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Relation Formulas for Protoalgebraic Equality Free Quasivarieties; Pałasińska's Theorem Revisited

Abstract. We provide a new proof of the following Pałasińska's theorem: Every finitely generated protoalgebraic relation distributive equality free quasivariety is finitely axiomatizable. The main tool we use are $\mathcal{Q}$-relation formulas for a protoalgebraic equality free quasivariety $\mathcal{Q}$. They are the counterparts of the congruence formulas used for describing the generation of congruences in algebras. Having this tool in hand, we prove a finite axiomatization theorem for $\mathcal{Q}$ when it has definable principal $\mathcal{Q}$-subrelations. This is a property obtained by carrying over the definability of principal subcongruences, invented by Baker and Wang for varieties, and which holds for finitely generated protoalgebraic relation distributive equality free quasivarieties.

Keywords: Equality free quasivariety, Protoalgebraicity, Relation distributivity, Finite axiomatization, Relation formulas, Definable principal subrelations.

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1. Introduction

In abstract algebraic logic the following theorem of Katarzyna Pałasińska is remarkable [15]: Every protoalgebraic and filter distributive multidimensional deductive system determined by a finite set of finite matrices can be presented by finitely many inference rules and axioms. By reformulating it into the context of equality free quasivarieties we have (see Section 2 for definitions).

Pałasińska's Theorem 1.1 ([22, 23]). *Every finitely generated protoalgebraic relation distributive equality free quasivariety is finitely axiomatizable.*

The aim of this paper is to provide a new proof of this theorem. For this purpose we apply the technique of definable principal $\mathcal{Q}$-subrelations. This is equality free quasivariety counterpart of the definable principal subcongruences technique invented by Kirby Baker and Ju Wang [2]. They used it

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for providing a very elegant and short proof of the celebrated Baker's theorem: Every finitely generated congruence distributive variety of algebras is finitely axiomatizable [1]. This technique was also successfully applied by authors of this article for quasivarieties. In [21] we obtained a short proof of Pigozzi's theorem: Every finitely generated relative congruence distributive quasivariety of algebras is finitely axiomatizable [24]. Here we go one step further. To this end we first need to fill a gap in the theory of equality free quasivarieties: the lack of a counterpart of the notion of congruence formula. We do it by introducing the notion of a $\mathcal{Q}$-relation formula without equality for a protoalgebraic equality free quasivariety $\mathcal{Q}$. Let us add that this notion is more subtle than that of a congruence formula. Indeed, it works properly only under additional assumptions summarized in Better Universe Theorem 5.2. Here the key property is the definability of Leibniz equalities in $\mathcal{Q}$ by a positive formula.

Let us write few words about Pałasińska's theorem from the perspective of deductive systems. Note that deductive systems correspond to equality free quasivarieties in a language with one relation symbol which is unary [5]. Models in such a language are called *matrices* and their relations *filters*. In this context the assumptions of Pałasińska's theorem are very natural. Namely, filter distributivity may be guaranteed by the existence of disjunction [13] or by the satisfaction of deduction theorem [9, Corrolary 2.6]. In fact, the latter yields also protoalgebraicity: a generalized form of deduction theorem is equivalent to protoalgebraicity [11].

Substantial effort was made to prove finite axiomatization results for deductive systems before Pałasińska obtained her theorem. Willem Blok and Don Pigozzi proved that if a deductive system $\mathcal{S}$ with finitely many nonaxiomatic inference rules Λ is protoalgebraic, filter distributive, and the class of its finitely irreducible matrices is finitely axiomatizable then $\mathcal{S}$ can be presented by Λ and finitely many axioms [3, Theorem 4.1]. The last condition of this theorem holds when $\mathcal{S}$ is determined by a finite family of finite matrices, i.e., when the equality free quasivariety corresponding to $\mathcal{S}$ is finitely generated. Janusz Czelakowski proved that if the assumptions of protoalgebraicity and of finiteness of Λ are dropped, then $\mathcal{S}$ may be presented by finitely many axioms together with, possibly, infinitely many nonaxiomatic inference rules [8, Theorem 5.1]. Moreover, he showed that if $\mathcal{S}$ posses a disjunction, which implies filter distributivity, and the class of its finitely irreducible matrices is finitely axiomatizable, then $\mathcal{S}$ can be presented by finitely many axioms and nonaxiomatic inference rules [7, Theorem 3.2]. This means that the corresponding equality free quasivariety of matrices is finitely axiomatizable [7, Theorem 3.2].

Pałasińska's theorem yields a result for ordinary quasivarieties in a language containing operation and relation symbols. Let $\mathcal{Q} = \mathrm{Mod}(\Sigma)$ be such a quasivariety. Extend the language of $\mathcal{Q}$ by one binary relation symbol $x \sim y$, and consider the class $\widetilde{\mathcal{Q}} = \mathrm{Mod}(\widetilde{\Sigma})$ with axiomatization $\widetilde{\Sigma}$ obtained from Σ by replacing in it all occurrences of $t \approx s$ by $t \sim s$, where t and s are arbitrary terms, and by adding axioms guaranteeing that the interpretations of $x \sim y$ are strict congruences. Clearly, $\widetilde{\mathcal{Q}}$ is protoalgebraic (even finitely equivalential) equality free quasivariety and $\mathcal{Q}$ is finitely axiomatizable, or finitely generated, iff $\widetilde{\mathcal{Q}}$ is. Moreover, the relation distributivity of $\widetilde{\mathcal{Q}}$ translates to the relative congruence distributivity of $\mathcal{Q}$ in the sense of [18]. Hence the restriction to algebras in Pigozzi's theorem is not necessary.

There is a common opinion that techniques and ideas from general algebra (or rather from quasivariety theory) may carry over to abstract algebraic logic when deductive systems under consideration are protoalgebraic. However, from the perspective of this paper, we see that what is really necessary is the definability of Leibniz equalities by a positive formula (Theorem 5.2). Definability by a set of positive formulas (i.e., protoalgebraicity cf. Theorem 2.4) is not enough as first-order logic would be left and compactness theorem is lost. Note that in the construction described in the preceding paragraph we got the desired definability for free. In fact, in this case, the proof of Pałasińska's theorem may be simplified a bit. Indeed, Section 5 is then irrelevant, and some definitions may be simplified (Remarks 3.4 and 6.3).

The paper is organized as follows. In Section 2 we gather the needed information about equality free quasivarieties. In Sections 3, 4, 5 and 6 we develop a general theory needed later: Section 3 is devoted to finitely generated and locally finite equality free quasivarieties. In Section 4 we formulate and prove the analogue of Jónsson's Lemma. When we divide the set of equality free quasi-identities axiomatizing an investigated equality free quasivariety into the set of equality free identities and the rest, then Jónsson's Lemma shows how to reduce the first set to a finite one. In Section 6 we define $\mathcal{Q}$-relation formulas without equality, where $\mathcal{Q}$ is an equality free quasivariety. In Section 5 we describe conditions for $\mathcal{Q}$ under which this definition makes sense. Sections 7, 8 and 9 are devoted to the proof of Pałasińska's theorem: In Section 7 we say what it means that an equality free quasivariety $\mathcal{Q}$ has definable principal $\mathcal{Q}$-subrelations. In Section 8 we prove a finite axiomatization theorem for such equality free quasivarieties. In Section 9 it is showed that a finitely generated protoalgebraic relation distributive equality free quasivariety $\mathcal{Q}$ has definable principal $\mathcal{Q}$-subrelations, thus Pałasińska's theorem is obtained. Here a brilliant argument due to

Baker and Wang is used. Finally Appendix contains information about how to obtain the results when we have more than one relation symbols in the language. Most of the paper is written under the restriction that there is just one such symbol. We do so in order not to obscure the reasoning. It should bring the reader's attention to relevant aspects of the theory, not to notational technicalities.

To finish the introduction let us add that the novelty of this paper lies mainly in introducing the proper notion of $\mathcal{Q}$-relation formula for protoalgebraic equality free quasivarieties. With this tool in hand the results are obtained by translating the arguments from [2] and [21].

2. Toolbox

Here we collect the facts that we need in the paper. We also fix terminology and notation. The reader may consult the introductory paper [4]. It is focused on models with just one relation and is written from the perspective of deductive systems. However it is not difficult to generalize and translate the results obtained there to our setting of equality free quasivarieties. Moreover, results from [12, 14] are particularly important for us. Furthermore, there are books about abstract algebraic logic that may serve here [10, 16, 26]. Finally, basic knowledge about quasivarieties [18, 20] (axiomatization, freeness, generation, subdirect irreducibility) may help reading.

We fix a default first-order language $\mathcal{L}$. We assume that $\mathcal{L}$ is finite, i.e., it contains only finitely many operation and relation symbols. We also assume that $\mathcal{L}$ does not contain equality symbol $\approx$. By an *equality free formula* or a *formula without equality* we mean a first-order formula in $\mathcal{L}$. Sometimes we write that a model $\mathbf{M}$ in $\mathcal{L}$ satisfies a formula in $\mathcal{L} \cup \{\approx\}$. Then we consider $\mathbf{M}$ as a structure in $\mathcal{L} \cup \{\approx\}$ where $\approx$ is interpreted as the equality on the carrier of $\mathbf{M}$. A model $\mathbf{M} = (M, \mathcal{O}, \mathcal{R})$ in $\mathcal{L}$ will be written as $\mathbf{M} = (\mathbf{A}, \mathcal{R})$, where $\mathbf{A} = (M, \mathcal{O})$ is an algebra reduct of $\mathbf{M}$ and $\mathcal{R}$ are relations of $\mathbf{M}$. We do so because relations are more important than operations in our considerations. Notice that in abstract algebraic logic $\mathbf{M}$ is traditionally called a matrix, and $\mathcal{R}$ a filter, however we decided to stick with model theoretic terminology.

An *equality free quasivariety* is a class defined by *equality free quasi-identities*, i.e., by equality free sentences of the form

$$(\forall \bar{x})[\varphi_0(\bar{x}) \wedge \cdots \wedge \varphi_{n-1}(\bar{x}) \to \varphi(\bar{x})],$$

where n is a natural number and $\varphi_0, \ldots, \varphi_{n-1}, \varphi$ are atomic formulas. The name "quasi-identity" comes from [18, 20], however sentences of this kind are also called strict universal Horn sentences [10]. An *equality free variety* is a class defined by universally quantified equality free atomic formulas. Such formulas are also called *equality free identities*.

There is a characterization of equality free quasivarieties by the closure on some class operators. Beside commonly known P product, $\mathsf{P_U}$ ultraproduct, $\mathsf{P_{SD}}$ subdirect product and S submodel class operators we need to consider two more C contraction and E expansion class operators. A homomorphism $h\colon \mathbf{M} \to \mathbf{N}$ is *strict* provided $\mathbf{M} \models R(\bar{a})$ iff $\mathbf{N} \models R(h(\bar{a}))$ for every tuple $\bar{a} \in M$ and every relation symbol $R \in \mathcal{L}$. (Here and in many places we abuse the notation writing $\bar{a} \in M$ instead of $a_0, \dots, a_{k-1} \in M$.) Then $\mathbf{M} \in \mathsf{E}(\mathcal{C})$ if there is a surjective strict homomorphism $h\colon \mathbf{M} \to \mathbf{N}$ with $\mathbf{N} \in \mathcal{C}$, and $\mathbf{N} \in \mathsf{C}(\mathcal{C})$ if there is a surjective strict homomorphism $h\colon \mathbf{M} \to \mathbf{N}$ with $\mathbf{M} \in \mathcal{C}$. We say that $\mathbf{M}$ and $\mathbf{N}$ are *relatives* provided $\mathbf{N} \in \mathsf{EC}(\mathbf{M})$ (or equivalently $\mathbf{M} \in \mathsf{EC}(\mathbf{N})$). Note that if $\mathbf{M}$ and $\mathbf{N}$ are relatives, then they satisfy the same equality free sentences.

PROPOSITION 2.1 ([12, Theorem 9]). *A class $\mathcal{Q}$ is an equality free quasivariety if and only if it is closed under $\mathsf{E}, \mathsf{C}, \mathsf{S}, \mathsf{P}, \mathsf{P_U}$ class operators. The smallest equality free quasivariety containing a class $\mathcal{G}$, i.e., generated by $\mathcal{G}$, is given by $\mathsf{ECSPP_U}(\mathcal{G})$.*

A *strict congruence* of a model $\mathbf{M} = (\mathbf{A}, \mathcal{R})$ is a congruence α of the algebra $\mathbf{A}$ such that $\mathbf{M} \models R(\bar{a}) \leftrightarrow R(\bar{b})$ provided $\bar{a}\,\alpha\,\bar{b}$ for every relation symbol R and every pair of tuples $\bar{a}$, $\bar{b}$ of elements from M of the lengths equal the arity of R. We alert that this notion is different than the congruences introduced in [18]. The largest strict congruence of $\mathbf{M}$ is called *Leibniz equality* of $\mathbf{M}$ and is denoted by $\Omega(\mathbf{M})$. Note that $(a, b) \in \Omega(\mathbf{M})$ if we cannot distinguish a from b in the following sense: there is no equality free formula $\varphi(x, \bar{z})$ and a tuple $\bar{c} \in M$ such that $\mathbf{M} \models \varphi(a, \bar{c})$ and $\mathbf{M} \models \neg\varphi(b, \bar{c})$. (See e.g. [17] for the origin and philosophical aspects.) Thus Leibniz equalities are definable by a set of formulas. Note however, that the existence of the automorphism of $\mathbf{M}$ switching only two elements does not imply that these elements are indistinguishable in the above sense, and are Leibniz equality congruent, as the graph  shows. We say that a model $\mathbf{M}$ is *reduced* if $\Omega(\mathbf{M})$ is equal to the equality relation on M. We use the notation $\mathbf{M}^* = \mathbf{M}/\Omega(\mathbf{M})$, $a^* = a/\Omega(\mathbf{M})$ for $a \in M$, and $\mathcal{C}^* = \{\mathbf{M}^* \mid \mathbf{M} \in \mathcal{C}\}$. Models $\mathbf{M}$ and $\mathbf{N}$ are relatives iff the reduced models $\mathbf{M}^*$ and $\mathbf{N}^*$ are isomorphic.

On an algebra $\mathbf{A}$ we order all interpretations of relational part $\mathcal{L}_R$ of the default language $\mathcal{L}$ componentwise: $\mathcal{R} \subseteq_{\mathcal{L}} \mathcal{S}$ if for every $R \in \mathcal{L}_R$ the interpretation of R in $\mathcal{R}$ is contained in the interpretation of R in $\mathcal{S}$ (i.e., if the identity mapping is a homomorphism from $(\mathbf{A}, \mathcal{S})$ to $(\mathbf{A}, \mathcal{R})$). For an equality free quasivariety $\mathcal{Q}$ let $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{A})$ be the set all interpretations $\mathcal{R}$ of $\mathcal{L}_R$ such that $(\mathbf{A}, \mathcal{R}) \in \mathcal{Q}$. Note that $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{A})$ forms an algebraic lattice with componentwise intersections as meets. For a model $\mathbf{M} = (\mathbf{A}, \mathcal{R})$ define $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{M}) = \{\mathcal{S} \mid (\mathbf{A}, \mathcal{S}) \in \mathcal{Q} \text{ and } \mathcal{R} \subseteq_{\mathcal{L}} \mathcal{S}\}$. If all lattices $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$ are distributive, we say that $\mathcal{Q}$ is *relation distributive*.

A model $\mathbf{M} = (\mathbf{A}, \mathcal{R})$ in an equality free quasivariety $\mathcal{Q}$ is *completely irreducible relative to* $\mathcal{Q}$ iff $\mathcal{R}$ is completely meet irreducible in the lattice $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$. Let $\mathcal{Q}_{CI}$ stand for the class of all completely irreducible models relative to $\mathcal{Q}$. Because every lattice $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$ is algebraic, every model $\mathbf{M} \in \mathcal{Q}$ may be represented as $(\mathbf{A}, \bigcap \mathcal{R}_i)$, where all $(\mathbf{A}, \mathcal{R}_i) \in \mathcal{Q}_{CI}$. From this we obtain the following fact.

Lemma 2.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be equality free quasivarieties. If $\mathcal{P}_{CI} \subseteq \mathcal{Q}$, then $\mathcal{P} \subseteq \mathcal{Q}$.*

An equality free quasivariety $\mathcal{Q}$ is *protoalgebraic* if for every algebra $\mathbf{A}$ and $\mathcal{R}, \mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{A})$ the inclusion $\mathcal{R} \subseteq \mathcal{S}$ yields $\Omega(\mathbf{A}, \mathcal{R}) \subseteq \Omega(\mathbf{A}, \mathcal{S})$. In particular, the protoalgebraicity guarantees that complete irreducibility in equality free quasivarieties plays the same role as relative subdirect irreducibility in quasivarieties. The following fact may be treated as an exercise. Optionally, the reader may consult [4, Section 9].

Proposition 2.3. *Let $\mathcal{Q}$ be a protoalgebraic equality free quasivariety. Then*

(1) *If $\mathbf{M}$ and $\mathbf{N}$ are relatives, then $\mathbf{M} \in \mathcal{Q}_{CI}$ iff $\mathbf{N} \in \mathcal{Q}_{CI}$;*

(2) *$\mathcal{Q}^* = \mathsf{P}_{\mathsf{SD}}(\mathcal{Q}^*_{CI})$.*

Formulas of the form $(\forall \bar{z})\, \varphi(\bar{x}, \bar{z})$, where φ is atomic, are called *pseudo-atomic*. We will need the following nontrivial fact. We encourage the reader to check the brilliant proof.

Theorem 2.4 ([4, Theorem 13.5], [14, Theorem 7]). *An equality free quasivariety $\mathcal{Q}$ is protoalgebraic if and only if Leibniz equalities are definable in $\mathcal{Q}$ by a set of equality free pseudo-atomic formulas.*

Equality free quasivarieties with Leibniz equalities definable by a (finite) set of equality free atomic formulas are called *(finitely) equivalential* [6]. For instance, the equality free quasivariety $\tilde{\mathcal{Q}}$ constructed from an ordinary one $\mathcal{Q}$ in Introduction is finitely equivalential.

Finally, the free model in an equality free quasivariety $\mathcal{Q}$ over a set of variables X is constructed as $\mathbf{F}_{\mathcal{Q}}(X) = (\mathbf{T}(X), \bigcap \mathrm{Rel}_{\mathcal{Q}}(\mathbf{T}(X)))$, where $\mathbf{T}(X)$ is an algebra of terms over X in the algebraic part of the default language. Note that if $\mathcal{V}$ is the equality free variety generated by $\mathcal{Q}$, i.e., the class satisfying equality free identities true in $\mathcal{Q}$, then $\mathbf{F}_{\mathcal{Q}}(X) = \mathbf{F}_{\mathcal{V}}(X)$ for every X.

3. Finitely generated equality free quasivarieties

An equality free quasivariety $\mathcal{Q}$ is *locally finite* if all finitely generated submodels of every reduced model from $\mathcal{Q}$ are finite. Furthermore, $\mathcal{Q}$ is *finitely generated* if it is generated by a finite family of finite models.

LEMMA 3.1. *Let $\mathcal{Q}$ be an equality free quasivariety. If $\mathcal{Q}$ is finitely generated, then it is locally finite.*

PROOF. Assume that $\mathcal{G}$ is a finite family of finite models generating $\mathcal{Q}$. Let $\mathbf{M} = (\mathbf{A}, \mathcal{R})$ be a model from $\mathcal{Q}^*$. By Proposition 2.1, $\mathbf{M} \in (\mathsf{SP}(\mathcal{G}))^*$. Hence $\mathbf{A}$ belongs to the variety $\mathcal{V}$ generated by the algebra reducts of models from $\mathcal{G}$. By a standard argument in general algebra, $\mathcal{V}$ is locally finite, i.e., all finitely generated algebras in $\mathcal{V}$ are finite. Now let $\mathbf{N} = (\mathbf{B}, \mathcal{S}) \leqslant \mathbf{M}$ be finitely generated. Then $\mathbf{B} \leqslant \mathbf{A}$ is finitely generated. Thus $\mathbf{B}$ and $\mathbf{N}$ are finite. ■

LEMMA 3.2. *Let $\mathcal{Q}$ be a locally finite protoalgebraic equality free quasivariety. Then for every natural k there exists a natural m such that if $\mathbf{N} \leqslant \mathbf{M} \in \mathcal{Q}^*$ and $\mathbf{N}$ is k-generated, then $|N| \leqslant m$.*

PROOF. Let m be the cardinality of submodel $\mathbf{G}_k$ of $(\mathbf{F}_{\mathcal{Q}}(\mathbb{N}))^*$ generated by the set $\{0^*, 1^*, \ldots, (k-1)^*\}$. By local finiteness, m is finite.

Let $\mathbf{N} \leqslant \mathbf{M} \in \mathcal{Q}^*$ and assume that $\mathbf{N}$ is generated by $\{a_0, \ldots, a_{k-1}\}$. By the Löwenheim-Skolem Theorem there exists a countable elementary submodel $\mathbf{M}'$ of $\mathbf{M}$ containing $\mathbf{N}$. Because Leibniz equalities are definable by a set of formulas, $\mathbf{M}'$ is reduced. Let $h \colon \mathbf{F}_{\mathcal{Q}}(\mathbb{N}) \to \mathbf{M}' = (\mathbf{A}', \mathcal{R}')$ be a surjective homomorphism sending each $i < k$ onto a_i. Since $h^{-1}(\mathcal{R}')$ belongs to $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{F}_{\mathcal{Q}}(\mathbb{N}))$, the protoalgebraicity yields $\Omega(\mathbf{F}_{\mathcal{Q}}(\mathbb{N})) \subseteq \Omega(\mathbf{T}(X), h^{-1}(\mathcal{R}))$. Thus there is a surjective homomorphism $h^* \colon (\mathbf{F}_Q(\mathbb{N}))^* \to \mathbf{M}'$ sending each i^* onto a_i, where $i < k$. We have $h^*(G_k) = N$ and $|N| \leqslant m$. ■

LEMMA 3.3. *If $\mathcal{Q}$ is a finitely generated protoalgebraic equality free quasivariety, then there is a natural number which is an upper bound of the cardinalities of all models in $\mathcal{Q}^*_{CI}$.*

PROOF. Let $\mathcal{G}$ be a finite family of finite models that generates $\mathcal{Q}$. It is sufficient to show that $\mathcal{Q}^*_{CI} \subseteq \mathsf{S}(\mathcal{G})^*$. By Proposition 2.1 and Proposition 2.3

point (1), for every $\mathbf{M} \in \mathcal{Q}_{CI}$ there is $\mathbf{M}' = (\mathbf{A}', \mathcal{R}') \in \mathcal{Q}_{CI}$ a relative of $\mathbf{M}$ such that $\mathbf{M}' \in \mathsf{SP}(\mathcal{G})$. Thus $\mathbf{M}' \leqslant_{SD} \prod \mathbf{M}_i$ is a subdirect product of some models $\mathbf{M}_i \in \mathsf{S}(\mathcal{G})$. Then there are $\mathbf{M}_i' = (\mathbf{A}', \mathcal{R}_i') \in \mathsf{E}(\mathbf{M}_\mathsf{i})$ such that $\mathcal{R}' = \bigcap \mathcal{R}_i'$. Because $\mathbf{M}'$ is completely irreducible, $\mathbf{M}' = \mathbf{M}_j'$ for some j, and hence $\mathbf{M}$ and $\mathbf{M}_j$ are relatives. ■

Remark 3.4. In an equivalential equality free quasivariety submodels of reduced models are reduced. Hence for them being locally finite is equivalent to having all reduced finitely generated models finite.

4. Jónsson's Lemma

Jónsson's Lemma [19, Theorem 1.1] may be thought of as a road map for proving finite axiomatization theorems. Here we present its variant for equality free quasivarieties. Our proof is based on [25, Proof of Lemma 4.2].

Jónsson's Lemma 4.1. *Let $\mathcal{K}$ be an equality free quasivariety and $\mathcal{V}$ be an equality free variety. Assume that there are finitely axiomatizable classes $\mathcal{E}$ and $\mathcal{I}$ such that*

(1) $\mathcal{K} \cap \mathcal{V} \subseteq \mathcal{E}$;

(2) $\mathcal{K}_{CI} \cap \mathcal{V} \supseteq \mathcal{K} \cap \mathcal{V} \cap \mathcal{I}$;

(3) $\mathcal{K}_{CI} \cap \mathcal{E} \subseteq \mathcal{I}$.

If $\mathcal{K}_{CI} \cap \mathcal{V}$ is finitely axiomatizable, then $\mathcal{K} \cap \mathcal{V}$ is finitely axiomatizable relative to $\mathcal{K}$.

Note that the conditions (2) and (3) hold when $\mathcal{K}_{CI} \cap \mathcal{E} = \mathcal{I}$

Proof. By (1) and then by (2), we have

$$\mathcal{K} \cap \mathcal{V} \subseteq (\mathcal{E} - \mathcal{I}) \cup (\mathcal{K} \cap \mathcal{V} \cap \mathcal{I}) \subseteq (\mathcal{E} - \mathcal{I}) \cup (\mathcal{K}_{CI} \cap \mathcal{V}) =: \mathcal{C} \subseteq \mathcal{E}.$$

By assumption, $\mathcal{C}$ is finitely axiomatizable. Hence, by compactness theorem, there exists a finitely axiomatizable equality free variety $\mathcal{W} \supseteq \mathcal{V}$ such that $\mathcal{K} \cap \mathcal{W} \subseteq \mathcal{C}$. We will show that $\mathcal{K} \cap \mathcal{V} = \mathcal{K} \cap \mathcal{W}$ with the aid of Lemma 2.2. Let $\mathbf{M} \in (\mathcal{K} \cap \mathcal{W})_{CI}$. Because $\mathrm{Rel}_{\mathcal{K}}(\mathbf{M}) = \mathrm{Rel}_{\mathcal{K} \cap \mathcal{W}}(\mathbf{M})$, $\mathbf{M} \in \mathcal{K}_{CI}$. Hence by point (3)

$$\mathbf{M} \in \mathcal{K}_{CI} \cap \mathcal{W} \subseteq \mathcal{K}_{CI} \cap \mathcal{E} \subseteq \mathcal{I}.$$

This and $\mathbf{M} \in \mathcal{C}$ yield $\mathbf{M} \in \mathcal{K}_{CI} \cap \mathcal{V} \subseteq \mathcal{K} \cap \mathcal{V}$. ■

5. Better universe

LEMMA 5.1. *Let $\mathcal{Q}$ be a protoalgebraic equality free quasivariety and $\mathcal{C} \subseteq \mathcal{Q}^*$. If $\mathcal{C}$ is axiomatizable, then there exists an equality free positive formula $x \sim y$ such that*

(1) *$x \sim y$ defines Leibniz equalities in $\mathsf{E}(\mathcal{C})$;*

(2) *for every $\mathbf{M} \in \mathcal{Q}$, $\Omega(\mathbf{M})$ is contained in the interpretation of $x \sim y$ in $\mathbf{M}$.*

PROOF. Let $\mathcal{C} = \mathrm{Mod}(\Sigma)$ and $x \asymp y$ be a set of equality free pseudo-atomic formulas from Theorem 2.4 defining Leibniz equalities in $\mathcal{Q}$. Then

$$\Sigma,\ x \asymp y \models x \approx y.$$

By compactness

$$\Sigma,\ x \asymp_f y \models x \approx y$$

for some finite $x \asymp_f y \subseteq x \asymp y$. The formula $x \sim y = \bigwedge x \asymp_f y$, which is equality free and positive, defines equalities in $\mathcal{C}$. Thus it defines Leibniz equalities in $\mathsf{E}(\mathcal{C})$. ■

BETTER UNIVERSE THEOREM 5.2. *Let $\mathcal{Q}$ be a protoalgebraic equality free quasivariety with $\mathcal{Q}^*$ axiomatizable. Then there exists a better universe $\mathcal{U}$: an equality free quasivariety such that*

(1) *$\mathcal{Q} \subseteq \mathcal{U}$;*

(2) *$\mathcal{U}$ is finitely axiomatizable;*

(3) *$\mathcal{U}$ is protoalgebraic;*

(4) *there is a positive equality free formula $x \sim y$ defining Leibniz equalities in $\mathcal{U}$.*

PROOF. Let $x \sim y$ be the formula from Lemma 5.1 when $\mathcal{C} = \mathcal{Q}^*$. Let χ be a sentence such that $\mathbf{M} \models \chi$ iff the interpretation of $x \sim y$ in $\mathbf{M}$ is a strict congruence. We have $\mathcal{Q} \models \chi$, thus there exists a finitely axiomatizable equality free quasivariety $\mathcal{U} \supseteq \mathcal{Q}$ such that $\mathcal{U} \models \chi$. We will show that for $\mathbf{M} \in \mathcal{U}$ the interpretation of $x \sim y$ in $\mathbf{M}$ coincides with $\Omega(\mathbf{M})$. Recall that the Leibniz equality $\Omega(\mathbf{M})$ is the largest strict congruence of $\mathbf{M}$. Thus $\mathbf{M} \models \chi$ guarantees that the interpretation of $x \sim y$ in $\mathbf{M}$ in contained in $\Omega(\mathbf{M})$. Conversely, let $(a, b) \in \Omega(\mathbf{M})$. Then $\mathbf{M}^* \models a^* \approx b^*$. Hence, because the interpretation of $x \sim y$ in $\mathbf{M}^*$ is reflexive, $\mathbf{M}^* \models a^* \sim b^*$. Now the fact that $x \sim y$ is equality free gives us $\mathbf{M} \models a \sim b$. Thus (4) is proved. Finally, the positivity of $x \sim y$ yields (3). ■

We indicate the case when Better Universe Theorem is applicable.

PROPOSITION 5.3. *Let $\mathcal{Q}$ be a protoalgebraic equality free quasivariety and $\mathcal{Q}^*_{CI}$ be axiomatizable. Then $\mathcal{Q}^*$ is axiomatizable and Better Universe Theorem holds for $\mathcal{Q}$.*

PROOF. Let $x \sim y$ be the formula from Lemma 5.1 when $\mathcal{C} = \mathcal{Q}^*_{CI}$. We claim that $\mathcal{Q}^*$ is definable relative to $\mathcal{Q}$ by $\sigma = (\forall x, y)[x \sim y \to x \approx y]$. By point (2) in Lemma 5.1, every model from $\mathcal{Q}$ satisfying σ is reduced. Conversely, the positivity of $x \sim y$ yields preservation of satisfaction of σ under taking subdirect products. Thus, by Propositions 2.3 point (2), $\mathcal{Q}^* \models \sigma$ follows from $\mathcal{Q}^*_{CI} \models \sigma$. ∎

Everywhere from now on $\mathcal{Q}$ is always assumed to satisfy conditions of Better Universe Theorem, and $\mathcal{U}$, $x \sim y$ are as there.

6. $\mathcal{Q}$-Relation formulas

Congruence formulas are a key tool in general algebra. Most standard proofs of finite axiomatization results for (quasi)varieties use them. However their counterparts for protoalgebraic equality free quasivarieties were overlooked. The situation is a bit more complicated here and we need to consider two notions: $\mathcal{Q}$-relation formulas with and without equality. We will introduce them in the case when there is only one relation symbol R in the default language because the general case is a bit cumbersome. We will identify the interpretation $\mathcal{R}$ of R in $\mathbf{M}$ with the subset of $M^{\mathrm{arity}(R)}$. Then $(\mathbf{A}, \mathcal{R}) \models R(\bar{a})$ means exactly $\bar{a} \in \mathcal{R}$. We will present the adjustment to the general case in Appendix.

For tuples $\bar{a}_0, \dots, \bar{a}_{n-1} \in M$ of the lengths arity(R), let

$$\mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{a}_0, \dots, \bar{a}_{n-1}) = \bigcap \{\mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{M}) \mid \bar{a}_0, \dots, \bar{a}_{n-1} \in \mathcal{S}\}.$$

Note that relations $\mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{a}_0, \dots, \bar{a}_{n-1})$ are compact elements in the lattice $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$. Relations of the form $\mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{a})$ will be called *principal $\mathcal{Q}$-relations*. Observe that a model $\mathbf{M} \in \mathcal{Q}$ is completely irreducible relative to $\mathcal{Q}$ iff there exists a tuple $\bar{c} \in M$ such that $\mathbf{M} \models \neg R(\bar{c})$ and whenever $\mathbf{M} \models \neg R(\bar{a})$ for some $\bar{a} \in M$, then $\bar{c} \in \mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{a})$. Studying definability of principal $\mathcal{Q}$-relations is one of our main concerns in this paper, as is studying definability of principal congruences in general algebra.

A Formula $\Gamma(\bar{y}, \bar{x})$ is *$\mathcal{Q}$-relation formula with equality* if it is (equivalent to) an existential positive formula (or a disjunction of primitive positive formulas), possibly with equality, such that

$$\mathcal{Q} \models (\forall \bar{x}, \bar{y})\ [\Gamma(\bar{y}, \bar{x}) \wedge R(\bar{x})] \to R(\bar{y}). \qquad (\triangledown_\Gamma)$$

A *$\mathcal{Q}$-relation formula without equality* is any formula $\widetilde{\Gamma}$ that may be obtained obtained from a $\mathcal{Q}$-relation formula with equality Γ by replacing all occurrences of $t \approx s$ in Γ by $t \sim s$, where t and s are arbitrary terms. Note that $\mathcal{Q}$-relation formulas without equality do not have to be equivalent to existential formulas, but, due to the protoalgebraicity, they are positive. This is an important observation that will be useful later. The simplest $\mathcal{Q}$-relation formulas with(out) equality are $\bar{x} \approx \bar{y}$ ($\bar{x} \sim \bar{y}$) and $R(\bar{y})$.

Lemma 6.1. *Let $\widetilde{\Gamma}$ be a $\mathcal{Q}$-relation formula without equality. Then*

$$\mathcal{Q} \models (\forall \bar{x}, \bar{y})\ [\widetilde{\Gamma}(\bar{y}, \bar{x}) \wedge R(\bar{x})] \to R(\bar{y}). \qquad (\triangledown_{\widetilde{\Gamma}})$$

Proof. Assume that $\mathbf{M} \in \mathcal{Q}$ and $\mathbf{M} \models \widetilde{\Gamma}(\bar{b}, \bar{a}) \wedge R(\bar{a})$. Then we have $\mathbf{M}^* \models \Gamma(\bar{b}^*, \bar{a}^*) \wedge R(\bar{a}^*)$. Hence, by $(\triangledown_\Gamma)$, $\mathbf{M}^* \models R(\bar{b}^*)$ and $\mathbf{M} \models R(\bar{b})$. ■

Proposition 6.2. *Let $\bar{a}, \bar{b} \in M$, $\mathbf{M} \in \mathcal{U}$. The following conditions are equivalent.*

(1) $\bar{b} \in \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})$*;*

(2) $\mathbf{M} \models \Gamma(\bar{b}, \bar{a})$ *for some $\mathcal{Q}$-relation formula Γ with equality;*

(3) $\mathbf{M} \models \widetilde{\Gamma}(\bar{b}, \bar{a})$ *for some $\mathcal{Q}$-relation formula $\widetilde{\Gamma}$ without equality.*

Proof.
(3)⇒(1) Assume that $\mathbf{M} = (\mathbf{A}, \mathcal{R}) \models \widetilde{\Gamma}(\bar{b}, \bar{a})$. The positivity of $\widetilde{\Gamma}$ implies $(\mathbf{A}, \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})) \models \widetilde{\Gamma}(\bar{b}, \bar{a})$. Moreover, $(\mathbf{A}, \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})) \models R(\bar{a})$. Thus, by $(\triangledown_{\widetilde{\Gamma}})$, $(\mathbf{A}, \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})) \models R(\bar{b})$.
(1)⇒(3) Let

$$\mathcal{S} = \{\bar{c} \mid \mathbf{M} \models \widetilde{\Gamma}(\bar{c}, \bar{a}) \text{ for some } \mathcal{Q}\text{-relation formula without equality } \widetilde{\Gamma}\}.$$

It is enough to show that $\bar{a} \in \mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$. Because $\bar{y} \sim \bar{x}$ is a $\mathcal{Q}$-relation formula without equality, $\bar{a} \in \mathcal{S}$. Similarly, the fact that $R(\bar{y})$ is a $\mathcal{Q}$-relation formula without equality yields $\mathcal{R} \subseteq \mathcal{S}$. In order to see that $\mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$ consider an equality free quasi-identity true in $\mathcal{Q}$

$$q = (\forall \bar{z}) \left[\bigwedge_i R(\bar{t}_i(\bar{z})) \to R(\bar{t}(\bar{z})) \right].$$

We need to verify that $(\mathbf{A},\mathcal{S}) \models q$. So assume that $\mathbf{M} \models \widetilde{\Gamma}_i(\bar{t}_i(\bar{d}),\bar{a})$ for some $\mathcal{Q}$-relation formulas without equality $\widetilde{\Gamma}_i$ and some $\bar{d} \in M$. Put

$$\widetilde{\Gamma}(\bar{y},\bar{x}) = (\exists \bar{u}) \left[\bar{y} \sim \bar{t}(\bar{u}) \wedge \bigwedge_i \widetilde{\Gamma}_i(\bar{t}_i(\bar{u}),\bar{x}) \right].$$

Then $\widetilde{\Gamma}$ is equivalent to a $\mathcal{Q}$-relation formula without equality and moreover $\mathbf{M} \models \widetilde{\Gamma}(t(\bar{d}),\bar{a})$. Hence $(\mathbf{A},\mathcal{S}) \models q$.

The proof of the equivalence (1)$\Leftrightarrow$(2) is analogous. Note however that this equivalence is more general. It holds for arbitrary equality free quasivarieties. ■

Remark 6.3. Note that when $\mathcal{Q}$ is finitely equivalential $\mathcal{Q}$-relation formulas without equality are (equivalent to) existential positive equality free formulas satisfying $(\triangledown_\Gamma)$. Thus then they form a subclass of $\mathcal{Q}$-relation formulas with equality.

7. Definable principal $\mathcal{Q}$-subrelations

A $\mathcal{Q}$-relation formula with or without equality Υ *defines principal $\mathcal{Q}$-subrelations* in a class $\mathcal{C} \subseteq \mathcal{Q}$ if for every $\mathbf{M} \in \mathcal{C}$, and $\bar{a} \in M$ such that $\mathbf{M} \models \neg R(\bar{a})$, there exists $\bar{b} \in M$ satisfying

$$\mathbf{M} \models \neg R(\bar{b}), \quad \mathbf{M} \models \Upsilon(\bar{b},\bar{a}) \quad \text{and} \quad \mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{b}) = \{\bar{c} \in M \mid \mathbf{M} \models \Upsilon(\bar{c},\bar{b})\}.$$

We say that $\mathcal{Q}$ has *definable principal subrelations* (DPSR in short) if there exists a $\mathcal{Q}$-relation formula without equality $\widetilde{\Gamma}$ defining principal $\mathcal{Q}$-subrelations in $\mathcal{Q}$.

Lemma 7.1. *$\mathcal{Q}$ has DPSR if and only if there is a $\mathcal{Q}$-relation formula with equality Γ defining principal $\mathcal{Q}$-subrelations in $\mathcal{Q}^*$.*

Proof. This is so because $\mathbf{M} \models \widetilde{\Gamma}(\bar{b},\bar{a})$ iff $\mathbf{M}^* \models \Gamma(\bar{b}^*,\bar{a}^*)$. ■

Proposition 7.2. *Assume that there exists a natural number m such that for every $\mathbf{M} \in \mathcal{Q}^*$ and $\bar{a} \in M$, $\mathbf{M} \models \neg R(\bar{a})$*

(1) *there exist $\bar{b} \in M$ and $\mathbf{N} \leqslant \mathbf{M}$, such that $\bar{a},\bar{b} \in N$, $|N| \leqslant m$, $\mathbf{M} \models \neg R(\bar{b})$ and $\bar{b} \in \mathrm{rel}^{\mathbf{N}}_{\mathcal{Q}}(\bar{a})$;*

(2) *for every $\bar{c} \in \mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{b})$ there exists $\mathbf{K} \leqslant \mathbf{M}$ such that $\bar{b},\bar{c} \in K$, $|K| \leqslant m$ and $\bar{c} \in \mathrm{rel}^{\mathbf{K}}_{\mathcal{Q}}(\bar{b})$.*

Then $\mathcal{Q}$ has DPSR. The converse is true provided $\mathcal{Q}$ is locally finite.

Proof. We will verify the condition from Lemma 7.1. By the finiteness of the default language it follows that there are only finitely many models of cardinality at most m. Thus, by Proposition 6.2, there is a $\mathcal{Q}$-relation formula with equality Γ such that for every $\mathbf{N}$ with $|N| \leqslant m$ and for all $\bar{a}, \bar{b} \in N$

$$\bar{b} \in \mathrm{rel}_{\mathcal{Q}}^{\mathbf{N}}(\bar{a}) \quad \text{iff} \quad \mathbf{N} \models \Gamma(\bar{b}, \bar{a}).$$

Now let $\bar{a} \in M$, where $\mathcal{Q}^* \ni \mathbf{M} \models \neg R(\bar{a})$. Let $\bar{b}$ and $\mathbf{N}$ be as in point (1). Then $\mathbf{N} \models \Gamma(\bar{b}, \bar{a})$ and, because Γ is existential, $\mathbf{M} \models \Gamma(\bar{b}, \bar{a})$. We analogically verify that $\bar{c} \in \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{b})$ iff $\mathbf{M} \models \Gamma(\bar{c}, \bar{b})$ for $\bar{c} \in M$.

Conversely, by Lemma 7.1 there exists a $\mathcal{Q}$-relation formula with equality $\Gamma(\bar{y}, \bar{x}) = (\exists \bar{z})\, \gamma(\bar{x}, \bar{y}, \bar{z})$, where γ is quantifier free, defining principal $\mathcal{Q}$-subrelations in $\mathcal{Q}^*$. Take m to be the number from Lemma 3.2 for $k = (\text{length of } \bar{z}) + 2\,\mathrm{arity}(R)$. ■

8. Finite axiomatization theorem

Lemma 8.1. *Let $\mathcal{Q}$ and $\mathcal{P}$ be equality free quasivarieties generating the same equality free variety $\mathcal{V}$. If for every $\bar{a} \in M$ and $\mathbf{M} \in \mathcal{Q}$, we have $\mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a}) = \mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a})$, then $\mathcal{Q} = \mathcal{P}$.*

Proof. First notice that every model has a relative of the form $(\mathbf{T}(X), \mathcal{R})$ for some set X, this is with algebra of terms as its algebra reduct. Moreover, the assumption gives us $\mathbf{F}_{\mathcal{Q}}(X) = \mathbf{F}_{\mathcal{P}}(X) =: \mathbf{F}$. Thus it would be enough to show that $\mathrm{Rel}_{\mathcal{P}}(\mathbf{F}) = \mathrm{Rel}_{\mathcal{Q}}(\mathbf{F})$. But even less is needed. Because both lattices $\mathrm{Rel}_{\mathcal{P}}(\mathbf{F})$ and $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{F})$ are algebraic, we just need to show that they have the same compact elements, i.e.,

$$\mathrm{rel}_{\mathcal{Q}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1}) = \mathrm{rel}_{\mathcal{P}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1})$$

for all $\bar{t}_0, \ldots, \bar{t}_{n-1} \in F$, $n \in \mathbb{N}$. We will verify this equality by induction on n. For $n = 0$ the equation clearly holds. So assume that it holds for n. Then

$$\begin{aligned}
\mathrm{rel}_{\mathcal{Q}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1}, \bar{t}_n) &= \mathrm{rel}_{\mathcal{Q}}^{(\mathbf{T}(X), \mathrm{rel}_{\mathcal{Q}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1}))}(\bar{t}_n) \\
&= \mathrm{rel}_{\mathcal{P}}^{(\mathbf{T}(X), \mathrm{rel}_{\mathcal{Q}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1}))}(\bar{t}_n) \\
&= \mathrm{rel}_{\mathcal{P}}^{(\mathbf{T}(X), \mathrm{rel}_{\mathcal{P}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1}))}(\bar{t}_n) = \mathrm{rel}_{\mathcal{P}}^{\mathbf{F}}(\bar{t}_0, \ldots, \bar{t}_{n-1}, \bar{t}_n).
\end{aligned}$$

Here the first and the last equalities hold by the definition, the second follows from the assumption of the lemma, and the third from the induction assumption. ■

Lemma 8.2. *Assume that $\mathcal{Q}$ has DPSR. Then $\mathcal{Q}$ is finitely axiomatizable relative to the equality free variety $\mathcal{V}$ it generates.*

Proof. Let $\widetilde{\Gamma}$ be a $\mathcal{Q}$-relation formula without equality witnessing DPSR for $\mathcal{Q}$. By compactness theorem, there is a finitely axiomatizable equality free quasivariety $\mathcal{K}$ such that $\mathcal{Q} \subseteq \mathcal{K} \subseteq \mathcal{U}$, where $\mathcal{U}$ is from Better Universe Theorem and $\widetilde{\Gamma}$ is a $\mathcal{K}$-relation formula without equality. We will prove that $\mathcal{P} := \mathcal{K} \cap \mathcal{V} = \mathcal{Q}$ by verifying the condition from Lemma 8.1. So we want to check that $\mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a}) = \mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a})$ for $\bar{a} \in M$, $\mathbf{M} = (\mathbf{A}, \mathcal{R}) \in \mathcal{Q}$.

The inclusion $\mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a}) \subseteq \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})$ follows from the containment $\mathcal{Q} \subseteq \mathcal{P}$. In order to prove the converse one we construct a sequence $(\mathcal{R}_\kappa)_{\kappa<\rho}$, where ρ is an ordinal, of relations on M with the following properties:

- $|\operatorname{Rel}_Q(\mathbf{M})| < |\rho|$;
- if $\lambda \leqslant \kappa < \rho$, then $\mathcal{R}_\lambda \subseteq \mathcal{R}_\kappa$;
- for $\kappa < \rho$, $\mathcal{R}_\kappa \in \operatorname{Rel}_{\mathcal{Q}}(\mathbf{M})$;
- for $\kappa < \rho$, $\mathcal{R}_\kappa \subseteq \mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a})$;
- for $\kappa < \rho$, $\mathcal{R}_\kappa = \mathcal{R}_{\kappa+1}$ implies $\mathcal{R}_\kappa = \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})$;

The first condition yields that $(\mathcal{R}_\kappa)_{\kappa<\rho}$ is not strictly increasing. Hence, by the last two conditions, we obtain $\mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a}) \subseteq \mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a})$. We start by putting $\mathcal{R}_0 := \mathcal{R}$. Let $\kappa = \lambda + 1$ and assume that $\mathcal{R}_\lambda$ is already defined. If $\bar{a} \in \mathcal{R}_\lambda$, then $\mathcal{R}_\lambda = \mathrm{rel}_{\mathcal{Q}}^{\mathbf{M}}(\bar{a})$ since $\bar{a} \in \mathcal{R}_\lambda \in \operatorname{Rel}_Q(\mathbf{M})$. In this case we simply put $\mathcal{R}_\kappa := \mathcal{R}_\lambda$. Otherwise there exists $\bar{b} \notin \mathcal{R}_\lambda$ such that

$$(\mathbf{A}, \mathcal{R}_\lambda) \models \widetilde{\Gamma}(\bar{b}, \bar{a}), \quad \mathrm{rel}_{\mathcal{Q}}^{(\mathbf{A},\mathcal{R}_\lambda)}(\bar{b}) = \{\bar{c} \in M \mid (\mathbf{A}, \mathcal{R}_\lambda) \models \widetilde{\Gamma}(\bar{c}, \bar{b})\},$$

and we define $\mathcal{R}_\kappa := \mathrm{rel}_{\mathcal{Q}}^{(\mathbf{A},\mathcal{R}_\lambda)}(\bar{b})$. Since $\widetilde{\Gamma}$ is a $\mathcal{K}$-, and hence a $\mathcal{P}$-relation formula without equality, by Proposition 6.2 we have $\mathcal{R}_\kappa \subseteq \mathrm{rel}_{\mathcal{P}}^{(\mathbf{A},\mathcal{R}_\lambda)}(\bar{b})$ and $\bar{b} \in \mathrm{rel}_{\mathcal{P}}^{(\mathbf{A},\mathcal{R}_\lambda)}(\bar{a})$. This and $\mathcal{R}_\lambda \subseteq \mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a})$ yield $\mathcal{R}_\kappa \subseteq \mathrm{rel}_{\mathcal{P}}^{\mathbf{M}}(\bar{a})$. If κ is a limit ordinal we define $\mathcal{R}_\lambda := \bigcup_{\lambda \leqslant \kappa} \mathcal{R}_\lambda$. ■

Lemma 8.3. *Let $\mathcal{K}$ be an equality free quasivariety contained in $\mathcal{U}$ and $\widetilde{\Gamma}$ be a $\mathcal{K}$-relation formula without equality. Assume further that $\widetilde{\Gamma}$ defines principal $\mathcal{K}$-subrelations in $\mathbf{M} \in \mathcal{K}$. Then $\mathbf{M} \in \mathcal{K}_{CI}$ if and only if $\mathbf{M}$ satisfies the sentence*

$$\sigma = (\exists \bar{z})\Big[\neg R(\bar{z}) \wedge \big[(\forall \bar{x})[\neg R(\bar{x}) \to \widetilde{\Gamma}^2(\bar{z}, \bar{x})]\big]\Big],$$

where $\widetilde{\Gamma}^2(\bar{z}, \bar{x}) = (\exists \bar{y})[\widetilde{\Gamma}(\bar{y}, \bar{x}) \wedge \widetilde{\Gamma}(\bar{z}, \bar{y})]$.

PROOF. Assume that $\mathbf{M} \models \sigma$. Because $\widetilde{\Gamma}^2$ is equivalent to a $\mathcal{K}$-relation formula, there exists a tuple $\bar{c}$ such that $\mathbf{M} \models \neg R(\bar{c})$ and $\bar{c} \in \mathrm{rel}_{\mathcal{K}}^{\mathbf{M}}(\bar{a})$ whenever $\mathbf{M} \models \neg R(\bar{a})$. Hence $\mathbf{M} \in \mathcal{K}_{CI}$. Conversely, let $\mathbf{M}$ be in $\mathcal{K}_{CI}$ and $\bar{c} \in M$ be a tuple such that $\mathrm{rel}_{\mathcal{K}}^{\mathbf{M}}(\bar{c})$ is the only atom in the lattice $\mathrm{Rel}_{\mathcal{K}}(\mathbf{M})$. Let $\bar{a} \in M$ be a tuple such that $\mathbf{M} \models \neg R(\bar{a})$. Because $\widetilde{\Gamma}$ defines principal $\mathcal{K}$-subrelations in $\mathcal{K}$, there exists $\bar{b} \in M$ such that $\mathbf{M} \models \neg R(\bar{b})$, $\mathbf{M} \models \widetilde{\Gamma}(\bar{b}, \bar{a})$ and $\mathrm{rel}_{\mathcal{K}}^{\mathbf{M}}(\bar{b}) = \{\bar{d} \in M \mid \mathbf{M} \models \widetilde{\Gamma}(\bar{d}, \bar{b})\}$. We have $\mathrm{rel}_{\mathcal{K}}^{\mathbf{M}}(\bar{c}) \subseteq \mathrm{rel}_{\mathcal{K}}^{\mathbf{M}}(\bar{b})$ and hence $\mathbf{M} \models \widetilde{\Gamma}(\bar{c}, \bar{b})$. This way we proved that $\mathbf{M} \models \sigma$. ■

THEOREM 8.4. *Assume that $\mathcal{Q}$ has DPSR witnessed by $\widetilde{\Gamma}$. Then $\mathcal{Q}$ is finitely axiomatizable if and only if $\mathcal{Q}_{CI}$ or $\mathcal{Q}^*_{CI}$ is finitely axiomatizable.*

PROOF. Because we implicitly assume that $\mathcal{Q}$ satisfies the conditions of Better Universe Theorem, $\mathcal{Q}_{CI}$ is finitely axiomatizable iff $\mathcal{Q}^*_{CI}$ is finitely axiomatizable.
($\Rightarrow$) It follows from Lemma 8.3.
($\Leftarrow$) By Lemma 8.2, there exists an equality free quasivariety $\mathcal{K} \subseteq \mathcal{U}$, with a finite axiomatization Σ, such that $\mathcal{Q} = \mathcal{K} \cap \mathcal{V}$, where $\mathcal{V}$ is an equality free variety generated by $\mathcal{Q}$, and $\widetilde{\Gamma}$ is a $\mathcal{K}$-relation formula without equality.

Let $\delta(\bar{y})$ be a formula such that for every $\mathbf{M} \in \mathcal{U}$ and $\bar{b} \in M$

$$\mathbf{M} \models \delta(\bar{b}) \quad \text{iff} \quad \{\bar{c} \in M \mid \mathbf{M} \models \widetilde{\Gamma}(\bar{c}, \bar{b})\} = \mathrm{rel}_{\mathcal{K}}^{\mathbf{M}}(\bar{b}). \tag{$\square$}$$

We may construct δ as follows. For

$$q = (\forall \bar{z}) \left[\bigwedge_i R(\bar{t}_i(\bar{z})) \to R(\bar{t}(\bar{z})) \right] \in \Sigma$$

let

$$\delta_q(\bar{y}) := (\forall \bar{z}) \left[\bigwedge_i \widetilde{\Gamma}(\bar{t}_i(\bar{z}), \bar{y}) \to \widetilde{\Gamma}(\bar{t}(\bar{z}), \bar{y}) \right]$$

and

$$\delta(\bar{y}) := \widetilde{\Gamma}(\bar{y}, \bar{y}) \wedge \bigwedge_{q \in \Sigma} \delta_q(\bar{y}).$$

The construction gives us the equivalence

$$\mathbf{M} \models \delta(\bar{b}) \quad \text{iff} \quad \bar{b} \in \{\bar{c} \in M \mid \mathbf{M} \models \widetilde{\Gamma}(\bar{c}, \bar{b})\} \in \mathrm{Rel}_{\mathcal{K}}(\mathbf{M}).$$

Thus, by Proposition 6.2, ($\square$) holds.

Now we apply Jónsson's Lemma. For this purpose we need to define classes $\mathcal{E}$ and $\mathcal{I}$ satisfying conditions stated there. The class $\mathcal{E}$ is the subclass of $\mathcal{K}$ satisfying

$$(\forall\bar{x})\Big[\neg R(\bar{x}) \to \big[(\exists\bar{y})[\neg R(\bar{y}) \wedge \widetilde{\Gamma}(\bar{y},\bar{x}) \wedge \delta(\bar{y})]\big]\Big].$$

In other words, $\mathcal{E}$ is the subclass of $\mathcal{K}$ where $\widetilde{\Gamma}$ defines principal $\mathcal{K}$-subrelations. In particular, $\mathcal{Q} \subseteq \mathcal{E}$ and the condition (1) from Jónsson's Lemma holds. The class $\mathcal{I}$ is the subclass of $\mathcal{E}$ satisfying the sentence σ from Lemma 8.3. The satisfaction of the conditions (2) and (3) from Jónsson's Lemma follows from Lemma 8.3. Finally, by the assumption, $\mathcal{K}_{CI} \cap \mathcal{V} = \mathcal{Q}_{CI}$ is finitely axiomatizable. This proves that $\mathcal{Q}$ is finitely axiomatizable relative to $\mathcal{K}$. Thus, since $\mathcal{K}$ is finitely axiomatizable, $\mathcal{Q}$ is finitely axiomatizable. ■

9. Pałasińska's theorem

For a relation $\mathcal{D}$ of arity r on the carrier set of $\mathbf{M}$ and a submodel $\mathbf{N}$ of $\mathbf{M}$ with the algebra reduct $\mathbf{B}$ we will use the notation $\mathcal{D}|_{\mathbf{B}}$ for $\mathcal{D} \cap N^r$. In particular, if $\mathbf{M} = (\mathbf{A}, \mathcal{R})$, then $\mathbf{N} = (\mathbf{B}, \mathcal{R}|_{\mathbf{B}})$.

LEMMA 9.1. *Let* $\mathcal{R}, \mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{A})$ *and* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$*. Assume that*

(1) $|(\mathbf{A},\mathcal{S})^*| \leqslant |(\mathbf{A},\mathcal{R})^*|$;

(2) $\mathbf{B}$ *contains a representative of each class of* $\Omega(\mathbf{A},\mathcal{R})$;

(3) $\mathcal{S}|_{\mathbf{B}} \subseteq \mathcal{R}|_{\mathbf{B}}$;

(4) $(\mathbf{A},\mathcal{S})^*$ *is finite.*

Then $\mathbf{B}$ *contains a representative of each class of* $\Omega(\mathbf{A},\mathcal{S})$.

PROOF. We have

$$|(\mathbf{B},\mathcal{S}|_{\mathbf{B}})^*| \leqslant |(\mathbf{A},\mathcal{S})^*| \leqslant |(\mathbf{A},\mathcal{R})^*| = |(\mathbf{B},\mathcal{R}|_{\mathbf{B}})^*| \leqslant |(\mathbf{B},\mathcal{S}|_{\mathbf{B}})^*|.$$

Indeed, the first inequality may be deduced from the definition of Leibniz equality. The second inequality is assumed as point (1). The equality follows from point (2) and the observation that then $\Omega(\mathbf{B},\mathcal{R}|_{\mathbf{B}}) = \Omega(\mathbf{A},\mathcal{R})|_{\mathbf{B}}$. And the last inequality follows from the protoalgebraicity and point (3). Hence $|(\mathbf{B},\mathcal{S}|_{\mathbf{B}})^*| = |(\mathbf{A},\mathcal{S})^*|$, and the conclusion follows from point (4). ■

THEOREM 9.2. *Let* $\mathcal{Q}$ *be a finitely generated protoalgebraic relation distributive equality free quasivariety. Then* $\mathcal{Q}$ *has DPSR.*

Proof. By Lemma 3.3, there is a natural number l which is an upper bound of the cardinality of models in $\mathcal{Q}^*_{CI}$. In particular, $\mathcal{Q}^*_{CI}$ is axiomatizable, and, by Proposition 5.3, Better Universe Theorem holds for $\mathcal{Q}$.

We will verify the condition from Proposition 7.2. By Lemma 3.1, $\mathcal{Q}$ is locally finite, and we may use Lemma 3.2. Let m be the number from this lemma when $k = l + 2\,\mathrm{arity}(R)$.

Let $\mathbf{M} = (\mathbf{A}, \mathcal{R}) \in \mathcal{Q}^*$. By the algebraicity of the lattice $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$ there are $\mathcal{R}_i \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{M})$ such that $\mathcal{R} = \bigwedge_i \mathcal{R}_i$ and $(\mathbf{A}, \mathcal{R}_i) \in \mathcal{Q}_{CI}$. Let $\bar{a} \notin \mathcal{R}$. Let j be such that $(\mathbf{A}, \mathcal{R}_j)^*$ has the largest possible cardinality with respect to the condition $\bar{a} \notin \mathcal{R}_j$.

Let $\mathbf{N} = (\mathbf{B}, \mathcal{R}|_{\mathbf{B}})$ be a submodel of $\mathbf{M}$ generated by a tuple $\bar{a}$ and at most l-element set of representatives of all classes of $\Omega(\mathbf{A}, \mathcal{R}_j)$. Then $\mathbf{N}$ has generating set with at most $l + \mathrm{arity}(R) \leqslant k$ elements, and hence $|N| \leqslant m$. Because $(\mathbf{A}, \mathcal{R}_j)^* \cong (\mathbf{B}, \mathcal{R}_j|_{\mathbf{B}})^*$, by Proposition 2.3 point (1), $\mathcal{R}_j|_{\mathbf{B}}$ is meet irreducible in $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{N})$. By the distributivity of $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{N})$, $\mathcal{R}_j|_{\mathbf{B}}$ is also meet prime. Hence, since $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{N})$ is a finite lattice, there is the least relation $\mathcal{R}' \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{N})$ not contained in $\mathcal{R}_j|_{\mathbf{B}}$. Moreover, $\mathcal{R}'$ is principal, i.e., it equals $\mathrm{rel}^{\mathbf{N}}_{\mathcal{Q}}(\bar{b})$ for some $\bar{b} \in N$. Summarizing, we have the following splitting

$$\big(\forall \mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{N})\big)\ \big[\mathcal{S} \leqslant \mathcal{R}_j|_{\mathbf{B}} \ \text{ xor } \ \mathrm{rel}^{\mathbf{N}}_{\mathcal{Q}}(\bar{b}) \leqslant \mathcal{S}\big]. \qquad (\bowtie)$$

Because $\bar{a} \notin \mathcal{R}_j$, $\mathrm{rel}^{\mathbf{N}}_{\mathcal{Q}}(\bar{b}) \leqslant \mathrm{rel}^{\mathbf{N}}_{\mathcal{Q}}(\bar{a})$. This means exactly $\bar{b} \in \mathrm{rel}^{\mathbf{N}}_{\mathcal{Q}}(\bar{a})$.

Now consider a tuple $\bar{c} \in \mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{b})$. Let $\mathbf{K} = (\mathbf{C}, \mathcal{R}|_{\mathbf{C}})$ be the submodel of $\mathbf{M}$ generated by $\mathbf{B}$ and $\bar{c}$. Thus defined $\mathbf{K}$ has generating set with at most k elements, and hence $|K| \leqslant m$. We are going to verify that $\bar{c} \in \mathrm{rel}^{\mathbf{K}}_{\mathcal{Q}}(\bar{b})$. By the distributivity of the finite lattice $\mathrm{Rel}_{\mathcal{Q}}(\mathbf{K})$

$$\mathrm{rel}^{\mathbf{K}}_{\mathcal{Q}}(\bar{b}) = \mathrm{rel}^{\mathbf{K}}_{\mathcal{Q}}(\bar{b}) \vee \bigwedge_i \mathcal{R}_i|_{\mathbf{C}} = \bigwedge_i \big(\mathrm{rel}^{\mathbf{K}}_{\mathcal{Q}}(\bar{b}) \vee \mathcal{R}_i|_{\mathbf{C}}\big) = \bigwedge_i \mathrm{rel}^{(\mathbf{C}, \mathcal{R}_i|_{\mathbf{C}})}_{\mathcal{Q}}(\bar{b}).$$

Hence it is enough to show that $\bar{c} \in \mathrm{rel}^{(\mathbf{C}, \mathcal{R}_i|_{\mathbf{C}})}_{\mathcal{Q}}(\bar{b})$ for all i.
Case $\bar{b} \in \mathcal{R}_i$: Then $\bar{c} \in \mathcal{R}_i$ and in particular $\bar{c} \in \mathrm{rel}^{(\mathbf{C}, \mathcal{R}_i|_{\mathbf{C}})}_{\mathcal{Q}}(\bar{b})$.
Case $\bar{b} \notin \mathcal{R}_i$: By ($\bowtie$), $\mathcal{R}_i|_{\mathbf{B}} \subseteq \mathcal{R}_j|_{\mathbf{B}}$, and we may apply Lemma 9.1 with $\mathcal{R} = \mathcal{R}_j$ and $\mathcal{S} = \mathcal{R}_i$. This proves that $\mathbf{B}$ contains a representative of each class of $\Omega(\mathbf{A}, \mathcal{R}_i)$. Hence $\mathbf{C}$ contains a representative of each class of $\Omega(\mathbf{A}, \mathcal{R}_i)$. Recall that, by Proposition 6.2, $\bar{c} \in \mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}(\bar{b})$ implies that $(\mathbf{A}, \mathcal{R}_i) \models \gamma(\bar{c}, \bar{b}, \bar{d})$, where $(\exists \bar{z})\, \gamma(\bar{x}, \bar{y}, \bar{z})$ is some $\mathcal{Q}$-relation formula without equality, γ is quantifier free, and $\bar{d} \in M$. Let $\bar{e}$ be a tuple in K such that $(\bar{d}, \bar{e}) \in \Omega(\mathbf{A}, \mathcal{R}_i)$. Then $(\mathbf{C}, \mathcal{R}_i|_{\mathbf{C}}) \models \gamma(\bar{c}, \bar{b}, \bar{e})$, and hence $\bar{c} \in \mathrm{rel}^{(C, \mathcal{R}_i|_{\mathbf{C}})}_{\mathcal{Q}}(\bar{b})$.
∎

Proof of Pałasińska's Theorem. Combine Theorem 8.4 and Theorem 9.2. ■

Appendix

Here we briefly show what we need to modify in the presented reasoning in order to obtain the proof of Pałasińska's theorem when the set $\mathcal{L}_R$ of relation symbols is of any finite cardinality.

It is convenient to see the interpretation $\mathcal{R}$ of $\mathcal{L}_R$ in a given model $\mathbf{M}$ as a subset of $\bigcup_{R\in\mathcal{L}_R} M^{\mathrm{arity}(R)} \times \{R\}$. For relation symbols R_i and tuples $\bar{a}_i \in M$ of the lengths $\mathrm{arity}(R_i)$, $i < n$, now we may define

$$\mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}((\bar{a}_0, R_0), \dots, (\bar{a}_{n-1}, R_{n-1})) := \bigcap\{\mathcal{S} \in \mathrm{Rel}_{\mathcal{Q}}(\mathbf{M}) \mid (\bar{a}_0, R_0), \dots, (\bar{a}_{n-1}, R_{n-1}) \in \mathcal{S}\}.$$

A *$\mathcal{Q}$-relation formula with equality* is a family $\Gamma = \{\Gamma_{S,R} \mid S, R \in \mathcal{L}_R\}$ of existential positive formulas, possibly with equality, such that

$$\mathcal{Q} \models (\forall \bar{x}, \bar{y})\ [\Gamma_{S,R}(\bar{y}, \bar{x}) \wedge R(\bar{x})] \to S(\bar{y}). \qquad (\nabla\Gamma_{S,R})$$

holds for all $R, S \in \mathcal{L}_R$. A *$\mathcal{Q}$-relation formula without equality* is the family of formulas $\widetilde{\Gamma} = \{\widetilde{\Gamma}_{S,R} \mid S, R \in \mathcal{L}_R\}$ obtained from Γ, a $\mathcal{Q}$-relation formula with equality, by replacing all occurrences of $t \approx s$ by $t \sim s$, where t and s are arbitrary terms in all $\Gamma_{S,R}$.

A $\mathcal{Q}$-relation formula with or without equality $\Upsilon = \{\Upsilon_{S,R} \mid R, S \in \mathcal{L}_R\}$ *defines principal $\mathcal{Q}$-subrelations* in a class $\mathcal{C} \subseteq \mathcal{Q}$ if for every $\mathbf{M} \in \mathcal{C}$, and $\bar{a} \in M$ such that $\mathbf{M} \models \neg R(\bar{a})$, there exists $S \in \mathcal{L}_R$ and $\bar{b} \in M$ of the length equals $\mathrm{arity}(S)$ such that

$$\mathbf{M} \models \neg S(\bar{b}), \quad \mathbf{M} \models \Upsilon_{S,R}(\bar{b}, \bar{a}), \quad \mathrm{rel}^{\mathbf{M}}_{\mathcal{Q}}((\bar{b}, S)) = \{(\bar{c}, T) \mid \mathbf{M} \models \Upsilon_{T,S}(\bar{c}, \bar{b})\}.$$

We say that $\mathcal{Q}$ has *definable principal subrelations* (DPSR in short) if there exists a $\mathcal{Q}$-relation formula without equality $\widetilde{\Gamma}$ defining principal $\mathcal{Q}$-subrelations in $\mathcal{Q}$.

With the modified definitions the reader may reformulate the statements and the proofs of facts obtained in Sections 6,7, 8 and 9. Let us describe two places where the changes are not completely straightforward.

The definition of the sentence σ from Lemma 8.3 is now more complicated. Let

$$\widetilde{\Gamma}^2_{T,R}(\bar{z}, \bar{x}) := \bigvee_{S\in\mathcal{L}_R} (\exists \bar{y}_S)[\widetilde{\Gamma}_{T,S}(\bar{z}, \bar{y}_S) \wedge \widetilde{\Gamma}_{S,R}(\bar{y}_S, \bar{x})],$$

where $\bar{y}_S$ is a tuple of variables of the length equals arity(S), and

$$\sigma := \bigvee_{T \in \mathcal{L}_R} (\exists \bar{z}_T)\Big[\neg T(\bar{z}_T) \wedge \bigwedge_{R \in \mathcal{L}_R} [(\forall \bar{x}_R)[\neg R(\bar{x}) \rightarrow \widetilde{\Gamma}^2_{T,R}(\bar{z}_T, \bar{x}_R)]]\Big],$$

where $\bar{x}_R$ and $\bar{z}_T$ are tuples of variables of the lengths equal arity(R) and arity(T) respectively.

In the proof of Theorem 8.4 we need to redefine the sentence axiomatizing the class $\mathcal{E}$ relative to $\mathcal{K}$. For

$$q = (\forall \bar{z}) \left[\bigwedge_i T_i(\bar{t}_i(\bar{z})) \rightarrow T(\bar{t}(\bar{z}))\right] \in \Sigma$$

let

$$\delta_{S,q}(\bar{y}_S) := (\forall \bar{z}) \left[\bigwedge_i \widetilde{\Gamma}_{T_i,S}(\bar{t}_i(\bar{z}), \bar{y}_S) \rightarrow \widetilde{\Gamma}_{T,S}(\bar{t}(\bar{z}), \bar{y}_S)\right]$$

and

$$\delta_S(\bar{y}_S) := \widetilde{\Gamma}_{S,S}(\bar{y}_S) \wedge \bigwedge_{q \in \Sigma} \delta_{S,q}(\bar{y}_S).$$

Now let $\mathcal{E}$ be the subclass of $\mathcal{K}$ satisfying

$$\bigwedge_{R \in \mathcal{L}_R} (\forall \bar{x}_R)\Big[\neg R(\bar{x}_R) \rightarrow \bigvee_{S \in \mathcal{L}_R} [(\exists \bar{y}_S)[\neg S(\bar{y}_S) \wedge \widetilde{\Gamma}_{S,R}(\bar{y}_S, \bar{x}_R) \wedge \delta_S(\bar{y}_S)]]\Big].$$

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