. Furthermore, for any mapping k such that $(k\bigcup)$ is satisfied, $k_{k[A]} = k$. On the other hand, for any dual quasipartition $\mathcal{R}$ of A we have $k_{\mathcal{R}}[A] = \mathcal{R}$. This means that there exists a 1-1 correspondence between the class of all the dual quasipartitions and the class of all mappings k fulfilling $(k\bigcup)$.

Now, one can simply rewrite the condition $(k\bigcup)$ in terms of binary relations ρ corresponding to the maps k which satisfy $(k\bigcup)$, in the following way: for any $a, b \in A$, $b\rho a$ iff $\exists x \in A$ $(\neg a\rho x \ \&\ b\rho x)$. This condition is equivalent to the following one:

$$\forall a, b \in A\ (\neg b\rho a \text{ iff } \forall x \in A\ (b\rho x \Rightarrow a\rho x)), \qquad (aqo)$$

which in turn is equivalent to the conjunction of two conditions:

(i) ρ is irreflexive,

(ii) $\forall a, b \in A\ (a\rho b \Rightarrow \forall x \in A\ (a\rho x \text{ or } x\rho b))$.

Let us call such a binary relation ρ on A fulfilling $(i), (ii)$ an *antiquasiorder*, which is justified by the following:

FACT 4. *The class of all antiquasiorders defined on a set A is exactly the class of all the complements of quasiorders on A.*

PROOF. Take any quasiorder ρ defined on A. Then the following condition (equivalent to the statement that ρ is a quasiorder) holds: $\forall a, b \in A$ $(a\rho b$ iff $\forall x \in A\ (b\rho x \Rightarrow a\rho x))$. This is equivalent to the following: $\forall a, b \in A$ $(b\rho a$ iff $\forall x \in A\ (\neg b\rho x \Rightarrow \neg a\rho x))$, which means that the complement $-\rho$ $(= A^2 - \rho)$ fulfils the condition (aqo). Conversely, consider any antiquasiorder ρ on A. Then from (aqo) it follows that for any $a, b \in A:\ b(-\rho)a$ iff $\forall x \in A\ (a(-\rho)x \Rightarrow b(-\rho)x)$ which is equivalent to the fact that $-\rho$ is a quasiorder on A, that is ρ is the complement of a quasiorder. ■

One can show that the class of all binary relations ρ fulfilling the condition (ii) is just the class of all complements of transitive relations defined on A. So let us call a relation ρ from that class *intransitive*.

Notice that a binary relation ρ defined on A may be both transitive and intransitive:

FACT 5. *Let ρ be any quasiorder on A. Then ρ is connected* $(\forall a, b \in A\ (a\rho b$ *or* $b\rho a))$ *iff* $-\rho$ *is transitive (ρ is intransitive).*

PROOF. Straightforward. ■

Notice also that the poset $(AQOrd(A), \subseteq)$ of all antiquasiorders defined on a set A is a complete lattice such that $(A \times A) - id_A$ and the empty relation are the greatest and the least elements, respectively, and for any nonempty $\Theta \subseteq AQOrd(A)$ we have $sup\Theta = \bigcup\Theta$, $inf\Theta = \underline{\bigcap\Theta}$, where for any $\rho \subseteq A \times A:$ $\underline{\rho}$ is the greatest intransitive relation contained in ρ, that is, $\underline{\rho} = -(\overline{-\rho})$.

COROLLARY 6. *The complete lattices* $(AQOrd(A), \subseteq)$ *of all the antiquasiorders of A and* $(DQPart(A), \leq)$ *of all dual quasipartitions of A are isomorphic. Here for any dual quasipartitions $\mathcal{R}, \mathcal{S}$ of A we have $\mathcal{R} \leq \mathcal{S}$ iff* $\forall a \in A$, $\bigcup\mathcal{R}^a \subseteq \bigcup\mathcal{S}^a$. *(The poset* $(DQPart(A), \leq)$ *is a complete lattice for it is isomorphic to the poset* $(AQOrd(A), \subseteq)$ *which is a complete lattice.)*

Given any antiquasiorder $-\rho$, where ρ is a quasiorder on A (cf. Fact 4) for any $a \in A$ we have $(a]_{-\rho} = -(a]_\rho$. Therefore, the canonical mapping $k_{-\rho}$ corresponding to the relation $-\rho$ is of the form: for each $a \in A$, $k_{-\rho}(a) = -(a]_\rho$. In this way the dual quasipartition corresponding to the antiquasiorder $-\rho$ is of the form: $\{k_{-\rho}(a) : a \in A\} = \{-(a]_\rho : a \in A\}$ (cf. Fact 3). In turn, given any dual quasipartition $\mathcal{R}$ of A let $\mathcal{R}' = \{-X : X \in \mathcal{R}\}$ be the corresponding quasipartition of A. Moreover, let $-\rho_1$ be the antiquasiorder corresponding to $\mathcal{R}$, that is $\mathcal{R} = \{(a]_{-\rho_1} : a \in A\}$. Then obviously, $\mathcal{R}' = \{(a]_{\rho_1} : a \in A\}$. Suppose similarly that a dual quasipartition $\mathcal{S} = \{(a]_{-\rho_2} : a \in A\}$, where ρ_2 is a quasiorder. Then for the relation of refinement of dual quasipartitions the following holds: $\mathcal{R} \leq \mathcal{S}$ iff $\forall a \in A$, $(a]_{-\rho_1} \subseteq (a]_{-\rho_2}$ iff $\forall a \in A$, $(a]_{\rho_2} \subseteq (a]_{\rho_1}$ iff $\mathcal{S}' \leq \mathcal{R}'$.

Finally, one can also consider all the complements of equivalence relations on A. They form the class of all irreflexive, intransitive and symmetric relations on A hereafter called *antiequivalence* relations. Furthermore, the class $AE(A)$ of all the antiequivalence relations on A forms a complete sublattice of the lattice $(AQOrd(A), \subseteq)$.

Given any antiequivalence relation defined on A, let us say $-\rho$, where ρ is an equivalence relation on A, the dual quasipartition corresponding to $-\rho$, hereafter called a *dual partition* of A, is of the form: $\{(a]_{-\rho} : a \in A\} = \{-[a]_\rho : a \in A\}$. So, given two dual partitions $\mathcal{R}, \mathcal{S}$ of A, the corresponding quasipartitions $\mathcal{R}', \mathcal{S}'$ are ordinary partitions of A and $\mathcal{R} \leq \mathcal{S}$ iff $\mathcal{S}' \leq \mathcal{R}'$ iff $\forall X \in \mathcal{S}' \exists Y \in \mathcal{R}', X \subseteq Y$ iff $\forall X \in \mathcal{S} \exists Y \in \mathcal{R}, Y \subseteq X$.

Restricting the isomorphism from Corollary 6 to the class $AE(A)$ one obtains the following:

COROLLARY 7. *The complete lattice* $(AE(A), \subseteq)$ *of all the antiequivalence relations of* A *is isomorphic to the lattice* $(DPart(A), \leq)$ *of all dual partitions. Here the ordering of dual partitions is given by* $\mathcal{R} \leq \mathcal{S}$ *iff* $\forall a \in A, \bigcup \mathcal{R}^a \subseteq \bigcup \mathcal{S}^a$ *iff* $\forall X \in \mathcal{S}\ \exists Y \in \mathcal{R},\ Y \subseteq X$.

5. Semiequivalence Relations and Pseudopartitions

DEFINITION. A family $\mathcal{R}$ of subsets of a given set A will be said to be a *pseudopartition of the set* A iff

(1) $\forall a \in A,\ \bigcup \mathcal{R}_a \in \mathcal{R}$,

(2) $\forall X \in \mathcal{R} \exists a \in A,\ X = \bigcup \mathcal{R}_a$,

where, as previously, for each $a \in A$, $\mathcal{R}_a = \{X \in \mathcal{R} : a \in X\}$. In other words, a family $\mathcal{R} \subseteq P(A)$ is a pseudopartition of A iff $\mathcal{R} = \{\bigcup \mathcal{R}_a : a \in A\}$.

This definition leads to the following characterization of a pseudopartition of a set A:

FACT 8. *For any pseudopartition* $\mathcal{R}$ *of* A : $\forall a \in A\ (\mathcal{R}_a \neq \emptyset \ \Rightarrow\ \mathcal{R}_a = \{\bigcup \mathcal{R}_a\})$.

PROOF. Let $\mathcal{R}$ be any pseudopartition of a set A and $a \in A$. Suppose that $X \in \mathcal{R}_a$.

$(\supseteq)$: Since $a \in X$, so $a \in \bigcup \mathcal{R}_a$ and, finally, $\bigcup \mathcal{R}_a \in \mathcal{R}_a$.

$(\subseteq)$: Obviously, $X \subseteq \bigcup \mathcal{R}_a$. So it is enough to show the converse inclusion. Since $X \in \mathcal{R}$, so $X = \bigcup \mathcal{R}_b$ for some $b \in A$. Then $b \in X$, therefore $b \in \bigcup \mathcal{R}_a$, as well. Thus $\bigcup \mathcal{R}_a \in \mathcal{R}_b$. This means that $\bigcup \mathcal{R}_a \subseteq \bigcup \mathcal{R}_b$, that is $\bigcup \mathcal{R}_a \subseteq X$. ■

It is evident that given any element of A, in the family of all members of a given pseudopartition of A containing that element, either there is exactly one set or there is none. So each element of A either belongs to exactly one set in the pseudopartition or does not belong to any of its sets. The difference between *partition* and *pseudopartition* of a set A consists only in the fact that the empty set may be an element of a pseudopartition while it cannot be a member of a partition.

Now consider the class of all the mappings $k : A \longrightarrow P(A)$ fulfilling the following condition:

$$k(a) = \bigcup\{k(x) : x \in A \ \&\ a \in k(x)\}, \text{ for each } a \in A. \qquad (-k\bigcup)$$

It is evident that given a function k from that class, its counterdomain $k[A]$ is a pseudopartition of A. Conversely, a function $k_{\mathcal{R}} : A \longrightarrow P(A)$ defined by a pseudopartition $\mathcal{R}$ by $\forall a \in A,\ k_{\mathcal{R}}(a) = \bigcup \mathcal{R}_a$, fulfils the condition $(-k\bigcup)$ because for any $a \in A$ we have $\mathcal{R}_a = \{\bigcup \mathcal{R}_x : x \in A\ \&\ a \in \bigcup \mathcal{R}_x\}$.

Furthermore, for any mapping k such that the condition $(-k\bigcup)$ is satisfied, for any $a \in A$, we have $k_{k[A]}(a) = \bigcup k[A]_a = \bigcup\{k(x) : x \in A\ \&\ a \in k(x)\} = k(a)$. Hence, $k_{k[A]} = k$. On the other hand, for any pseudopartition $\mathcal{R}$ of A we have $k_{\mathcal{R}}[A] = \{k_{\mathcal{R}}(x) : x \in A\} = \{\bigcup \mathcal{R}_x : x \in A\} = \mathcal{R}$. This means that there exists a 1-1 correspondence between the class of all the pseudopartitions and the class of all mappings k fulfilling $(-k\bigcup)$.

Now, the condition $(-k\bigcup)$ can be expressed in terms of binary relations as follows:

$$\forall a, b \in A\ (b\rho a \text{ iff } \exists x \in A\ (b\rho x\ \&\ a\rho x)). \qquad (se)$$

Let us call a binary relation ρ defined on A *semiequivalence* iff ρ is symmetric, transitive and *semireflexive*, where the semireflexive property is:

$$\forall a \in A\ ((\exists x \in A, a\rho x \text{ or } x\rho a)\ \Rightarrow\ a\rho a). \qquad (sr)$$

Obviously, a semiequivalence relation defined on a set A, restricted to the set $\{a \in A : a\rho a\}$ (of all elements of A which are not isolated with respect to ρ) is an ordinary equivalence relation on that set. Furthermore, we have the following

FACT 9. *For any binary relation ρ on A, ρ is a semiequivalence iff the condition (se) holds for ρ.*

PROOF. $(\Rightarrow)$: Suppose that ρ is a semiequivalence on A. Then the condition $(se)(\Rightarrow)$ follows due to semireflexivity of ρ. The converse condition: $(se)(\Leftarrow)$ holds due to symmetry and transitivity of ρ.

$(\Leftarrow)$: Assume (se). Then the symmetry of ρ follows directly. Next we have that for each $b \in A$: $b\rho b$ iff $\exists x \in A, b\rho x$, which together with symmetry implies semireflexivity of ρ. For transitivity, suppose that $b\rho a$ and $a\rho c$. Then we have $c\rho a$ from symmetry. In this way, $\exists x \in A\ (b\rho x\ \&\ c\rho x)$. So $b\rho c$ follows from (se). ■

One can check that the class $SE(A)$ of all the semiequivalence relations defined on A forms a complete lattice. Its greatest element is $A \times A$, and its least element is $\emptyset$; for any nonempty family $\Theta \subseteq SE(A)$, we have $inf\Theta = \bigcap \Theta$ and $sup\Theta = \overline{\bigcup \Theta}$. Thus $(E(A), \subseteq)$ forms a complete sublattice of $(SE(A), \subseteq)$.

Notice also that applying Fact 8, given two functions k_1, k_2 fulfilling the condition $(-k\bigcup)$ one can easily show that $k_1 \leq k_2$ iff $\forall a \in A,\ k_1(a) \subseteq k_2(a)$ iff $\forall a \in A,\ \bigcup \mathcal{R}_a \subseteq \bigcup \mathcal{S}_a$ iff $\forall X \in \mathcal{R} \exists Y \in \mathcal{S}, X \subseteq Y$, where $\mathcal{R}, \mathcal{S}$ are the pseudopartitions of A which are counterdomains of k_1 and k_2, respectively.

Taking into account the considerations of the section one can formulate the following result.

Corollary 10. *The complete lattice $(SE(A), \subseteq)$ of all the semiequivalence relations of A is isomorphic to the lattice $(PsPart(A), \leq)$ of all pseudopartitions of A. Here the ordering of pseudopartitions is given by $\mathcal{R} \leq \mathcal{S}$ iff $\forall a \in A,\ \bigcup \mathcal{R}_a \subseteq \bigcup \mathcal{S}_a$ iff $\forall X \in \mathcal{R}\ \exists Y \in \mathcal{S}, X \subseteq Y$.*

6. Preliminaries Concerning the Second Method

The second method used to establish the counterparts of partitions related with tolerance relations is based on two tools. The first one is a general correspondence between the family of all downward subsets with maximal elements in a given poset and the family of all antichains of the poset:

Lemma 11. *Given any poset $(A, \leq)$, let $U \subseteq A$ fulfil the following conditions:*

(b) $\forall x, y \in A\ (x \leq y\ \&\ y \in U \ \Rightarrow x \in U)$,

(c) $\forall x \in U \exists a \in Max(U),\ x \leq a$,

where for any $B \subseteq A$, $Max(B)$ is the set of all maximal elements in $(B, \leq)$. Then $U = \bigcup\{(a]_{\leq} : a \in Max(U)\}$.

Proof. Straightforward. ∎

Lemma 12. *Given any poset $(A, \leq)$ let $B \subseteq A$ be any antichain contained in $(A, \leq)$. Then $B = Max(\bigcup\{(a]_{\leq} : a \in B\})$.*

Proof. Let B be any antichain contained in $(A, \leq)$.

$(\subseteq)$: Assume that $b \in B,\ x \in \bigcup\{(a]_{\leq} : a \in B\}$ and $b \leq x$. So $b \leq a$ for some $a \in B$ such that $x \leq a$. Thus, $b = a$ for different elements of B are incomparable with respect to $\leq$. Therefore, $b = x$ which proves that b is a maximal element in $\bigcup\{(a]_{\leq} : a \in B\}$.

($\supseteq$) : Suppose that b is a maximal element in $\bigcup\{(a]_\leq : a \in B\}$. Then naturally, $b \leq x$ for some $x \in B$. However also $x \in \bigcup\{(a]_\leq : a \in B\}$. Therefore from the maximality of b it follows that $b = x$. Hence, $b \in B$. ∎

Given a poset $(A, \leq)$ let $DWM(A)$ be the family of all the downward subsets U of A with maximal elements, that is all the sets $U \subseteq A$ which fulfil the conditions (b) and (c) of Lemma 11. Moreover, let $Ach(A)$ be the family of all the antichains contained in the poset $(A, \leq)$. In the next proposition we consider the partial ordering $\sqsubseteq$ defined on $Ach(A)$ by $B_1 \sqsubseteq B_2$ iff $\forall x \in B_1 \exists y \in B_2,\ x \leq y$.

PROPOSITION 13. *For any poset $(A, \leq)$, the mapping $\phi : DWM(A) \longrightarrow Ach(A)$ defined by $\phi(U) = Max(U)$, is an isomorphism of the posets: $(DWM(A), \subseteq)$, $(Ach(A), \sqsubseteq)$. The inverse isomorphism $\psi : (Ach(A), \sqsubseteq) \longrightarrow (DWM(A), \subseteq)$ is of the form: $\psi(B) = \bigcup\{(a]_\leq : a \in B\}$.*

PROOF. Since for any $B \subseteq A$, $Max(B)$ is an antichain in $(A, \leq)$, so the mapping ϕ is well defined. Furthermore, it is obvious that for any $B \subseteq A$ the set $\bigcup\{(a]_\leq : a \in B\}$ is a downward one, that is fulfils the condition (b) of Lemma 11. When B is an antichain, by Lemma 12 one can see that the set $\bigcup\{(a]_\leq : a \in B\}$ also fulfils the condition (c). Thus the map ψ is well defined too.

Now, for any $B \in Ach(A)$ we have $\phi(\psi(B)) = \phi(\bigcup\{(a]_\leq : a \in B\}) = Max(\bigcup\{(a]_\leq : a \in B\}) = B$, due to Lemma 12. On the other hand, for any $U \in DWM(A), \psi(\phi(U)) = \psi(Max(U)) = \bigcup\{(a]_\leq : a \in Max(U)\} = U$ due to Lemma 11. This means that ϕ is 1-1 and onto.

Now it is sufficient to show that for any $U_1, U_2 \in DWM(A)$ we have $U_1 \subseteq U_2$ iff $Max(U_1) \sqsubseteq Max(U_2)$.

($\Rightarrow$): When $U_1 \subseteq U_2$ and $x \in Max(U_1)$ then from (c) applied to U_2 it follows that there exists $a \in Max(U_2)$ such that $x \leq a$.

($\Leftarrow$): Suppose that $\forall x \in Max(U_1) \exists y \in Max(U_2),\ x \leq y$, and let $x \in U_1$. Then $x \leq a$ for some $a \in Max(U_1)$ by (c). Hence and from the assumption it follows that $x \leq b$ for some $b \in Max(U_2)$. So $x \in U_2$ by Lemma 11. ∎

The second tool of the method is the concept of a *residuated pair* of mappings. We are going to apply it first, in order to provide one of the counterparts of partition corresponding to the so-called semitolerance relation. Let us recall briefly the notion. Given any posets $(A, \leq_A)$, $(B, \leq_B)$, a pair of mappings $f : A \longrightarrow B,\ g : B \longrightarrow A$ is called *residuated* iff for each $a \in A, b \in B$:

$$b \leq_B f(a) \text{ iff } g(b) \leq_A a, \tag{\textit{res}}$$

or equivalently, f and g are monotone and for any $a \in A, b \in B$:

$$g(f(a)) \leq_A a \text{ and } b \leq_B f(g(b)).$$

Some authors use the adjective "residuated" in case the converse orderings instead of $\leq_A,\ \leq_B$ are applied. A function $f : A \longrightarrow B$ for which there exists a map $g : B \longrightarrow A$ such that (f, g) is residuated pair, is also called *residuated*. Given a residuated function there is a unique g such that (f, g) is a residuated pair. Given a residuated pair (f, g) of functions, their compositions I and C of the form: $Ia = g(f(a)),\ Cb = g(f(b))$, any $a \in A,\ b \in B$, are interior and closure operations on A and B, respectively. The following condition is also satisfied:

$$\{a \in A : a = Ia\} = g[B] \text{ and } \{b \in B : b = Cb\} = f[A].$$

In this way it follows that the mapping f restricted to the image $g[B]$ is a bijection from that image onto $f[A]$. Moreover, for any $a_1, a_2 \in g[B]$: $a_1 \leq_A a_2$ iff $f(a_1) \leq_B f(a_2)$, so in fact the posets $(g[B], \leq_A),\ (f[A], \leq_B)$ (of all open and closed elements respectively) are isomorphic.

The important examples of residuated pairs arise when the posets $(A, \leq_A),\ (B, \leq_B)$ are complete lattices. Then the following holds: any mapping $f : A \longrightarrow B$ is residuated iff for each $A' \subseteq A,\ f(inf_A A') = inf_B f[A']$ (cf. for example [7,8]).

7. Semitolerance Relations and Corresponding "Partitions"

Now, given any set A let us consider two mappings, $f : P(A \times A) \longrightarrow P(P(A))$ and $g : P(P(A)) \longrightarrow P(A \times A)$ defined as follows: for any $\rho \subseteq A \times A,\ f(\rho) = \{X \subseteq A : X^2 \subseteq \rho\}$ and for each $\mathcal{S} \subseteq P(A),\ g(\mathcal{S}) = \bigcup\{X^2 : X \in \mathcal{S}\}$.

Lemma 14. *The mappings f, g form a residuated pair for the complete lattices $(P(A \times A), \subseteq),\ (P(P(A)), \subseteq)$.*

Proof. It is obvious that both mappings are monotone. Moreover, for each $\rho \subseteq A \times A,\ g(f(\rho)) = \bigcup\{X^2 : X \subseteq A \ \&\ X^2 \subseteq \rho\} \subseteq \rho$. On the other hand, for any $\mathcal{S} \subseteq P(A)$ we have $\mathcal{S} \subseteq \{X \subseteq A : X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}\} = f(g(\mathcal{S}))$. ■

Now, the interior operation $I : P(A\times A) \longrightarrow P(A\times A)$ determined by the residuated pair (f, g) is of the form: for any binary relation ρ defined on A, for each $a, b \in A$, $(a, b) \in I(\rho)$ iff $(a, b) \in g(f(\rho))$ iff $\exists X \subseteq A\ (X^2 \subseteq \rho \,\&\, a, b \in X)$. The closure operation $C : P(P(A)) \longrightarrow P(P(A))$ is of the form: for any $\mathcal{S} \subseteq P(A)$, $C(\mathcal{S}) = f(g(\mathcal{S})) = \{X \subseteq A : X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}\}$.

Let us call a *semitolerance* any binary relation on A which is semireflexive and symmetric. The name takes its origin from the tolerance relations introduced in [10] as reflexive and symmetric relations, and considered in many papers.

Notice that for any binary relation ρ on A, the relation $I(\rho)$ is a semitolerance. Moreover, given any semitolerance ρ on A, one may easily show that $\rho \subseteq I(\rho)$. In this way the following lemma holds:

LEMMA 15. *For any binary relation ρ defined on A, ρ is an open element, that is $I(\rho) = \rho$, iff ρ is a semitolerance relation.*

In order to characterize the closed elements in the lattice $(P(P(A)), \subseteq)$, first let us formulate the following obvious fact.

FACT 16. *Given any $\mathcal{S} \subseteq P(A)$ and $X \subseteq A$ the following conditions are equivalent:*

(i) $X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$,

(ii) $\forall Z \in P_2(X)\ \exists Y \in \mathcal{S},\ Z \subseteq Y$,

where $P_2(X) = \{\{a, b\} : a, b \in X\}$.

LEMMA 17. *For any $\mathcal{S} \subseteq P(A)$, $\mathcal{S}$ is closed, that is $C(\mathcal{S}) = \mathcal{S}$, iff the following two conditions hold:*

(a) *for any* $X \subseteq A\ (P_2(X) \subseteq \mathcal{S} \ \Rightarrow\ X \in \mathcal{S})$,

(b) *for any* $X, Y \subseteq A\ (X \subseteq Y \,\&\, Y \in \mathcal{S} \ \Rightarrow\ X \in \mathcal{S})$.

PROOF. Notice that according to the definition of the closure operation C, the condition that $\mathcal{S}$ is closed is equivalent to the following one: for each $X \subseteq A\ (X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\} \ \Rightarrow\ X \in \mathcal{S})$.

($\Rightarrow$): Assume that $\mathcal{S}$ is closed. In order to show (a) let $\forall Z \in P_2(X), Z \in \mathcal{S}$, where $X \subseteq A$. Then $\forall Z \in P_2(X) \exists Y \in \mathcal{S},\ Z \subseteq Y$. So $X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$ due to Fact 16 and consequently, $X \in \mathcal{S}$ by the assumption. In order to show (b) suppose that $X \subseteq Y$ and $Y \in \mathcal{S}$. Then obviously $X^2 \subseteq Y^2$, therefore $X^2 \subseteq \bigcup\{U^2 : U \in \mathcal{S}\}$, thus $X \in \mathcal{S}$ due to the assumption.

($\Leftarrow$): Assume that (a) and (b) hold and suppose that $X^2 \subseteq \bigcup\{Y^2 : Y \in \mathcal{S}\}$. Having (a), in order to complete the proof, it is enough to show that $P_2(X) \subseteq \mathcal{S}$. So let $Z \in P_2(X)$. Then from Fact 16 and the third assumption it follows that for some $Y \in \mathcal{S}$, $Z \subseteq Y$, which due to (b) leads to the result $Z \in \mathcal{S}$. ■

COROLLARY 18. *Let A be any set. Then*

- (i) *the poset $(ST(A), \subseteq)$ of all the semitolerances defined on the set A is isomorphic to the poset $(\Sigma(A), \subseteq)$ composed of all the families $\mathcal{S} \subseteq P(A)$ fulfilling both the conditions (a) and (b) from Lemma 17. The mapping $f : ST(A) \longrightarrow \Sigma(A)$ such that $f(\rho) = \{X \subseteq A : X^2 \subseteq \rho\}$, is the desired isomorphism. The inverse isomorphism $g : \Sigma(A) \longrightarrow ST(A)$, is defined by $g(\mathcal{S}) = \bigcup\{X^2 : X \in \mathcal{S}\}$.*
- (ii) $ST(A) = \{\bigcup\{X^2 : X \in \mathcal{S}\} : \mathcal{S} \subseteq P(A)\}$,
- (iii) $\Sigma(A) = \{\{X \subseteq A : X^2 \subseteq \rho\} : \rho \subseteq A \times A\}$.

PROOF. This is an immediate corollary to Lemmas 14, 15 and 17. (Note that we do not distinguish in symbols the mappings f, g from their appropriate restrictions.) ■

One can check that the poset $(ST(A), \subseteq)$ of all the semitolerances defined on A is a complete lattice such that given any $\Theta \subseteq ST(A)$ we have $sup\Theta = \bigcup \Theta$ and when $\Theta \neq \emptyset$, $inf\Theta = \bigcap \Theta$, and $inf\emptyset = A \times A$. So, in fact, f is an isomorphism of the complete lattices $(ST(A), \subseteq)$, $(\Sigma(A), \subseteq)$. In this way, a candidate to be a counterpart of partition corresponding to a given semitolerance ρ defined on A is just the family $f(\rho) = \{X \subseteq A : X^2 \subseteq \rho\}$. In order to see the difference between an ordinary partition and its counterpart, let us consider the latter when it corresponds to an equivalence relation:

PROPOSITION 19. *For any equivalence relation θ on A we have $f(\theta) = \bigcup\{P([a]_\theta) : a \in A\}$.*

PROOF. Let θ be an equivalence relation on A.

($\subseteq$) : Let $X \in f(\theta)$, i.e., $X^2 \subseteq \theta$. In case $X = \emptyset$, obviously $X \in \bigcup\{P([a]_\theta) : a \in A\}$. If $X \neq \emptyset$, then, naturally, $X \subseteq [a]_\theta$, for any $a \in X$. So we get a conclusion.

($\supseteq$) : Suppose that $X \subseteq [a]_\theta$ for some $a \in A$. Let $x, y \in X$. Then $x, y \in [a]_\theta$, so $(x, y) \in \theta$. This means that $X^2 \subseteq \theta$, that is $X \in f(\theta)$. ■

For example, $f(id_A) = \{\{a\} : a \in A\} \cup \{\emptyset\}$, where id_A is the identity relation of A, and $f(A \times A) = P(A)$.

Evidently, the ordinary partition related with an equivalence relation θ is the family of all the maximal elements of $f(\theta)$. We are going to generalize this result for all the semitolerances, introducing the second counterpart of partition which coincides with the ordinary partition in case of the equivalence relations. As is well known (cf. e.g. [5,9]), in case of a tolerance relation θ defined on a given lattice, such maximal elements of the family $f(\theta)$ play the role of the blocks of a "partition" (they are elements of a quotient lattice modulo a tolerance).

Now let us use the first tool. In place of the poset $(A, \leq)$ from Proposition 13 let us consider a poset $(P(A), \subseteq)$, where A is any set. Then we have an isomorphism $\phi : (DWM(P(A)), \subseteq) \longrightarrow (Ach(P(A)), \sqsubseteq)$ such that $\phi(\mathcal{S}) = Max(\mathcal{S})$, and the inverse isomorphism $\psi : (Ach(P(A)), \sqsubseteq) \longrightarrow (DWM(P(A)), \subseteq)$ defined by $\psi(\mathcal{B}) = \bigcup\{P(X) : X \in \mathcal{B}\}$. $DWM(P(A))$ is the family of all the subsets $\mathcal{S} \subseteq P(A)$ fulfilling the conditions

(b) $\forall X, Y \subseteq A\ (X \subseteq Y\ \&\ Y \in \mathcal{S} \ \Rightarrow\ X \in \mathcal{S})$,

(c) $\forall X \in \mathcal{S}\ \exists Y \in Max(\mathcal{S}),\ X \subseteq Y$,

while $Ach(P(A))$ is the family of all antichains included in the poset $(P(A), \subseteq)$. Moreover, for any $\mathcal{B}_1, \mathcal{B}_2 \in Ach(P(A))$ we have $\mathcal{B}_1 \sqsubseteq \mathcal{B}_2$ iff $\forall X \in \mathcal{B}_1\ \exists Y \in \mathcal{B}_2,\ X \subseteq Y$.

It turns out that $\Sigma(A) \subseteq DWM(P(A))$:

Lemma 20. *Any closed family $\mathcal{S} \subseteq P(A)$ from the class $\Sigma(A)$ (satisfying the conditions (a), (b) from Lemma 17) also fulfils the condition (c) from Lemma 11, that is $\forall X \in \mathcal{S}\ \exists Y \in Max(\mathcal{S}),\ X \subseteq Y$.*

Proof. Let $\mathcal{S} \in \Sigma(A)$. Then Corollary 18(*iii*) yields that there exists a binary relation ρ defined on A such that $\mathcal{S} = \{Y \subseteq A : Y^2 \subseteq \rho\}$. Let $X \in \mathcal{S}$. Consider any nonempty chain $\mathcal{L}$ in the nonempty poset $(\{Y \subseteq A : X \subseteq Y\ \&\ Y^2 \subseteq \rho\}, \subseteq)$. Then it is straightforward that $X \subseteq \bigcup \mathcal{L}$ and $(\bigcup \mathcal{L})^2 \subseteq \rho$. So, by the Kuratowski-Zorn lemma it follows that there exists a maximal element in that poset. It is obviously a maximal element in $\{Y \subseteq A : Y^2 \subseteq \rho\}$, that is in $\mathcal{S}$. ■

Now, one can restrict the isomorphism $\phi : DWM(P(A)) \longrightarrow Ach(P(A))$ to the class $\Sigma(A)$ (one may show that $\Sigma(A) \neq DWM(P(A))$), obtaining the new isomorphism onto some partially ordered subset of the poset $(Ach(P(A)), \sqsubseteq)$. Taking into account Corollary 18(*i*), it is easily seen that the composition of that new isomorphism and the isomorphism f from the corollary, is the isomorphism of the complete lattice $(ST(A), \subseteq)$ of all the semitolerance relations on A onto that partially ordered subset. So the antichains from the subset form just the counterparts (of the second kind) of partitions corresponding to semitolerances. We will focus our attention on showing that this subset is composed of all the antichains $\mathcal{B}$ of $(P(A), \subseteq)$ which fulfil the following condition:

$$\text{for all } Z \subseteq A \ (P_2(Z) \subseteq \bigcup\{P_2(X) : X \in \mathcal{B}\} \ \Rightarrow \ \exists X \in \mathcal{B}, \ Z \subseteq X). \qquad (*)$$

It coincides with the condition (2) of the definition of τ-covering of a set in [2] whenever the expression "two-element subset" used in [2] also allows singleton sets to be included (cf. the next section 8). Let us provide some simple examples showing how this condition works. Suppose that $a, b, c, d, e, f, g, h, u, v, x, y$ are the pairwise different elements of a set A. The antichain $\{\{a, b\}, \{b, c\}, \{a, c\}\}$ does not fulfil (*), while the antichain $\{\{a, b, c\}\}$ does. One may show that any 1-element and 2-element antichains fulfil (*). This condition is not satisfied for the antichain $\{\{a, b, c, d\}, \{a, b, c, e\}, \{d, e\}\}$ since any pair which is a subset of the set $\{a, b, c, d, e\}$ is also a subset of some member of the antichain. However, the set $\{a, b, c, d, e\}$ is included in no member of the antichain. The following antichain: $\{\{a, b, c, d\}, \{c, x, e, f\}, \{d, x, g, h\}, \{x, y, u, v\}\}$, does not fulfil (*) since any pair which is a subset of $\{c, d, x\}$ is a subset of some member of the antichain. However, the set $\{c, d, x\}$ is not a subset of any member of the antichain.

LEMMA 21. *For any $\mathcal{S} \in \Sigma(A)$, $Max(\mathcal{S})$ is an antichain fulfilling the condition (*).*

PROOF. According to Lemma 20, any element $\mathcal{S}$ of $\Sigma(A)$ satisfies the conditions (b) and (c) of Lemma 11 for the poset $(P(A), \subseteq)$, so $\mathcal{S} = \bigcup\{P(X) : X \in Max(\mathcal{S})\}$, due to the lemma. In order to show that $Max(\mathcal{S})$ fulfils (*) for any such $\mathcal{S}$, suppose that $P_2(Z) \subseteq \bigcup\{P_2(X) : X \in Max(\mathcal{S})\}$, where $Z \subseteq A$. Since for each $X \subseteq A$, $P_2(X) \subseteq P(X)$ so $\bigcup\{P_2(X) : X \in Max(\mathcal{S})\} \subseteq \bigcup\{P(X) : X \in Max(\mathcal{S})\}$. Therefore from the assumption we obtain that $P_2(Z) \subseteq \mathcal{S}$. However, the condition (a) from Lemma 17 is satisfied for $\mathcal{S}$. Thus $Z \in \mathcal{S}$ which implies that $Z \subseteq X$ for some $X \in Max(\mathcal{S})$. ■

LEMMA 22. *For any antichain $\mathcal{B} \subseteq P(A)$ for which the condition (*) holds true, the family $\mathcal{S} = \bigcup\{P(X) : X \in \mathcal{B}\}$ fulfils the conditions $(a), (b)$ of Lemma 17, that is, $\mathcal{S}$ belongs to the class $\Sigma(A)$.*

Proof. Assume that $\mathcal{B}$ is an antichain in $(P(A), \subseteq)$ meeting (*). Obviously, $\mathcal{S} = \bigcup\{P(X) : X \in \mathcal{B}\} = \psi(\mathcal{B})$, where $\psi : Ach(P(A)) \longrightarrow DWM(P(A))$, is the inverse isomorphism from Proposition 13. It means that the condition (b) of Lemma 17 is satisfied for $\mathcal{S}$, since it is satisfied for any member of the class $DWM(P(A))$. So it is sufficient to show that $\mathcal{S}$ fulfils the condition (a), which is obvious: when $P_2(Z) \subseteq \mathcal{S}$ then $P_2(Z) \subseteq \bigcup\{P(X) : X \in \mathcal{B}\}$ so $P_2(Z) \subseteq \bigcup\{P_2(X) : X \in \mathcal{B}\}$. Hence, and from (*), it follows that $Z \subseteq X$ for some $X \in \mathcal{B}$, and, finally, $Z \in \mathcal{S}$, due to the form of $\mathcal{S}$. ∎

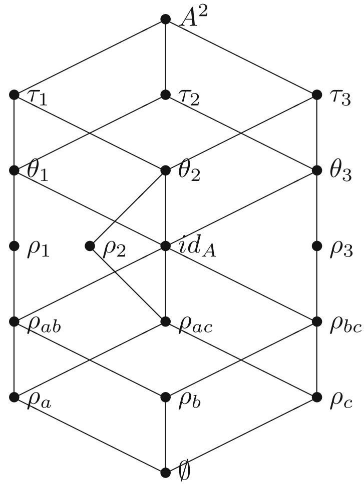
Summing up one may formulate the expected result.

Corollary 23. *The complete lattice $(ST(A), \subseteq)$ of all the semitolerance relations defined on A is isomorphic to the lattice $(Ach^*(P(A)), \sqsubseteq)$ of all the antichains in the poset $(P(A), \subseteq)$ fulfilling the condition (*). Here for any $\mathcal{B}_1, \mathcal{B}_2 \in Ach^*(P(A))$: $\mathcal{B}_1 \sqsubseteq \mathcal{B}_2$ iff $\forall X \in \mathcal{B}_1\ \exists Y \in \mathcal{B}_2,\ X \subseteq Y$. The mapping $F : ST(A) \longrightarrow Ach^*(P(A))$ of the form: $F(\rho) = Max(\{X \subseteq A : X^2 \subseteq \rho\})$ is this isomorphism. The inverse isomorphism, $G : Ach^*(P(A)) \longrightarrow ST(A)$, is defined by $G(\mathcal{B}) = \bigcup\{X^2 : X \in \mathcal{B}\}$.*

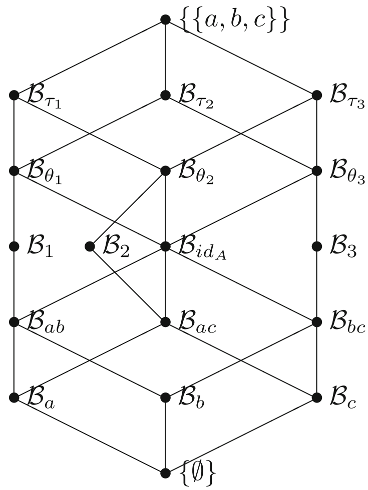
Proof. According to Corollary 18(i) the mappings $f : ST(A) \longrightarrow \Sigma(A)$ of the form: $f(\rho) = \{X \subseteq A : X^2 \subseteq \rho\}$, and $g : \Sigma(A) \longrightarrow ST(A)$ defined as follows: $g(\mathcal{S}) = \bigcup\{X^2 : X \in \mathcal{S}\}$, are the isomorphisms of the complete lattices $(ST(A), \subseteq)$ and $(\Sigma(A), \subseteq)$. Furthermore, from Proposition 13 applied for the poset $(P(A), \subseteq)$ and Lemmas 20, 21, 22, the mappings $\phi : \Sigma(A) \longrightarrow Ach^*(P(A))$ defined by $\phi(\mathcal{S}) = Max(\mathcal{S})$, and $\psi : Ach^*(P(A)) \longrightarrow \Sigma(A)$ of the form: $\psi(\mathcal{B}) = \bigcup\{P(X) : X \in \mathcal{B}\}$, are the isomorphisms of the complete lattices $(\Sigma(A), \subseteq)$, $(Ach^*(P(A)), \sqsubseteq)$. So the compositions: $F : ST(A) \longrightarrow Ach^*(P(A))$ defined as: $F(\rho) = \phi(f(\rho)) = \phi(\{X \subseteq A : X^2 \subseteq \rho\}) = Max(\{X \subseteq A : X^2 \subseteq \rho\})$, and $G : Ach^*(P(A)) \longrightarrow ST(A)$, of the form: $G(\mathcal{B}) = g(\psi(\mathcal{B})) = g(\bigcup\{P(X) : X \in \mathcal{B}\}) = \bigcup\{Y^2 : \exists X \in \mathcal{B},\ Y \subseteq X\} = \bigcup\{X^2 : X \in \mathcal{B}\}$, are the isomorphisms of the complete lattices $(ST(A), \subseteq)$ and $(Ach^*(P(A)), \sqsubseteq)$. ∎

In order to illustrate the corollary let us consider the example of the lattice of all semitolerances defined on 3-element set $A = \{a, b, c\}$ and the lattice of all antichains $\mathcal{B}$ in the poset $(P(\{a, b, c\}), \subseteq)$ satisfying the condition: $\forall Z \subseteq \{a, b, c\}\ (P_2(Z) \subseteq \bigcup\{P_2(X) : X \in \mathcal{B}\} \Rightarrow \exists X \in \mathcal{B},\ Z \subseteq X)$:

$\rho_a = \{(a,a)\}$
$\rho_b = \{(b,b)\}$
$\rho_c = \{(c,c)\}$
$\rho_{ab} = \{(a,a),(b,b)\}$
$\rho_{ac} = \{(a,a),(c,c)\}$
$\rho_{bc} = \{(b,b),(c,c)\}$
$id_A = \{(a,a),(b,b),(c,c)\}$
$\rho_1 = \{(a,b),(b,a),(a,a),(b,b)\}$
$\rho_2 = \{(a,c),(c,a),(a,a),(c,c)\}$
$\rho_3 = \{(b,c),(c,b),(b,b),(c,c)\}$
$\theta_1 = \{(a,b),(b,a)\} \cup id_A$
$\theta_2 = \{(a,c),(c,a)\} \cup id_A$
$\theta_3 = \{(b,c),(c,b)\} \cup id_A$
$\tau_1 = \{(a,b),(b,a),(a,c),(c,a)\} \cup id_A$
$\tau_2 = \{(a,b),(b,a),(b,c),(c,b)\} \cup id_A$
$\tau_3 = \{(a,c),(c,a),(b,c),(c,b)\} \cup id_A$



$\mathcal{B}_a = \{\{a\}\}$
$\mathcal{B}_b = \{\{b\}\}$
$\mathcal{B}_c = \{\{c\}\}$
$\mathcal{B}_{ab} = \{\{a\},\{b\}\}$
$\mathcal{B}_{ac} = \{\{a\},\{c\}\}$
$\mathcal{B}_{bc} = \{\{b\},\{c\}\}$
$\mathcal{B}_{id_A} = \{\{a\},\{b\},\{c\}\}$
$\mathcal{B}_1 = \{\{a,b\}\}$
$\mathcal{B}_2 = \{\{a,c\}\}$
$\mathcal{B}_3 = \{\{b,c\}\}$
$\mathcal{B}_{\theta_1} = \{\{a,b\},\{c\}\}$
$\mathcal{B}_{\theta_2} = \{\{a,c\},\{b\}\}$
$\mathcal{B}_{\theta_3} = \{\{b,c\},\{a\}\}$
$\mathcal{B}_{\tau_1} = \{\{a,b\},\{a,c\}\}$
$\mathcal{B}_{\tau_2} = \{\{a,b\},\{b,c\}\}$
$\mathcal{B}_{\tau_3} = \{\{a,c\},\{b,c\}\}$



It is not difficult to notice that Corollary 23 is a generalization of the standard theorem on the isomorphisms of the complete lattice $(E(A), \subseteq)$ of all the equivalence relations on A and the lattice $(Part(A), \leq)$ of all the partitions of the set A. That is, the standard isomorphism from

$(E(A), \subseteq)$ onto $(Part(A), \leq)$ and its inverse are extended to the isomorphism F from $(ST(A), \subseteq)$ onto $(Ach^*(P(A)), \sqsubseteq)$ and G from $(Ach^*(P(A)), \sqsubseteq)$ onto $(ST(A), \subseteq)$, respectively. Simply, taking into account Proposition 19 one can see that in case ρ is an equivalence relation on A we have (cf. the proof of the corollary) $F(\rho) = \phi(f(\rho)) = Max(\bigcup\{P([a]_\rho) : a \in A\}) = \{[a]_\rho : a \in A\} = A/\rho$. Conversely, when Π is an ordinary partition of A then obviously, $\Pi \in Ach^*(P(A))$, and $G(\Pi) = \bigcup\{X^2 : X \in \Pi\}$ which is the equivalence relation induced by the partition Π.

In order to establish a connection between a "partition" $F(\rho)$, given a semitolerance ρ on A, and an equivalential class $(a]_\rho = \{x \in A : x\rho a\}$ let us extend the mapping F to the whole class $P(A \times A)$ of binary relations on A : $F(\rho) = Max(\{X \subseteq A : X^2 \subseteq \rho\}$, and formulate some simple facts:

FACT 24. *Let $\rho \subseteq A \times A$ and $X \subseteq A$.*

(1) $X = \bigcap\{(a]_\rho : a \in X\}$ *implies that* $X \in F(\rho)$,

(2) $X \in F(\rho)$ *implies that* $X \subseteq \bigcap\{(a]_\rho : a \in X\}$,

(3) $\rho \in ST(A)$ *and* $\rho \neq \emptyset$ *imply that* $(X = \bigcap\{(a]_\rho : a \in X\}$ *iff* $X \in F(\rho))$.

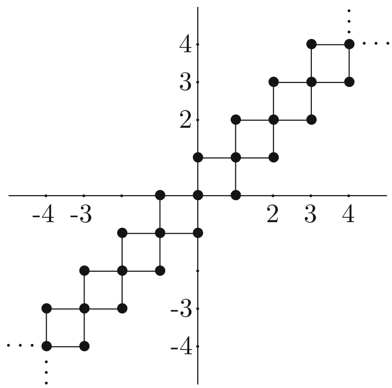
PROOF. For (1): Suppose that $X = \bigcap\{(a]_\rho : a \in X\}$. Then directly: $X^2 \subseteq \rho$. In order to show that X is a maximal set with the property: $X^2 \subseteq \rho$, assume, conversely, that this does not hold. Then there is a $Y \subseteq A$ such that $X \subseteq Y$, $X \neq Y$ and $Y^2 \subseteq \rho$. So $b \notin X$ for some $b \in Y$. Thus according to the assumption there exists an $a \in X$ such that $b\rho a$ does not hold. However, $a \in Y$ so $b\rho a$ is true; a contradiction.

For (2): When $X \subseteq A$ is such that $X^2 \subseteq \rho$, then obviously, $X \subseteq \bigcap\{(a]_\rho : a \in X\}$.

For (3): Assume that ρ is a nonempty semitolerance relation. Due to (1), (2) it suffices to show the implication: $X \in F(\rho)$ implies that $\bigcap\{(a]_\rho : a \in X\} \subseteq X$. So suppose that $X \in F(\rho)$ and conversely that there is an element $b \in \bigcap\{(a]_\rho : a \in X\}$ such that $b \notin X$. Then for each $a \in X$, $b\rho a$, and from the symmetry of ρ : for each $a \in X$, $a\rho b$. Moreover, since $\rho \neq \emptyset$ so $\emptyset \notin F(\rho)$ which together with assumption imply that $X \neq \emptyset$. Therefore $b\rho b$, due to the condition (*sr*) of semireflexivity. From the very assumption it follows that $X^2 \subseteq \rho$. All this means that $(X \cup \{b\})^2 \subseteq \rho$ and consequently, $X \notin F(\rho)$. ∎

8. Tolerance Relations and Corresponding "Partitions"

First let us consider some particular simple examples of tolerance relations to illustrate Fact 24. Let $\rho = \{(a,b),(b,a)(a,c),(c,a),(a,a),(b,b),(c,c)\}$. It is a tolerance on $A = \{a,b,c\}$. Then $F(\rho) = \{\{a,b\},\{a,c\}\}$ and $\{a,b\} = (a]_\rho \cap (b]_\rho$, $\{a,c\} = (a]_\rho \cap (c]_\rho$, where $(a]_\rho = \{a,b,c\}$, $(b]_\rho = \{a,b\}$, $(c]_\rho = \{a,c\}$. Notice that here any block from $F(\rho)$ is an "equivalence" class. We now show another example in which no block is an "equivalence" class. Let A be the set of all integers and $\mathcal{B} = \{\{x,x+1\} : x \in A\}$ be an antichain in $(P(A),\subseteq)$. Then obviously, $\mathcal{B} \in Ach^*(P(A))$. The relation ρ induced on A by the "partition" $\mathcal{B}$ is a tolerance relation of the form: $\rho = G(\mathcal{B}) = \bigcup\{X^2 : X \in \mathcal{B}\} = \bigcup\{\{(x,x+1),(x+1,x),(x,x),(x+1,x+1)\} : x \in A\}$. Here for any $x \in A :\ (x]_\rho = \{x-1,x,x+1\}$ and for each $x \in A$ a block from $\mathcal{B}$ $(= F(\rho))$ is of the form: $\{x,x+1\} = (x]_\rho \cap (x+1]_\rho$ (cf. the figure below).



Obviously, when ρ is an equivalence relation, Fact 24(3) implies that any equivalence class $(a]_\rho$ is an element of $F(\rho)$ (for $(a]_\rho = \bigcap\{(x]_\rho : x \in (a]_\rho\}$), and conversely, for any $X \in F(\rho)$ we have $X \neq \emptyset$ and for each $a \in X$, $X = (a]_\rho$ (the conditions $X = \bigcap\{(a]_\rho : a \in X\}$ and $X \nsubseteq (a]_\rho$ or $(a]_\rho \nsubseteq X$, for some $a \in X$ lead to contradiction).

Now let $\overline{\Sigma(A)}$ be the family of all the subsets $\mathcal{S}$ of $P(A)$ for which the following three conditions hold:

(a) for any $X \subseteq A$ $(P_2(X) \subseteq \mathcal{S} \ \Rightarrow\ X \in \mathcal{S})$,

(b) for any $X, Y \subseteq A$ $(X \subseteq Y \ \&\ Y \in \mathcal{S} \ \Rightarrow\ X \in \mathcal{S})$,

(d) $\bigcup \mathcal{S} = A$,

Obviously, $\overline{\Sigma(A)} \subseteq \Sigma(A)$. One may show that the value $g(\mathcal{S})$ of the isomorphism g from Corollary 18(i) on the family $\mathcal{S} \in \overline{\Sigma(A)}$ is just a tolerance relation. Conversely, given any $\rho \in T(A)$, where $T(A)$ is the class of all the tolerances defined on A, we have $f(\rho) \in \overline{\Sigma(A)}$, where f is the isomorphism from that corollary. So remembering that $T(A)$ forms a complete lattice $(T(A), \subseteq)$ such that for each nonempty $\Theta \subseteq T(A), sup\Theta = \bigcup\Theta$ and $inf\Theta = \bigcap\Theta,\ sup\emptyset = id_A,\ inf\emptyset = A^2$, one obtains the following

Corollary 25. *A mapping $f : T(A) \longrightarrow \overline{\Sigma(A)}$ of the form: $f(\rho) = \{X \subseteq A : X^2 \subseteq \rho\}$, is an isomorphism of the complete lattices $(T(A), \subseteq)$, $(\overline{\Sigma(A)}, \subseteq)$. The inverse isomorphism, $g : \overline{\Sigma(A)} \longrightarrow T(A)$, is defined by $g(\mathcal{S}) = \bigcup\{X^2 : X \in \mathcal{S}\}$.*

Similarly, one obtains a counterpart of Corollary 23 for the tolerance relations. Namely, consider a subset $\overline{Ach^*(P(A))}$ of $Ach^*(P(A))$ consisting of all the antichains $\mathcal{B}$ included in $(P(A), \subseteq)$ which not only fulfil the condition (*) but also the condition:

(d) $\bigcup\mathcal{B} = A$.

Then $G(\mathcal{B}) \in T(A)$, where $G : (Ach^*(P(A)), \sqsubseteq) \longrightarrow (ST(A), \subseteq)$ is the isomorphism from Corollary 23. Conversely, given a tolerance $\rho \subseteq A^2$, the antichain $F(\rho)$ fulfils the condition (d), where $F : (ST(A), \subseteq) \longrightarrow (Ach^*(P(A)), \sqsubseteq)$, is the isomorphism from that corollary. Summing up let us formulate the following corollary.

Corollary 26. *A mapping $F : T(A) \longrightarrow \overline{Ach^*(P(A))}$ of the form: $F(\rho) = Max(\{X \subseteq A : X^2 \subseteq \rho\})$ is an isomorphism of the complete lattices $(T(A), \subseteq)$, $(\overline{Ach^*(P(A))}, \sqsubseteq)$. The inverse isomorphism, $G : (\overline{Ach^*(P(A))}, \sqsubseteq) \longrightarrow (T(A), \subseteq)$, is defined by $G(\mathcal{B}) = \bigcup\{X^2 : X \in \mathcal{B}\}$.*

In [2] a 1-1 and onto correspondence between $T(A)$ and the class of all so-called τ-coverings of A, was established. In our notation, it is just the correspondence G from Corollary 26: a tolerance ρ corresponds to a τ-covering $\mathcal{B}$ of A iff $\rho = G(\mathcal{B})$. Indeed, it is evident that the set $\overline{Ach^*(P(A))}$ and the family of all τ-coverings of A, coincide. In our notation, a *τ-covering of a nonempty set* A is a family $\mathcal{B}$ of subsets of A such that the conditions (*) and (d) are satisfied, as well as the condition

(1) $\forall\emptyset \neq \mathcal{B}' \subseteq \mathcal{B}\ \forall X \in \mathcal{B}\ (X \subseteq \bigcup\mathcal{B}' \Rightarrow \bigcap\mathcal{B}' \subseteq X)$.

(Instead of (*) the following clause occurs in [2]:

(2) if $N \subseteq A$ and N is not contained in any set from $\mathcal{B}$, then N contains a two-element subset of the same property.

However, when (d) is satisfied as well, the conditions (*) and (2) are equivalent.)

Then each τ-covering of A is a member of $\overline{Ach^*(P(A))}$, since (as the authors of [2] noticed) (1) implies that $\mathcal{B}$ is an antichain in the poset $(P(A), \subseteq)$ - put a 1-element family $\mathcal{B}'$ in (1). On the other hand, every antichain $\mathcal{B} \subseteq P(A)$ fulfilling the condition (*) (so $\mathcal{B} \in Ach^*(P(A))$) satisfies the condition (1), too. In order to show this, assume that $\mathcal{B} \in Ach^*(P(A))$ and that (1) does not hold. Therefore there are a nonempty $\mathcal{B}' \subseteq \mathcal{B}$ and an $X \in \mathcal{B}$ such that (i) $X \subseteq \bigcup \mathcal{B}'$ and (ii) $\bigcap \mathcal{B}' \nsubseteq X$. Consider $Z = X \cup \bigcap \mathcal{B}'$. Then for any $a, b \in A$ such that $\{a, b\} \in P_2(Z)$ there is a $Y \in \mathcal{B}$ such that $\{a, b\} \subseteq Y$. Having $a \in \bigcap \mathcal{B}'$ and $b \in X$, it follows from (i) that $b \in Y$ for some $Y \in \mathcal{B}'$, so $a \in Y$, as well. Applying (*) to the set Z we have that $X \cup \bigcap \mathcal{B}' \subseteq U$ for some $U \in \mathcal{B}$ which implies that $X \subseteq U$, therefore $X = U$ (X, U are the elements of antichain), and consequently, $\bigcap \mathcal{B}' \subseteq X$, a contradiction with (ii).

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Marek Nowak
Department of Logic
University of Lodz
Kopcińskiego 16/18
90-232 Lodz, Poland
marnowak@filozof.uni.lodz.pl