Inclosure and Intolerance*

ATTENTION: This is not the final version of the paper. The final version can be found in *Notre Dame Journal of Formal Logic* 62: 201–20. DOI: 10.1215/00294527-2021-0020.

Abstract

Graham Priest has influentially claimed that the Sorites paradox is an Inclosure paradox, concluding that his favoured dialetheic solution to the Inclosure paradoxes should be extended to the Sorites paradox. We argue that, given Priest's dialetheic solution to the Sorites paradox, the argument purporting to show that that paradox is an Inclosure is unsound, and discuss some issues surrounding this fact.

1 Introduction

Graham Priest has long been arguing (Priest [1994]; [2003]) that many paradoxes arising from the *use of ascending or descending principles* (e.g. the principle that P iff 'P' is true) *for certain properties* (e.g. the property of being true) *in the presence of the selfapplication of such properties* (e.g. the sentence saying of itself that it is not true; for brevity, we'll henceforth call such paradoxes 'paradoxes of *selfreference*[1]') *apparently*[2] *instantiate a certain structure*—the *Inclosure schema* ('IS' for short), thereby being *Inclosure paradoxes*—and,

*This paper grew out of a discussion the two authors had back in 2013 at the seminar *Semantic Paradoxes and Vagueness* (University of Barcelona), it was essentially ready in 2016 and has been circulating since then. Earlier versions of the material in the paper were presented in 2018 at the 93rd Joint Session of the Aristotelian Society and the Mind Association (Durham University) and at the LanCog MELL Seminar (University of Lisbon); in 2019, at the Studia Logica International Conference Trends in Logic XIX *Current Issues in Philosophical Logic* in Moscow (Higher School of Economics) and at the 9th SEFA Conference (University of Valencia); in 2020, at the Workshop *Non-Classical Validity and Logical Pluralism* (University of Barcelona). We'd like to thank all these audiences for very stimulating comments and discussions. Special thanks go to Eduardo Barrio, Bogdan Dicher, Vitalij Dolgorukov, Elena Dragalina, Ole Hjortland, José Martínez, Diogo Santos, Ricardo Santos and several anonymous referees. We'd also like to record a special debt of gratitude to Graham Priest, who throughout the years has patiently and inspiringly taught us—among many other things—to appreciate the subtleties of dialetheism and whose generous and open-minded comments on earlier versions of the material in the paper have greatly improved it. With regard to the first author (the order being merely alphabetical), at different stages, the paper was supported by the Project FFI2011-25626 of the Spanish Ministry of Science and Innovation *Reference, Self-Reference and Empirical Data*, by the Project FFI2015-70707-P of the Spanish Ministry of Economy, Industry and Competitiveness *Localism and Globalism in Logic and Semantics* and by the Project 2019PIDPID-107667GB-I00 of the Spanish Ministry of Science and Innovation *Worlds and Truth Values: Challenges to Formal Semantics*. With regard to the second author, at different stages, the paper was supported by the Marie Skłodowska-Curie Intraeuropean Research Fellowship 301493 *A Noncontractive Theory of Naive Semantic Properties: Logical Developments and Metaphysical Foundations*, by the FCT Research Fellowship IF/01202/2013 *Tolerance and Instability: The Substructure of Cognitions, Transitions and Collections* and by the Ramón y Cajal Reserach Fellowship RYC-2017-22883. Additionally, support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged. The second author also benefited from the Project CONSOLIDER-INGENIO 2010 CSD2009-00056 of the Spanish Ministry of Science and Innovation *Philosophy of Perspectival Thoughts and Facts*, from the FP7 Marie Curie Initial Training Network 238128 *Perspectival Thoughts and Facts*, from the Project FFI2012-35026 of the Spanish Ministry of Economy and Competition *The Makings of Truth: Nature, Extent, and Applications of Truthmaking*, from the FCT Project PTDC/FER-FIL/28442/2017 *Companion to Analytic Philosophy 2*, from the Project PID2019-105746GB-I00 of the Spanish Ministry of Science and Innovation *Linguistic Relativity and Experimental Philosophy* and from the projects already mentioned with regard to the first author.

[1]The label is problematic for some of the target paradoxes (e.g. Yablo [1985], p. 340); for Priest's defence of the claim that *Yablo's paradox* does involve selfreference, see Priest [1997]. Having flagged the issue, for want of a less problematic label, in this paper we stick to 'selfreference' (see the discussion in Zardini [2021b]).

[2]Throughout, by 'Apparently, ϕ' and its relatives we understand the same as we do by 'There is a *prima facie* justification for believing that ϕ' and its relatives. Such understanding might need some fine-tuning (see e.g. the discussion in Priest [2003], pp. 277–278), but it works well enough for the purposes of this paper.

in so doing, *they form a single (natural logical) kind*.[3] Since Priest (e.g. in the works just mentioned) has also endorsed the *principle of uniform solution* ('PUS' for short) according to which *paradoxes of the same kind should receive a solution of the same kind*,[4] he has inferred from all this that all such paradoxes should receive a solution of the same kind, which in turn he has argued to be a *dialetheic* one which consequently involves the adoption of a *paraconsistent* logic (as developed e.g. in Priest [2006]). More recently, Priest [2010] has claimed that the *Sorites paradox* too is an Inclosure paradox (see also Priest [2013]; [2019], which, however, for the purposes of this paper, do not substantially add to Priest [2010]; see Oms and Zardini [2019b] for a recent comprehensive guide to the Sorites paradox), and such a claim has become well-entrenched in subsequent discussions.[5] He has then concluded that his favoured dialetheic solution to the Inclosure paradoxes should be extended to the Sorites paradox.

In this paper, after providing the relevant background, we argue that, given Priest's dialetheic solution to the Sorites paradox, *the argument purporting to show that that paradox instantiates the IS is unsound*. After expanding on some noteworthy consequences of this fact, we examine some possible ways to resist our point and find them wanting.

2 Inclosure and Tolerance

An *Inclosure paradox* is *any paradox that apparently instantiates the schema*:

IS. There are two 1-place properties ϕ and ψ and a 1-place function δ such that:[6]

- There is a set Ω such that $\Omega = \{x : \phi(x)\}$ and $\psi(\Omega)$ holds (*Existence*);
- If $X \subseteq \Omega$ and $\psi(X)$ holds, then:
  - $\delta(X) \notin X$ (*Transcendence*);
  - $\delta(X) \in \Omega$ (*Closure*).

[3] The reader will likely wonder why we write 'many' and not 'all' (thereby diverging from claims to the effect that the Inclosure analysis "[. . .] aspires to explain all the usual paradoxes of self-reference", Weber et al. [2014], p. 821). That is because of *Curry's paradox*, which, besides being very usual, is evidently selfreferential but for which there is controversy as to whether it is an Inclosure paradox (we'll actually contribute to this controversy in section 2, providing reasons for thinking that it is). Priest himself is well-aware of this issue: he takes sides in the controversy arguing that Curry's paradox is not an Inclosure paradox and sticks to his guns concluding that the Liar paradox and Curry's paradox are not of the same kind (Priest [1994], p. 33). In the light of the *theoretically conspicuous similarities between the two paradoxes at several levels of analysis* (e.g. Zardini [2015]; [2019a]), the conclusion strikes us as deeply problematic, but this is not the place to pursue the issue further.

[4] Notice that, plausible as it may be, the PUS is not uncontroversial: for example, it becomes problematic from the point of view of the nowadays fashionable economistic approach to paradox (and, more generally, to philosophical theorising) focusing on *cost-benefit analyses*, since, even though two paradoxes are of the same kind, the most lucrative solution in one case might be of a different kind from the most lucrative solution in the other case. Whether this reflects badly on the PUS or on the economistic approach is a question that we leave in this paper to the reader's judgement (as a contrast, see Zardini [2021b] for a PUS-friendly approach). Thanks to Bogdan Dicher for pushing us on the PUS.

[5] To the best of our knowledge, the hypothesis that the Sorites paradox is an Inclosure paradox has first been aired in print by Colyvan [2009] (Beall [2014a], p. 793, fn 3; Weber et al. [2014], p. 821, fn 10 provide background on the preprint history of the hypothesis). The hypothesis would seem to be now accepted by most theorists favouring a dialetheic approach to vagueness (a sociological claim which, given the scarcity of such theorists, the existence of Weber et al. [2014] would seem to suffice to make true!). Indeed, the correctness of the hypothesis is a crucial premise in one of the main arguments in favour of a dialetheic approach to the Sorites paradox (in essence: "A dialetheic approach is the right one for many paradoxes of selfreference, and so, by the PUS and the assumptions that those paradoxes are Inclosure paradoxes and that Inclosure paradoxes are specific enough to form one single kind, the right one for the Inclosure paradoxes. By the hypothesis, the Sorites paradox is an Inclosure paradox. Therefore, a dialetheic approach is the right one for the Sorites paradox", see e.g. Weber et al. [2014], pp. 821–824), and would seem granted even by those who, while sympathetic to dialetheism in other cases, have criticised a dialetheic approach to the Sorites paradox (e.g. Beall [2014a], pp. 798–802; see however fn 21 for a more complete presentation of Beall's views). Therefore, our point to the effect that, given what is in fact the standard dialetheic solution to the Sorites paradox, the argument purporting to show that the paradox instantiates the IS is unsound would, if correct, seem to affect—in ways that we'll elaborate on in section 4—a crucial and common assumption in an important contemporary debate in the philosophy of logic. Having noted all this, for concreteness we mostly focus on Priest's own defence of the hypothesis, since it is the most sustained one we know of. Thanks to an anonymous referee for feedback on these matters.

[6] Throughout, we're totally—and totally innocuously—cavalier about the distinction between a property and a predicate expressing it as well as about similar distinctions.

Notice that, supposing that the IS is instantiated, the limit case where $X = \Omega$ produces a contradiction, for then, by Transcendence, $\delta(\Omega) \notin \Omega$ and, by Closure, $\delta(\Omega) \in \Omega$.

To warm up, let's see how the Liar paradox, a paradigmatic paradox of selfreference, instantiates the IS. Recall that the Liar paradox can be understood as the apparently valid derivation of the apparently false conclusion that the Liar sentence is true and the Liar sentence is not true from the apparently true naive truth-theoretic principles.

Consider now the following interpretation for ϕ, ψ and δ in the IS:

- ϕ is the property of being true;
- ψ is the property of being definable (in the usual sense that X is definable iff there is a 1-place predicate χ of the language such that, for every x, $x \in X$ iff $\chi(x)$ holds);
- δ is a function that, given a definable subset X of Ω, yields a sentence tantamount to $\delta(X) \notin X$.[7]

On this interpretation, Ω is the set of true sentences, which is definable if the language contains a truth predicate, so that Existence is satisfied. Suppose next that $X \subseteq \Omega$ and X is definable. Then, firstly, suppose for *reductio* that $\delta(X) \in X$. Since $X \subseteq \Omega$, it follows that $\delta(X) \in \Omega$, and so $\delta(X)$ is true, and hence, since $\delta(X) = \delta(X) \notin X$, by naive truth $\delta(X) \notin X$. Therefore, the supposition that $\delta(X) \in X$ entails that $\delta(X) \notin X$, and so, by *reductio*, $\delta(X) \notin X$, so that Transcendence is satisfied. Secondly, we've just established that $\delta(X) \notin X$, and so, since $\delta(X) = \delta(X) \notin X$, by naive truth $\delta(X)$ is true, and hence $\delta(X) \in \Omega$, so that Closure is satisfied. Moreover, on the current interpretation, $\delta(\Omega)$ is tantamount to $\delta(\Omega) \notin \Omega$, and so what in effect it says is, of itself, that it is not true: therefore, the Inclosure contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ corresponds to the contradictory conclusion of the Liar paradox that the Liar sentence is not true and the Liar sentence is true. The Liar paradox is an Inclosure paradox.

As an important twist, recall that Curry's paradox can be understood as the apparently valid derivation of the consequent of the Curry sentence from the apparently true naive truth-theoretic principles. Now, if one takes e.g. the Curry sentence saying of itself that, if it is true, Graham is tall (Tg) and tries to fit the resulting version of Curry's paradox into the IS *by following exactly the same pattern*, in the argument for Transcendence from the supposition that $\delta(X) \in X$ it only follows that Tg holds, which, as things stand, provides no trigger for *reductio* (*cf* Weber et al. [2014], pp. 821–824).[8,9]

One can easily fix this by considering the following interpretation for ϕ, ψ and δ in the IS (which can then be taken to provide an IS-based framework for the analysis of *every* version of Curry's paradox):

- ϕ is the property of [being true if it were the case that (everything is just the same save for the consequences of the fact that) Tg fails to hold] (we'll henceforth use 'G' to express this supposition; for the purposes of this paper, the property of failing to hold (fn 8) can be represented as the property of implying absurdity);
- ψ is the property of being definable;

[7] Since X is definable, standard diagonalisation procedures—for one thing—guarantee the existence of some such sentence and so the existence of some such function.

[8] If one takes instead a Curry sentence with a consequent that fails to hold (where it might help at least some to note that, throughout, we understand '*fail* to ϕ' and its relatives as *ruling out* ϕing), the resulting version of Curry's paradox does fit into the IS by following exactly the same pattern (*cf* e.g. Zardini [2014], pp. 356–357, fn 11; in that paper as well as in this one, we don't delve into the question of what conclusions should be drawn if it were really the case that different versions of Curry's paradox diverge in whether they are Inclosure paradoxes, although we'd be inclined to say that, in that case, Inclosure paradoxes are not general enough to form a kind).

[9] Quite obviously, this kind of version of Curry's paradox refutes the traditional definition of paradox as a *situation where apparently true premises apparently entail an apparently false conclusion*, since it is no less paradoxical to prove the *clearly true* Tg simply by the naive truth-theoretical principles than it is to prove a sentence that *clearly fails to hold* simply by those principles (see López de Sa and Zardini [2007], p. 246; [2011], pp. 472–473; Zardini [2021b]; Oms [2023] for elaborations of the point and proposals concerning a better definition of paradox). Indeed, we'll see in the fourth next paragraph that there is a kind of version of the Sorites paradox that analogously refutes the traditional definition of paradox.

- δ is a function that, given a definable subset X of Ω, yields a sentence tantamount to $\delta(X) \in X \to Tg$.

On this interpretation, Ω is the set of sentences that, under G, would be true, which is definable if the language contains certain standard expressive resources, so that Existence is satisfied.[10] Suppose next that $X \subseteq \Omega$ and X is definable. Then, firstly, suppose for *reductio* that $\delta(X) \in X$. Since $X \subseteq \Omega$, it follows that $\delta(X) \in \Omega$, and so, working now under G, $\delta(X)$ is true, and hence, by naive truth, $\delta(X) \in X \to Tg$ holds, and thus, by *modus ponens*, Tg holds. But, since the conclusion that, under G, Tg holds is absurd, it entails everything and in particular that $\delta(X) \notin X$. Therefore, the supposition that $\delta(X) \in X$ entails that $\delta(X) \notin X$, and so, by *reductio*, $\delta(X) \notin X$, so that Transcendence is satisfied. Secondly, working again under G, by the assumption that membership facts for sets of sentences are *de re* necessary, the supposition that $\delta(X) \in X$ still implies that $\delta(X) \in \Omega$ and, by the same assumption, that still implies that (also outside of G $\delta(X) \in \Omega$ and so that, working again under G,) $\delta(X)$ is true, which in turn we've just established to entail that Tg holds. By the deduction theorem, under G, $\delta(X) \in X \to Tg$ holds, and so, by naive truth, under G, $\delta(X)$ is true, and hence $\delta(X) \in \Omega$, so that Closure is satisfied. Moreover, on the current interpretation, $\delta(\Omega)$ is tantamount to $\delta(\Omega) \in \Omega \to Tg$, and so, under G, what in effect it says is, of itself, that, if it is true, Tg holds: therefore, the Inclosure contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ corresponds to the contradictory (and indeed absurd) conclusion of the Tg-version of Curry's paradox that, under G, the Curry sentence is not true and the Curry sentence is true. And that in turn captures well the idea that what is paradoxical in the Tg-version of Curry's paradox is that although it would appear that Tg should not be provable simply by the naive truth-theoretical principles—and so although it would appear that it should be in a broad sense possible that Tg fails to hold while the naive truth-theoretical principles hold—even supposing for the sake of argument that Tg failed to hold we could apparently still absurdly establish that the truth of the Curry sentence entails Tg and the Curry sentence is true by the naive truth-theoretical principles (see Zardini [2021b] for details on the underlying conception of paradox). Therefore, there are reasons for thinking that, after all, contrary to what Weber et al. [2014], pp. 821–824 claim, Curry's paradox too is an Inclosure paradox. (Yet, keeping fixed contraction, the naive truth-theoretic principles and *modus ponens*, it can hardly be solved by accepting that, under G, (the Curry sentence is not true and) the Curry sentence is true, for, even under G, that absurdly entails Tg, and so the paradox is not open to a dialetheic solution at the level of the IS.)

Let's now turn to the Sorites paradox. To set up the paradox, suppose that P is a 1-place vague property and $a_0, a_1, a_2 \ldots, a_n$ is a *soritical series* for P: Pa_0 holds, $\neg Pa_n$ holds and P is apparently *tolerant* over the series—that is, the principle of *tolerance*:

TOL. For every i such that $0 \leq i < n$, if Pa_i holds, so does Pa_{i+1}

is apparently true. The Sorites paradox is then the apparently valid derivation of the apparently false Pa_n from the apparently true Pa_0 and the apparently true TOL (typically proceeding by repeated applications of *universal instantiation* and *modus ponens*).

Consider now the following interpretation for ϕ, ψ and δ in the IS:

- ϕ is P (henceforth understood as restricted to $A = \{a_0, a_1, a_2 \ldots, a_n\}$);
- ψ is a trivial property (say, selfidentity);
- δ is the function that, given any subset X of Ω, yields the first object that comes after every member of X.[11]

[10] To cover versions of Curry's paradox with *logical truths as consequents*, we're assuming the availability of a notion of counterfactual implication allowing for *nonvacuous counterlogicals*, and, to guarantee nevertheless the entailments applied in this argument, we're assuming a specific understanding of counterfactual implication that *keeps those fixed*.

[11] An object a_i "comes after every member of X" iff, for every j such that $i \leq j \leq n$, $a_j \notin X$. Notice that, since $\neg Pa_n$ holds, for every $X \subseteq \Omega$ the set of objects that come after every member of X contains a_n and so is *nonempty*, and, since it is also *finite*, δ seems well-defined (*cf* however fns 13, 14, 33). Notice also that our interpretation of δ is slightly different from Priest's *official* one, according to which $\delta(X)$ is the first object that does not belong to X: for example, if $X = \{a_1\}$, according to our interpretation

On this interpretation, Ω is the set of P objects, which is definable if the language contains a predicate for P, so that Existence is satisfied. Suppose next that $X \subseteq \Omega$. Then, firstly, by definition, $\delta(X) \notin X$, so that Transcendence is satisfied. Secondly, if $\delta(X) = a_0$, since Pa_0 holds $\delta(X) \in \Omega$. If $\delta(X) \neq a_0$ instead, for some i such that $0 \leq i < n$ it follows that $\delta(X) = a_{i+1}$ and $a_i \in X$;[12] since $X \subseteq \Omega$, it follows that Pa_i holds, and so, *by tolerance*, Pa_{i+1}—that is, $P\delta(X)$—holds, and hence $\delta(X) \in \Omega$, so that Closure is satisfied. Moreover, on the current interpretation, $\delta(\Omega)$ is in effect the first case of $\neg P$: therefore, the Inclosure contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ corresponds to the contradictory conclusion of the Sorites paradox that there is a first object that is $\neg P$ and such an object is P.[13] The Sorites paradox too is an Inclosure paradox (let's call the argument in this paragraph '*Sorites-paradox-as-Inclosure argument*', 'SPIA' for short).

As an interesting twist, Beall [2014b], pp. 846–848; Oms and Zardini [2019a], p. 8, fn 14 independently point out that the Sorites paradox is no less paradoxical if used to establish of a clear positive (negative) case that it is positive (negative). For example, it is no less paradoxical to establish that Graham is tall

$\delta(X) = a_2$ whereas according to Priest's official interpretation $\delta(X) = a_0$. However, given both some of the details of Priest's *argumentation* ("... (if $X \neq \emptyset$) $\delta(X)$ comes immediately after something in $X \subseteq \Omega$, so $P\delta(X)$, by tolerance", Priest [2010], p. 71) and his general *idea* of δ as taking us closer and closer to the limits of Ω (Priest [2010], pp. 70–72), for the purposes of this paper, we assume that our interpretation is also his *operative* one. Finally, notice that, in a context—such as Priest's—where contradictions in membership in X are a possibility, not even our interpretation of δ ideally captures that general idea, since, if the last member a_i of X is such that (Pa_i holds but also) $\neg Pa_i$ holds and indeed $a_i \notin X$, and Pa_{i+1} (as well as $\neg Pa_{i+1}$) also holds, even on our interpretation $\delta(X)$ might *get stuck* at a_i, and so it would not seem to take us closer to the *limits* of Ω (indeed, for the same reason, as soon as a nonlast member a_i of X is such that (Pa_i holds but also) $\neg Pa_i$ holds and indeed $a_i \notin X$, even on our interpretation $\delta(X)$ might *fall back* to a_i; see section 5 for some further discussion on firstness). This hitch could be addressed by alternatively (and more consonantly with the meaning of 'after') understanding that an object a_i "comes after every member of X" iff, for every j such that $a_j \in X$, it is the case that $i > j$ (which would also seem to be in effect the understanding adopted by Priest [2019], p. 149). (Throughout, we assume the validity of universal instantiation for restricted universal quantification; see Beall et al. [2006]; Zardini [2016] for two extended discussions of nonclassical restricted quantification that confirm the assumption.) For Priest's purposes, however, one problem with this third interpretation of δ would be that, there being a last a_k such that Pa_k is not false-only, in the argument for Closure every Y whose last member is a_k is also relevant (since then $Y \subseteq \Omega$), but $\delta(Y) \geq a_{k+1}$, and $P\delta(Y)$ is false-only, so that it is false-only that $\delta(Y) \in \Omega$, so that Closure is *not* satisfied. Thanks to Bogdan Dicher and an anonymous referee for pressing us on these differences.

[12] On either our interpretation of δ or Priest's official one (fn 11), the second conjunct makes clear that, throughout, "the first ϕ object" should be understood as the object a_i such that $\phi(a_i)$ holds and, for every j such that $j < i$, $\neg\phi(a_j)$ holds (*mutatis mutandis* for "the last ϕ object"; see section 5 for some further discussion on firstness). Importantly, on this understanding, δ would not after all be *well-defined* if, for the relevant property ϕ^X, there were an object a_i such that *neither* $\phi^X(a_i)$ *nor* $\neg\phi^X(a_i)$ *hold* (for then there would no longer be a guarantee that there is some *single* object that satisfies *both* conditions on firstness). Indeed, if there were enough objects a_i such that neither Pa_i nor $\neg Pa_i$ hold, Priest's argumentative strategy for establishing that the Sorites paradox instantiates the IS would face an obstacle: if $X = \Omega$, Transcendence would require δ to take us *all the way* to the $\neg P$ objects—striding in one fell swoop over the swath of P-gappy objects—*too far* for tolerance on the P objects then to reach us, contrary to what Closure requires of δ. Just as you expect, Priest's argumentative strategy can overcome this obstacle by taking Ω to be instead the set of objects that are *either P or P-gappy* (and exploiting the *vagueness* and so *apparent tolerance* of the corresponding property). And, just as you expect, such a strategy would face an analogous obstacle if, as is natural to think from a gappist perspective, there were also a gap between the objects that are either P or P-gappy and the objects that aren't. And so it goes, but only for a little while. For let P^0 be P and, for every $i \geq 1$, let P^i be the property of being either P^{i-1} or P^{i-1}-gappy. Then, since A is finite and, for every i, the extension of P^i is a subset of the extension of P^{i+1}, it should follow by anyone's lights that, for some i, for every $j \geq i$, the extension of P^i is identical with the extension of P^j and so P^j is not gappy. However, such a *fixed-point* P^i *is presumably still vague on* A (and so *does presumably still not hold of some object in* A), in which case Priest's argumentative strategy will finally go through without gaps by taking Ω to be the set of P^i objects. Since we thus regard the possibility of gaps as a complication that does not ultimately lead to the failure of Priest's argumentative strategy (and since in his own discussion Priest—as well as, more generally, theorists favouring a dialetheic approach to vagueness—anyway assumes the nonexistence of gaps), we'll henceforth set it aside. Thanks to an anonymous referee for an observation that prompted this fn.

[13] As far as we know, the point spelt out in the last sentence in the main text is not explicitly made by Priest. Yet, it seems crucial for showing that it is *the Sorites paradox* that is an Inclosure paradox; without it, what is shown is only that there is an Inclosure paradox involving P and TOL. Even with the point in place, the "paradox" to which the *Inclosure* (i.e. instantiation of the IS) just established corresponds actually *sounds rather odd*, for it would go something like: "Let the first object coming after every member of the set of P objects be a_i. Then, $\neg Pa_i$ holds. Yet, since a_i is immediately preceded by a P object, by TOL Pa_{i+1} also holds. Contradiction." Such a "paradox" evidently assumes that *there is a first object in a soritical series that does not belong to the corresponding vague set*, which, contrary to what is assumed in a normal Sorites paradox, is *not* a particularly compelling assumption. A much more natural way of relating the IS with the Sorites paradox would be by taking δ to be the function that, given any subset X of Ω, yields a_n, thereby establishing that Transcendence is satisfied, and using flat-footed soritical reasoning to establish that Closure is satisfied. The resulting Inclosure corresponds neatly to a normal Sorites paradox. (Yet, with regard to that Inclosure, one can hardly accept that ($\delta(\Omega) \notin \Omega$ and) $\delta(\Omega) \in \Omega$, for that is tantamount to the absurd Pa_n, and so a normal Sorites paradox is not open to a dialetheic solution at the level of the IS.) Having flagged this issue, we'll mostly set it aside and grant that the Inclosure just established does correspond to the Sorites paradox (see fns 14, 33 for some related discussion).

via soritical reasoning from the premise that Richard Kiel is tall than it is to establish that Danny DeVito is tall *via* soritical reasoning from the same premise.

Similarly to the case of Curry's paradox, one can easily fix this by considering the following interpretation for ϕ, ψ and δ in the IS (which can then be taken to provide an IS-based framework for the analysis of *every* version of the Sorites paradox):

- ϕ is the property of being of the form Pa_i and [holding if a_c were the first object that fails to be P] (we'll henceforth use 'C' to express that supposition; a_c is an arbitrary case simply such that $c > 1$);
- ψ is a trivial property (say, selfidentity);
- δ is a function that, given any subset X of Ω, yields the first sentence Pa_i such that, for every j such that $i \leq j \leq n$, Pa_j fails to belong to X.

On this interpretation, Ω is the set of sentences of the form Pa_i that, under C, would hold, which is definable if the language contains certain standard expressive resources, so that Existence is satisfied. Suppose next that $X \subseteq \Omega$. Then, firstly, by definition, $\delta(X) \notin X$, so that Transcendence is satisfied. Secondly, if $\delta(X) = Pa_0$, since, under C, Pa_0 holds $\delta(X) \in \Omega$. If $\delta(X) \neq Pa_0$ instead, for some i such that $0 \leq i < c$ it follows that $\delta(X) = Pa_{i+1}$ and $Pa_i \in X$; since $X \subseteq \Omega$, it follows that, under C, Pa_i holds, and so, by tolerance, under C, Pa_{i+1}—that is, $\delta(X)$—holds, and hence $\delta(X) \in \Omega$, so that Closure is satisfied. Moreover, on the current interpretation, under C, $\delta(\Omega)$ is in effect the first case of failing to be P (that is, a_c): therefore, the Inclosure contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ corresponds to the contradictory (and indeed absurd) result of the arbitrary-case version of the Sorites paradox that, under C, a_c is $\neg P$ and a_c is P. And that in turn captures well the (right or wrong, see fn 14) idea that what is paradoxical in the arbitrary-case version of the Sorites paradox is that although it would appear that Pa_c should not be provable simply by TOL even if, for every j such that $j < c$, Pa_j held—and so although it would appear that it should be in a broad sense possible that a_c is the first object that fails to be P while TOL holds—even supposing for the sake of argument that a_c were the first object that fails to be P we could apparently still absurdly establish Pa_c by TOL. Therefore, there are reasons for thinking that, after all, contrary to what Beall [2014b], p. 847 claims, also the arbitrary-case version of the Sorites paradox is an Inclosure paradox. (Yet, it can hardly be solved by accepting that, under C, ($\neg Pa_c$ and) Pa_c holds, for, under C, that is absurd, and so the paradox is not open to a dialetheic solution at the level of the IS.)[14]

3 Tolerance and *Modus Ponens*

Appealing to the PUS and the assumption that Inclosure paradoxes are specific enough to form one single kind, Priest [2003] advocates a uniform solution to the Inclosure paradoxes, which in turn he argues to be a *dialetheic* one on which the contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$ holds. Obviously, any such solution had better adopt a *paraconsistent* logic where the principle of *explosion* ($\phi, \neg\phi \vdash \psi$) is invalid. Under minimal assumptions (which hold in Priest's favoured paraconsistent logic **LP**,[15] for which see Asenjo [1966]), explosion is entailed by the principle of *disjunctive syllogism* ($\phi, \neg\phi \vee \psi \vdash \psi$), and so this must be invalid too. Since disjunctive syllogism is tantamount to the principle of *modus ponens* for *material* implication ($\phi, \phi \supset \psi \vdash \psi$), that means that, on Priest's dialetheic solution, *modus ponens* for material implication is invalid. We'll eventually argue that it is this last fact that, on Priest's approach to the Sorites paradox, makes the SPIA unsound. But, to do that, we first need to zoom in briefly on Priest's approach—which, being shared by most theorists favouring an **LP**-approach to vagueness (see

[14] As in fn 13, also the "paradox" to which the Inclosure just established corresponds actually sounds rather odd, for it would go something like: "Suppose that the first object failing to be P is a_c. Then, $\neg Pa_c$ holds. Yet, since a_c is immediately preceded by a P object, by TOL Pa_c also holds. Absurdity." Importantly, then, *mutatis mutandis* all the points made in this paragraph still go through for the much more natural way of relating the IS with the Sorites paradox presented in fn 13.

[15] To wit, these are essentially the principle of *addition* ($\phi \vdash \phi \vee \psi$) and a version of the principle of *transitivity of logical consequence* (if $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \psi$ hold, $\Delta, \Gamma \vdash \psi$ holds). Thanks to an anonymous referee for help in straightening out this fn.

Weber et al. [2014], p. 813, fn 1 for references), may be called '*the standard* **LP**-*approach*'—and rehearse some of its main features.

On the standard **LP**-approach, it is first claimed that *borderline cases* of a vague property are cases of contradictions' holding. It is then observed that, in **LP**, that suffices for TOL to hold *in material-implication version*. It is finally argued that *the best version of TOL is indeed the material-implication one*, in that it is e.g. claimed (*pace* Priest [2019], p. 149) that "what tolerance is all about" is to guarantee that neighbouring cases always have "the same truth value" (Priest [2010], p. 73).[16,17] The paradox is then blocked because TOL is true only in material-implication version, in which case the derivation becomes invalid as it involves *modus ponens* for material implication. Notice that, by talking of "the standard **LP**-approach", there aren't really other approaches in the vicinity (*i.e. nonsubstructural* ones)[18] we're thereby ignoring. For what is essential for the purposes of this paper is only that, on the standard **LP**-approach, TOL holds only in a version with an implication (never mind whether material or not) that does not satisfy *modus ponens*: keeping fixed the structural principles of *contraction* (if $\Gamma, \phi, \phi \vdash \psi$ holds, $\Gamma, \phi \vdash \psi$ holds) and *transitivity* (fn 15) of logical consequence, such a kind of approach is no idiosyncrasy of **LP**-theorists—it is virtually *the only possible one compatible with the instances of TOL holding in their generality* (and with the facts concerning clear positive and negative cases of vague properties).[19,20]

[16] Assuming for the time being that TOL in material-implication version in **LP** does guarantee that neighbouring cases always have "the same truth value" (fn 17), the claim is extremely doubtful. A main point of tolerance would seem to be *avoidance of sharp boundaries* (Zardini [2008b], pp. 30–39, 51–65; [2019b], pp. 170–171; [2021a]): the existence of a magical nanosecond at which one stops being a child is utterly rebarbative. But such avoidance is not achieved by TOL in material-implication version in **LP**, since, in general, in **LP**, $\phi \supset \psi$ does not rule out $\phi \,\&\, \neg\psi$. Another main point of tolerance would seem to be *conclusive inference about similar cases* (Zardini [2008b], pp. 39–51; [2019b], p. 170; [2021a]): given the information that abortion at a certain time after conception is permissible, we want to be able to infer without the shadow of a doubt that abortion one nanosecond later is permissible. But such an inference is not licenced by TOL in material-implication version in **LP**, since, in general, in **LP**, ϕ and $\phi \supset \psi$ do not entail ψ. (Priest [2010], pp. 73–74 claims that TOL is *less plausible* in a version with an implication satisfying *modus ponens*, a claim that is in itself odd and that is even more improbable given that, in view of the considerations just advanced, it would rather seem that TOL is *even more plausible* in a version with an implication satisfying *modus ponens*. In most cases of implications Priest considers to substantiate his claim, he relies on the *unpalatable* assumption that there is a last P object immediately followed by an object that fails to be P (*cf* fns 13, 14, 33) to show how TOL in a version with that implication then fails to hold, which does not say much against the plausibility of TOL in that version (in general, to assume unpalatably that ϕ holds but ψ fails to hold and show how 'If ϕ, then ψ' then fails to hold does not say much against the plausibility of that implication!). In the only case where he does not so rely, he appeals instead to the idea that Pa_{i+1} does not "logically follow" from Pa_i (*cf* Beall and Colyvan [2001], p. 405; Weber [2010], p. 1041; Weber et al. [2014], p. 820). But, as the context of Priest's discussion makes clear, the relevant sense of 'logically follow' is a *broadly conceptual* one, which allows e.g. for ''Snow is white' is true' to "follow logically" from 'Snow is white', and it's hard to see why, *in that sense*, 'Abortion is permissible one nanosecond after t' should not "logically follow" from 'Abortion is permissible at t'—if there is a logic of truth, there surely is also a logic of permission!)

[17] Notice that it is actually not clear that TOL in material-implication version in **LP** does guarantee that neighbouring cases always have "the same truth value": since, in **LP**, material implication allows for the antecedent to have as truth values truth and falsity and for the consequent to have as truth value only falsity, one is hard pressed to find a natural sense in which TOL in material-implication version in **LP** guarantees that neighbouring cases always have "*the same truth value*" (as opposed to "*at least one truth value in common*"). (Similarly, Weber et al. [2014], p. 816 claim that "the key intuition driving the sorites paradox" is that "consecutive members of the sorites sequence are equally true/false"; however, since again, in **LP**, material implication allows for the antecedent to have as truth values truth and falsity and for the consequent to have as truth value only falsity, one is hard pressed to find a natural sense in which TOL in material-implication version in **LP** guarantees that neighbouring cases always are "*equally true/false*" (as opposed to "*equal in at least one truth value*").)

[18] A logic is *nonsubstructural* iff it is not substructural. In turn, a logic is *substructural* iff it denies some of the *structural* principles of classical logic (*i.e.*, roughly, those principles valid in classical logic that *do not concern specific object-language expressions*; see Zardini [2021b] for some philosophical discussion).

[19] It should be noted though that the idea of embracing tolerance without detachment would seem to come to grief, at the latest, when we move from *implication* to *restricted universal quantification*. Just as 'For every i, if i is a small number, i is immediately followed by a small number' is an example of implicational tolerance, 'Every small number is immediately followed by a small number' is an example of restricted-universal-quantificational tolerance. Yet, by universal instantiation for restricted universal quantification, 'i is a small number' and 'Every small number is immediately followed by a small number' entail '$i+1$ is a small number', so that, keeping fixed contraction and transitivity of logical consequence, restricted-universal-quantificational tolerance must fail to hold. One might then wonder what the value is of vindicating 'For every i, if i is a small number, i is immediately followed by a small number' while jettisoning 'Every small number is immediately followed by a small number'.

[20] We hasten to add that there are indeed substructural approaches that, by denying either contraction or transitivity of logical consequence, are compatible with the instances of TOL holding in their generality (see Slaney [2011]; Zardini [2008a] respectively; see fn 33 for some more indications concerning the latter kind of approach and Zardini [2019b], pp. 173–176 for a battery of considerations favouring it over the former kind of approach). Thanks to an anonymous referee for criticism that led to a revision of the last point in the main text.

4 *Modus Ponens* and Inclosure

Go back now to the SPIA. While it was immediate that Transcendence is satisfied, some argument was needed to establish that Closure is satisfied. In the more interesting case where $\delta(X) \neq a_0$, let's grant that we do manage to get up to the point where Pa_i holds (see fn 29 for discussion). How is Pa_{i+1} supposed to follow from that? Priest [2010], p. 71 says "by tolerance" (*cf* Priest [2019], p. 149: "that's just tolerance"), but, as we've seen in section 3, on the standard **LP**-approach, the only true version of TOL only yields the material implication $Pa_i \supset Pa_{i+1}$, and, because of the invalidity of *modus ponens* for material implication, Pa_{i+1} does *not* follow from Pa_i and $Pa_i \supset Pa_{i+1}$! On the one hand, obviously, the argument is valid only if TOL is in a version that satisfies *modus ponens*, but then, on the standard **LP**-approach, any such version of TOL fails to be true; on the other hand, on the standard **LP**-approach, the argument has true premises only if TOL is in material-implication version, but then, on the standard **LP**-approach, the argument is invalid. Either way, on the standard **LP**-approach, the argument is unsound.[21]

To be crystal clear, our point is *not* that, on the standard **LP**-approach, *the Sorites paradox is not an Inclosure paradox*: as per sections 1, 2, in Priest's proprietary sense, for something to be an "Inclosure paradox" it is only required that it *apparently* instantiate the IS, and that might still be the case for the Sorites paradox (though see fns 13, 14 for some initial reason for doubting this). Our point is rather that, on the standard **LP**-approach, *the SPIA is unsound* and, plausibly assuming that there are no substantially better arguments for the same conclusion, (the default position stands that) *the Sorites paradox does not instantiate the IS*. In other words, while the Sorites paradox might be an *Inclosure paradox*, it is not an *Inclosure*.

We take such a *logical-metaphysical* point to be in itself more significant than the *psychological-epistemological* point that the Sorites paradox apparently instantiates the IS (as a consequence of the more general fact that the point that something is not the case is more significant than the point that, apparently, it is). More specifically, the logical-metaphysical point shows that, while the standard **LP**-approach can still maintain that there are true contradictions flowing from the *symmetry* with respect to P exhibited by *borderline cases*, or even that there are true contradictions flowing from the *negativity* imposed by *TOL in material-implication version* on positive cases of P to avoid their spread to neighbouring cases,[22] it cannot maintain that there are true contradictions flowing from the *extensibility* of P afforded by *TOL in detachable-implication version*.[23]

The point also has an interesting repercussion for Priest's overall position (as well as for those positions sharing its relevant tenets, such as those of Weber [2010]; Weber et al. [2014]): although both the Liar paradox and the Sorites paradox are Inclosure paradoxes, and although Priest thinks that, by the PUS and the assumption that Inclosure paradoxes are specific enough to form one single kind, Inclosure

[21] While preparing the final version of this paper, we discovered a parenthetical remark in Beall [2014a], p. 801 to the effect that "[. . .] the [. . .] inclosure-structure argument for the sorites [. . .] relies—for the transcendence condition—on detachable tolerance". We suppose that 'transcendence' is a slip of the pen for 'closure', in which case Beall would in effect be making the same point we're making in this paragraph. Even so, however, Beall only mentions it in passing, neither expanding on the noteworthy consequences of this fact nor examining possible ways to resist the point, contrary to what we proceed to do in the rest of the paper.

[22] To elaborate, on the first scheme, one can take a borderline case a_i of P, note that there seem to be just as strong reasons in favour of its being P as there are in favour of its being $\neg P$, assume that such reasons are strong enough and therefore conclude to $Pa_i \,\&\, \neg Pa_i$. On the second scheme, one can start from the clearly true Pa_0 and from $Pa_0 \supset Pa_1$ (*i.e.* $\neg Pa_0 \vee Pa_1$) to infer $(Pa_0 \,\&\, \neg Pa_0) \vee Pa_1$, use analogous reasoning to develop that second disjunct into $(Pa_1 \,\&\, \neg Pa_1) \vee Pa_2$, use analogous reasoning to develop that second disjunct into $(Pa_2 \,\&\, \neg Pa_2) \vee Pa_3 \ldots$, use analogous reasoning to develop that second disjunct into $(Pa_{n-1} \,\&\, \neg Pa_{n-1}) \vee Pa_n$, use the clearly true $\neg Pa_n$ to develop that second disjunct into $Pa_n \,\&\, \neg Pa_n$ and therefore conclude to $(Pa_0 \,\&\, \neg Pa_0) \vee (Pa_1 \,\&\, \neg Pa_1) \vee (Pa_2 \,\&\, \neg Pa_2) \ldots \vee (Pa_n \,\&\, \neg Pa_n)$. Thanks to Graham Priest and an anonymous referee for reminding us of the importance of the second scheme in this context.

[23] Notice that, unsurprisingly, what plays a role in a few passages of Priest's and others' argumentation is the logical-metaphysical claim that the Sorites paradox instantiates the IS rather than the psychological-epistemological claim that it apparently does, and that, more specifically, that claim is appealed to as the explanation for why some contradictions hold along a soritical series (e.g. Priest [2010], p. 71: "Diagonalization takes us out of X, and tolerance keeps us within Ω. We see why a contradiction occurs at the limit of P things"; Weber [2010], p. 1038: "From a sorites chain, an inclosure contradiction arises"; Weber et al. [2014], p. 822: "In our glut-based theories, this is the role that contradictions arising out of the inclosure schema play. Inconsistency is an emergent property of an inclosure structure, and is itself the explanation for the entire sorites phenomenon. The structure of an inclosure explains why contradictions occur at the vague penumbra").

paradoxes should receive a solution of the same kind, in the case of the Liar paradox he accepts that Existence, Transcendence and Closure are satisfied—and so, in that case, accepts the contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$—whereas in the case of the Sorites paradox, as we've been arguing, he *lacks a good reason* to accept that Closure is satisfied—and so, in that case, *lacks a good IS-based reason* to accept the contradiction that $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$. Therefore, by maintaining that the Liar paradox and the Sorites paradox, *qua* being of the same kind *at the level of the IS*, should by the PUS receive a solution of the same kind, but then adopting a framework that cannot give the two paradoxes a solution of the same kind *precisely at the level of the IS*, Priest's overall position would seem to border on the incoherent.[24]

5 A Logical Fact and Two Failed Tweaks

Priest [2006], p. 119 proves the *classicality-or-contradiction fact* ('CCF' for short) to the effect that, if Γ entails ϕ in classical logic, then, for some sentence ψ, Γ entails $\phi \vee (\psi \,\&\, \neg\psi)$ in **LP**. There are at least two ways in which one could try to use the CCF to resist our point in section 4, which we take in turn.[25]

Firstly, and less interestingly, one could be inspired by the CCF to tweak the IS. Priest [2003], p. 130, fn 7 himself suggests, roughly, to reformulate Transcendence to the effect that, for some χ, $\delta(X) \notin X \vee (\chi \,\&\, \neg\chi)$ holds and Closure to the effect that, for some χ, $\delta(X) \in \Omega \vee (\chi \,\&\, \neg\chi)$ holds, respectively.[26] As far as we know, Priest himself has not insisted much on this reformulation. Setting aside its glaring *adhocness*[27] (and, relatedly, the facts that its instantiation would no longer explain *why contradictions arise in the first place* (*cf* Priest [2006], p. 135) and that, from the current point of view, since the Liar paradox instantiates the original IS whereas the Sorites paradox only instantiates its watered-down reformulation, that should mark *a deep logical-metaphysical difference* between the two anyway), one natural reason for this is that, if a dialetheic approach is correct of anything, there is indeed a χ such that $\chi \,\&\, \neg\chi$ holds (e.g., according to Priest, the Liar sentence), and so, since the truth of a disjunct suffices for the truth of a disjunction, Transcendence and Closure are satisfied by *absolutely any* choice of ϕ, ψ and δ, thereby utterly trivialising the IS.[28]

[24] Of course, at some other levels, Priest's overall position does give the two paradoxes the same solution: for one example, in the case of both paradoxes, the position accepts contradictions; for another example, in the case of both paradoxes, the position adopts an **LP**ish logic (though the starring implication is different in the two cases). These and suchlike levels are however much more general than the very specific level at which the IS works.

[25] As an anonymous referee suggested to us, one could also use the CCF to offer a version of the argument in fn 22 to the effect that, given TOL in material-implication version, some contradictions hold. We take no quarrel with such an argument's being available to the standard **LP**-approach: our point is that, from the point of view of that approach, the Sorites paradox is not an *Inclosure* structure—*not* that, from that point of view, it is not a *contradictory* structure. For what it's worth, we note that, if the aim is merely to offer an **LP**-*valid argument to the effect that, given TOL in material-implication version, some contradictions hold*, one could simply observe that, in **LP**, Pa_0 and $\neg Pa_n$ jointly entail as a matter of course the negation of TOL, and be already home and dry.

[26] Thanks to an anonymous referee for alerting us to the relevance of this passage.

[27] To be clear, the reformulation is not *ad hoc with respect to our point in section 4 in particular* (for one thing, it's much older than it is!). It is *ad hoc* rather in the following *general* sense. On the one hand, as far as we know, absolutely no reason has been given for such dramatic weakening in the conditions for instantiating the IS. On the other hand, if one were in the business of defending a dialetheic approach to a certain intended range of paradoxes while arguing that all the paradoxes in that range exhibit a certain feature, and were then worried that that might be too strong in that some paradoxes in the range might not exhibit that feature, the absolutely failsafe weakening to get around such a problem would be to allow that it is sufficient for exhibiting the feature that some contradiction or other holds (since that is still going to be the case no matter what the details of one's dialetheic approach to the problematic paradoxes are!)—and that's exactly what the weakening in question amounts to. It is this combination of—on the one hand—such *lack of reason* for a weakening that—on the other hand—so *brutally insulates the resulting theory from the relevant kind of counterexample* that makes the reformulation in question *ad hoc*. Thanks to an anonymous referee for a comment that brought about this clarification.

[28] A natural defensive move at this juncture would be to strengthen a bit the reformulation by requiring that χ *concern the paradox's subject matter*. Alas, such strengthening is still very far from being strong enough on at least two counts. Firstly, if a dialetheic approach is correct for a certain paradox, then, for some χ concerning the paradox's subject matter, $\chi \,\&\, \neg\chi$ holds, and so, even on the strengthened version of the reformulation, that paradox instantiates the IS. While on the original version of the reformulation *everything whatsoever*—insofar as a dialetheic approach is correct for *anything*—is an Inclosure, on the strengthened version of the reformulation *every paradox whatsoever*—insofar as a dialetheic approach is correct for *it*—is an Inclosure. Secondly, relinquishing provisos concerning the correctness of a dialetheic approach, according to the traditional definition—by which we'll here abide (see fn 9 for some critical remarks that hardly affect however the substance of the point we're about to make)—a paradox is a situation where apparently true premises apparently entail an apparently false conclusion χ (concerning the paradox's subject

Secondly, and more interestingly, one could be inspired by the CCF to tweak the SPIA. To warm up, notice that, henceforth assuming that *subsethood* is best understood in terms of material implication, *modus ponens* for material implication was already implicitly applied in the argument for Transcendence in the case of the Liar paradox (where the *reductio* argument inferred that $\delta(X) \in \Omega$ from the suppositions that $\delta(X) \in X$ and that $X \subseteq \Omega$). Therefore, that argument too was strictly speaking invalid. But it can easily be fixed by appealing to the CCF and to its more specific *modus-ponens* version ('MPCF' for short) to the effect that ϕ and $\phi \supset \psi$ entail $\psi \vee (\phi \,\&\, \neg\phi)$ in **LP**. For, by the MPCF, the suppositions that $\delta(X) \in X$ and that $X \subseteq \Omega$ entail that either $\delta(X) \in \Omega$ or $\delta(X) \in X \,\&\, \delta(X) \notin X$ holds. The first disjunct delivers $\delta(X) \notin X$ as per the original argument and the second disjunct delivers the same conclusion even more straightforwardly. Therefore, reasoning by cases (which does hold in **LP**), the suppositions that $\delta(X) \in X$ and that $X \subseteq \Omega$ still entail that $\delta(X) \notin X$. The argument for Transcendence can then still conclude by *reductio* that $\delta(X) \notin X$.

Can we similarly use the MPCF to fix the SPIA? Pa_i and $Pa_i \supset Pa_{i+1}$ now entail that either Pa_{i+1} holds or $Pa_i \,\&\, \neg Pa_i$ holds.[29] The question is now *how to rule out the second disjunct*. To our mind, the most tempting thought on how to do this might well go something like: "$Pa_i \,\&\, \neg Pa_i$ entails $\neg Pa_i$ which in turn may be assumed to entail that $a_i \notin \Omega$. Since $X \subseteq \Omega$, that entails that $a_i \notin X$, which cannot be the case since, by definition, $\delta(X)$ yields the *first* object (in $a_0, a_1, a_2 \ldots, a_n$) that comes after every member of X and, by supposition, that object is a_{i+1} rather than a_i". Alas, the dialetheist has two conclusive routine reasons for not yielding to such a thought (and so it is not surprising that, as far as we know, no dialetheist has in fact yielded to such a thought).

First, the thought crucially relies on an understanding of firstness such that the first such-and-such object "really is" the first such-and-such object, in the sense of *ruling out* that it is preceded by such-and-such objects (for the thought seeks to rule out something that implies that a_i comes after every member of X on the grounds that a_{i+1} is the first object that comes after every member of X). But, for better or worse, the dialetheist cannot understand firstness this way, since, even on that understanding (and barring the issue coming up in fns 13, 14, 33), the question of what object is the first $\neg P$ object would be *just as vague as* the question of what objects are P, but, contrary to the vagueness in the latter question, the vagueness in the former question cannot even start to be accounted for in terms of *contradictions*. For suppose that a_j is the first $\neg P$ object and a_j is not the first $\neg P$ object. Then, by the first conjunct, a_j is $\neg P$, which, together with the second conjunct, presumably entails that a_j is preceded by $\neg P$ objects, precisely what the first conjunct rules out.[30] Firstness is Boolean negation in sheep's clothing.[31]

matter). Therefore, every paradox is a situation where a sentence—χ—concerning its subject matter is apparently true (because of the apparently true premises and their apparent entailment to χ) and apparently false, and so, assuming that contradictions can apparently hold (as we must do for the reformulation of the IS to make dialectical sense), where $\chi \,\&\, \neg\chi$ apparently holds, and hence, even on the strengthened version of the reformulation, every paradox is an Inclosure paradox. While on the original version of the reformulation *everything whatsoever* is in effect an Inclosure paradox, on the strengthened version of the reformulation *every paradox whatsoever* is in effect an Inclosure paradox. A further natural defensive move at this further juncture would be to strengthen even more the reformulation by requiring that $\chi \,\&\, \neg\chi$ *be a genuine (i.e. relevant) alternative to its being the case that* $\delta(X) \notin X$ ($\delta(X) \in \Omega$). Alas, even such strengthening is still very far from being strong enough since a judicious choice of ϕ, ψ and δ allows us to reproduce the two problems just presented (for example, in the case of Closure, without going into many details, define δ so that it includes the proviso that, if $\chi \,\&\, \neg\chi$ fails to hold, $\delta(X) \in \Omega$, where the 'if' is relevant and χ is as in the two problems just presented). No doubt further epicycles could be tried out, but we take the foregoing considerations to be indicative of the formidable difficulties faced by the present defensive strategy. Thanks to Graham Priest for discussion of the move from which this fn starts off.

[29] Just before the conclusion that Pa_{i+1} holds, the SPIA also applies *modus ponens* for material implication in implicitly inferring from the suppositions that $a_i \in X$ and that $X \subseteq \Omega$ to the intermediate conclusion that $a_i \in \Omega$ (which is in turn what underwrites the explicit intermediate conclusion that Pa_i holds). We can indeed use the MPCF to fix *that* application of *modus ponens* for material implication, by appealing to the plausible assumption that, in the idealised situation of the Sorites paradox for P, the only contradictory cases of membership ultimately concern the set of P objects (*i.e.* Ω): by the MPCF, the suppositions that $a_i \in X$ and that $a_i \in X \supset a_i \in \Omega$ holds entail that either $a_i \in \Omega$ or $a_i \in X \,\&\, a_i \notin X$ holds, and so, by the assumption just mentioned and reasoning by cases, $a_i \in \Omega$.

[30] Essentially the same point can be made by defining '*P-only*' as 'P and preceding the first $\neg P$ object' and then considering the vagueness of 'P-only'.

[31] As a result, the dialetheist has to make do with an understanding of firstness under which *a* first such-and-such object can be preceded by *other* first such-and-such objects (something along the lines of this result—but not the argument for it we've just given in the main text—is extensively defended and philosophically exploited by Weber [2010]). While we (mostly) leave for further discussion the merits of such evangelical (*Matthew* 20:16) understanding of firstness (which, among other things, by universal

Second, the thought crucially relies on inferring that $a_i \notin X$ from the premise that $a_i \notin \Omega$ and the supposition that $X \subseteq \Omega$. Now, by the properties of material implication, the supposition that $X \subseteq \Omega$ does entail $a_i \notin \Omega \supset a_i \notin X$, but to get from that and the premise that $a_i \notin \Omega$ to the conclusion that $a_i \notin X$ requires again the validity of *modus ponens* for material implication.[32]

6 Conclusion

We conclude that, pending further argument, it cannot be shown by dialetheically acceptable means that Closure is satisfied in the case of the Sorites paradox. For all a dialetheist knows, the next grain of sand may well break the enclosure.[33]

References

Florencio Asenjo. A calculus of antinomies. *Notre Dame Journal of Formal Logic*, 7:103–105, 1966.

JC Beall. Finding tolerance without gluts. *Mind*, 123:791–811, 2014a.

JC Beall. End of inclosure. *Mind*, 123:829–849, 2014b.

JC Beall and Mark Colyvan. Heaps of gluts and Hyde-ing the sorites. *Mind*, 110:401–408, 2001.

JC Beall, Ross Brady, Allen Hazen, Graham Priest, and Greg Restall. Relevant restricted quantification. *Journal of Philosophical Logic*, 35:587–598, 2006.

instantiation for restricted universal quantification, must reject and cannot accept the platitude that *the first such-and-such object precedes every other such-and-such object*), we note that the ensuing "*plurality of first objects*" makes it problematic to *define a function* in terms of "the first such-and-such object" (as is done in the definition of δ in the SPIA).

[32] It might be useful to see how the glitch we've identified is realised in the situation of the Sorites paradox, and then observe how, in that situation, the tempting thought supposed to fix the glitch concretely fails twice to do so by dialetheic lights. As for the glitch, on the standard **LP**-approach, there is a last a_k such that Pa_k is not false-only. The glitch with the SPIA is then that, in the argument for Closure, every Y whose last member is a_k is also relevant (since then $Y \subseteq \Omega$), but it could very well be that $\delta(Y) = a_{k+1}$, and Pa_{k+1} is false-only, so that it could very well be false-only that $\delta(Y) \in \Omega$. (We write 'could very well be' instead of 'is' because of the plurality of first objects countenanced by the dialetheist (fn 31). However, we can indeed force that $\delta(Y) = a_{k+1}$ by considering the case where $Y = \{a_0, a_1, a_2 \ldots, a_k\}$ (or where Y is any similarly specified subset of A whose last member is a_k), since we can plausibly assume that, in the idealised situation of the Sorites paradox for P, the *enumeration-rules-out-not-belonging principle* ('ERBP' for short) holds according to which *it is false-only that an object does not belong to a set specified by an enumeration where the object occurs*.) In that case, $\delta(Y)$ breaks through the Inclosure, and, since Pa_k and $Pa_k \supset Pa_{k+1}$ are true, that's where *modus ponens* is not only *invalid*, but it also *effectively leads from true premises to a false-only conclusion*. As for the tempting thought supposed to fix the glitch, first, observe that, taking as a simple example the case where $X = \Omega$, for every j such that Pa_{j-1} and $\neg Pa_{j-1}$ hold, a_j comes after every member of X while, for every l such that $0 \leq l < j$, a_l does not, and so a_j is a first object that comes after every member of X. Therefore, $\delta(X)$ (*i.e.* a_{i+1}) could very well be any such a_j, all of which choices, including a_{k+1}, far from ruling out that a_i comes after every member of X or that $Pa_i \,\&\, \neg Pa_i$ holds, entail those claims. Second, observe that, taking as a simple example the case where $X = \{a_0, a_1, a_2 \ldots, a_k\}$, by the ERBP, $\delta(X)$ (*i.e.* a_{i+1}) is a_{k+1}, so that a_i is a_k. Therefore, $a_i \notin \Omega$ (since $\neg Pa_k$ holds) and $a_i \notin \Omega \supset a_i \notin X$ holds (since Pa_k holds), whereas, by the ERBP, it is false-only that $a_i \notin X$, so that $Pa_i \,\&\, \neg Pa_i$ does not entail that a_i comes after every member of X.

[33] In a sense, our considerations point towards a further respect, additional to those touched on in fn 16, in which it is crucial that tolerance is available *as a valid argument* rather than merely *as a true material implication*. There are indeed *nondialetheic* theories in which tolerance is so available, and which block the Sorites paradox essentially by restricting the transitivity of logical consequence rather than the law of noncontradiction (e.g. Zardini [2008a]). Since such theories are nondialetheic, the SPIA must break down in them, and yet, since tolerance is a valid argument in such theories, the relevant step in the argument for Closure in the SPIA is valid in them. So where does the SPIA break down in these theories? It presumably breaks down *at the very definition of* δ, since the set of P objects is an (improper) subset of Ω but, *by tolerance*, there is no last object that belongs to it, and so no first object that comes after every member of it (and, contrary to the case discussed in fn 12, there is no obvious move available to overcome this obstacle). This circumstance brings out the interesting fact that these theories would seem committed to the claim that *there is a subset of a set specified by enumeration that is different from any subset of that set that is specified by enumeration* (Priest [2010], p. 82, n. 17 would seem to agree), and, letting a totally ordered set be *bounded* iff it has a maximum member, to the claim that *there is a subset of a totally ordered finite (and so bounded) set that is unbounded*. Further investigation of these claims lies however beyond the limits of this paper.

Mark Colyvan. Vagueness and truth. In Heather Dyke, editor, *From Truth to Reality: New Essays in Logic and Metaphysics*, pages 29–40. Routledge, London, 2009.

Dan López de Sa and Elia Zardini. Truthmakers, knowledge and paradox. *Analysis*, 67:242–250, 2007.

Dan López de Sa and Elia Zardini. No-no. Paradox and consistency. *Analysis*, 71:472–478, 2011.

Sergi Oms. Some remarks on the notion of paradox. *Acta Analytica*, 38:211–228, 2023.

Sergi Oms and Elia Zardini. An introduction to the sorites paradox. In Sergi Oms and Elia Zardini, editors, *The Sorites Paradox*, pages 3–18. Cambridge University Press, Cambridge, 2019a.

Sergi Oms and Elia Zardini, editors. *The Sorites Paradox*. Cambridge University Press, Cambridge, 2019b.

Graham Priest. The structure of the paradoxes of self-reference. *Mind*, 103:25–34, 1994.

Graham Priest. Yablo's paradox. *Analysis*, 57:236–242, 1997.

Graham Priest. *Beyond the Limits of Thought*. Oxford University Press, Oxford, 2nd edition, 2003.

Graham Priest. *In Contradiction*. Oxford University Press, Oxford, 2nd edition, 2006.

Graham Priest. Inclosures, vagueness, and self-reference. *Notre Dame Journal of Formal Logic*, 51:69–84, 2010.

Graham Priest. Vague inclosures. In Koji Tanaka, Francesco Berto, Edwin Mares, and Francesco Paoli, editors, *Paraconsistency: Logic and Applications*, pages 367–377. Springer, Dordrecht, 2013.

Graham Priest. Dialetheism and the sorites paradox. In Sergi Oms and Elia Zardini, editors, *The Sorites Paradox*, pages 135–150. Cambridge University Press, Cambridge, 2019.

John Slaney. A logic for vagueness. *The Australasian Journal of Logic*, 8:100–134, 2011.

Zach Weber. A paraconsistent model of vagueness. *Mind*, 119:1025–1045, 2010.

Zach Weber, David Ripley, Graham Priest, Dominic Hyde, and Mark Colyvan. Tolerating gluts. *Mind*, 123:813–828, 2014.

Stephen Yablo. Truth and reflection. *Journal of Philosophical Logic*, 14:297–348, 1985.

Elia Zardini. A model of tolerance. *Studia logica*, 90:337–368, 2008a.

Elia Zardini. *Living on the Slippery Slope. The Nature, Sources and Logic of Vagueness*. PhD thesis, Department of Logic and Metaphysics, University of St Andrews, 2008b.

Elia Zardini. Naive truth and naive logical properties. *The Review of Symbolic Logic*, 7:351–384, 2014.

Elia Zardini. Getting one for two, or the contractors' bad deal. Towards a unified solution to the semantic paradoxes. In Theodora Achourioti, Kentaro Fujimoto, Henri Galinon, and José Martínez, editors, *Unifying the Philosophy of Truth*, pages 461–493. Springer, Dordrecht, 2015.

Elia Zardini. Restriction by noncontraction. *Notre Dame Journal of Formal Logic*, 57:287–327, 2016.

Elia Zardini. Instability and contraction. *Journal of Philosophical Logic*, 48:155–188, 2019a.

Elia Zardini. Non-transitivism and the sorites paradox. In Sergi Oms and Elia Zardini, editors, *The Sorites Paradox*, pages 168–186. Cambridge University Press, Cambridge, 2019b.

Elia Zardini. *Seconde naïveté*. In Otávio Bueno and Ali Abasnezhad, editors, *On the Sorites Paradox*. Springer, Dordrecht, 2021a. Forthcoming.

Elia Zardini. Substructural approaches to paradox. *Synthese*, 2021b. Forthcoming.