Inverse Operations With Transfinite Numbers And The Kalām Cosmological Argument

In 'Reply To Smith: On The Finitude Of The Past'[i], Professor William Craig writes: 'I reiterate that Smith has yet to deal with my strongest arguments in favour of the impossibility of the existence of an actual infinite, those based on inverse operations performed with transfinite numbers'[ii].

I think that this claim is mistaken; for: (i) there is no problem about allowing the inverse operations in question -- subtraction, division, extracting roots, etc. -- into transfinite ordinal arithmetic[iii]; and (ii) there is no problem about the exclusion of these operations from transfinite cardinal arithmetic. I shall take up these points in turn.

(i): In connection with transfinite ordinal arithmetic, we can begin by observing that in Cantorian formulations of transfinite arithmetic, the inverse operations in question are (conventionally) prohibited or excluded. But there are many non-Cantorian formulations of transfinite arithmetic in which the inverse operations are allowed. I shall briefly mention three.

In Abraham Robinson's Non-Standard Analysis[iv], the non-standard models which are introduced contain lots of transfinite elements -- i.e. elements which are larger than all the finite real numbers. Nonetheless, the models in question are (non-Archimedean) fields in which all non-zero elements have multiplicative inverses, and all elements have additive inverses.

In E. Nelson's Internal Set Theory[v], there are 'illimited numbers' which are larger than all the standard numbers; hence, there is a sense in which these numbers are transfinite. Nonetheless, the numbers in Nelson's theory all have all the relevant properties which are possessed by the standard numbers -- e.g. additive inverses and, except in the case of zero, multiplicative inverses.

To these examples, it might be objected that the numbers and elements in question are not really transfinite. In particular, it might be said, the classical claim that 'All numbers are finite' is still true within both of these theories.[vi] In my view, this suggestion is mistaken: there is no reason to suppose that the interpretation of the sentence 'All numbers are finite' within the context of these theories coincides with any intuitive interpretation of that claim. However, since the correct method for dealing with 'Skolem's Paradox' is perhaps still controversial, I shall not try to insist on this point here.[vii]

My third example is subject to no such difficulties. In J. H. Conway's On Numbers And Games, a construction for numbers is given which includes all of Cantor's ordinals, and yet under which the numbers form a totally ordered field. Among the numbers, in Conway's construction, there are such things as w-1, w/2, w1/2, w1/w, etc. Conway's elegant construction marries the work of Dedekind and Cantor in order to generate a transfinite ordinal arithmetic which is not subject to the 'deficiencies' which Craig decries in the standard Cantorian transfinite ordinal arithmetic. Moreover, since there is no doubt that this construction contains all of Cantor's transfinite ordinals, there is no question that it generates a genuinely transfinite ordinal arithmetic.[viii]

As Peirce noted long ago[ix], Cantor had a misguided prejudice against infinitesimals -- and, at least in part, it was this prejudice which prevented him from providing a truly adequate account of transfinite ordinal arithmetic. However, a modern defender of 'the possibility of the existence of an actual infinite' need not follow Cantor in this respect. Consequently, a modern defender of 'the possibility of the existence of an actual infinite' need not be swayed at all by this version of what Craig calls his 'strongest argument'.

(ii): In connection with transfinite cardinal arithmetic, we may again begin by observing that, in the Cantorian formulation of transfinite cardinal arithmetic, the inverse operations in question are indeed conventionally prohibited. Moreover, this time, we can also concede that there is no way in which the transfinite cardinals can be made to obey the conventional laws of arithmetic. But it is quite unclear why it should be supposed that this is any objection to the theory.

As far as I know, Craig nowhere explains what the difficulty is supposed to be. In Craig (1979), he seems to suppose -- though he offers no argument for this claim -- that the fact that the inverse operations must be conventionally prohibited in the case of transfinite cardinal numbers shows that the system is 'purely conceptual', i.e. has no application to the physical world.[x] But why should it be a requirement on transfinite cardinal numbers that these inverse operations be definable for them? That the operations can be defined for the finite case is irrelevant: there is no iron law which says that infinite numbers must behave just like finite numbers. Indeed, a defender of 'the possibility of the existence of an actual infinite' might well insist that this is just what one ought to expect; especially since it is not really clear that the standard arithmetic operations -- i.e. addition and multiplication -- apply to the transfinite cardinals. As Craig himself notes, 'addition' for transfinite cardinals is defined independently of the definition of addition for finite cardinals; consequently, it need hardly be surprising that there is no corresponding operation of 'subtraction'. In this respect, the transfinite cardinals are quite different from the transfinite ordinals: addition, multiplication, etc. are defined once and for all in Conway's system: i.e. transfinite ordinals are there added, divided, multiplied, etc. in exactly the same sense in which finite ordinals are added, divided, multiplied, etc.

Once again, I conclude that a defender of 'the possibility of the existence of an actual infinite' need not be swayed at all by this version of what Craig calls his 'strongest argument'. Consequently, I conclude that Craig's 'strongest argument' does nothing to advance his attempts to defend the claim that Kalām cosmological arguments are probative.

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[i] International Philosophical Quarterly 33.2, June 1993, pp.225-229; hereafter referred to as Craig (1993).

[ii] Craig (1993), p.230. Similar remarks may be found elsewhere in Craig's work: e.g., in 'Graham Oppy On The Kalām Cosmological Argument' Sophia 32.1, 1993, pp.1-11, at p.3: '[N]either Mackie nor Oppy has addressed the contradictions entailed by inverse arithmetic operations performed with transfinite numbers, operations which are conventionally prohibited in transfinite arithmetic in order to preserve logical consistency'.

[iii] It is not clear to me whether Craig wants to press his objection in connection with the transfinite ordinals. In The Kalām Cosmological Argument London: MacMillan, 1979 -- hereafter Craig (1979) -- at p.75, he says: 'The purely theoretical nature of the actual infinite becomes clear when one begins to perform arithmetic calculations with infinite numbers', and then goes on to discuss the transfinite ordinals. However, at p.80, op. cit., he writes: 'The purely conceptual nature of this system is nowhere clearer than when one begins to perform arithmetic operations with transfinite cardinals. ... The most interesting feature of transfinite cardinal arithmetic, however, is that the inverse operations ... subtraction and addition, cannot be performed.'

[iv] London: Academic Press, 1976

[v] 'Internal Set Theory: A New Approach To Non-Standard Analysis' Bulletin Of The American Mathematical Society 83, 1977, pp.1165-1198

[vi] See, for example, Alain Robert (1988) 'Non-Standard Analysis' Essex: John Wiley and Sons, at p.17: 'Indeed, the classical sentence 'All natural numbers are finite' is still true'.

[vii] See Paul Benacerraf (1985) 'Skolem and the Sceptic' Proceedings Of The Aristotelian Society (Supp. Vol.) 59, pp.85-115, for what I take to be an adequate treatment of these matters.

[viii] This indicates another advantage of Conway's system over traditional rivals, in which subsystems of the numbers -- e.g. positive integers, negative integers, positive rationals, negative rationals, positive reals, negative reals -- are defined separately, and the operations on these subsystems are defined by a kind of analogical extension.

[ix] 'The Law of Mind' Monist 2, 1891, pp.533-559, at pp.537ff.

[x] See, e.g., pp.80-82.