

# On superstable groups with residual properties

Abderezak OULD HOUCINE

## Abstract

Given a pseudovariety  $\mathcal{C}$ , it is proved that a residually- $\mathcal{C}$  superstable group  $G$  has a finite series  $G_0 \trianglelefteq G_1 \cdots \trianglelefteq G_n = G$  such that  $G_0$  is solvable and each factor  $G_{i+1}/G_i$  is in  $\mathcal{C}$  ( $0 \leq i \leq n-1$ ). In particular a residually finite superstable group is solvable-by-finite and if it is  $\omega$ -stable then it is nilpotent-by-finite. Given a finitely generated group  $G$ , we show that if  $G$  is  $\omega$ -stable and satisfies some residual properties (residual solvability, residual finiteness,...), then  $G$  is finite.

## 1 Introduction

In this paper we are concerned with groups satisfying some residual properties under model-theoretic assumptions as the  $\omega$ -stability and the superstability. There has been a considerable work in the study of residually finite groups and more generally residually- $\mathcal{C}$  groups when  $\mathcal{C}$  is a pseudovariety, but it seems that there is not a lot of literature on residually- $\mathcal{C}$  groups under model-theoretic considerations. We notice that it was shown in [3, Exercice 7, Chapter 5], that a residually finite group of finite Morley rank is abelian-by-finite.

A subgroup  $H \leq G$  is said *equationally-definable* if  $H$  is definable by a finite collection of equations, that is, if there exist words  $w_1(x), \dots, w_n(x)$ , with parameters from  $G$  such that  $H = \bigcap_{1 \leq i \leq n} \{g \in G \mid G \models w_i(g) = 1\}$ . Recall that a class of groups  $\mathcal{C}$  is called a pseudovariety if  $\mathcal{C}$  is closed under finite cartesian products, subgroups and quotients groups. We recall also that a group  $G$  is said to be residually- $\mathcal{C}$ , if for any nontrivial element  $g \in G$  there exists a homomorphism  $f : G \rightarrow A \in \mathcal{C}$  such that  $f(g) \neq 1$ . Our main result is as follows.

**Theorem 1.1** *Let  $\mathcal{C}$  be a pseudovariety and  $G$  a superstable residually- $\mathcal{C}$  group. Then  $G$  has a finite series  $G_0 \trianglelefteq G_1 \cdots \trianglelefteq G_n = G$  such that each  $G_i$  is equationally-definable,  $G_0$  is solvable and each factor  $G_{i+1}/G_i$  is in  $\mathcal{C}$ . If  $G$  is  $\omega$ -stable, then  $G_0$  is nilpotent; in addition if every  $\omega$ -stable residually- $\mathcal{C}$  abelian group is of bounded exponent, then  $G_0$  is of bounded exponent.*

**Corollary 1.2** *A residually finite superstable group  $G$  is solvable-by-finite and if it is  $\omega$ -stable then it is nilpotent-by-finite and of bounded exponent.*

**Proof**

The class  $\mathcal{C}$  of finite groups forms a pseudovariety and therefore by Theorem 1.1,  $G$  is solvable-by-finite. If  $G$  is  $\omega$ -stable, by Theorem 1.1, it is sufficient to show that every  $\omega$ -stable residually finite abelian group is of bounded exponent. But this is a consequence of Macintyre's Theorem on the structure of abelian  $\omega$ -stable groups, as there is no finite nontrivial divisible group.  $\square$

It should be noted that residual finiteness is not conserved by elementary equivalence. For instance the free group  $F_2$  of rank 2 is residually finite, whereas any nonprincipal ultrapower of it is not, as it contains  $\mathbb{Q}$  which is not residually finite.

A. Baudisch has shown that a superstable group  $G$  has definable subgroups  $1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r \trianglelefteq G$  such that  $H_{i+1}/H_i$  is infinite and either abelian or simple modulo a finite center and  $H_r$  is of finite index in  $G$  [1, Theorem 1.1]. The referee of the present paper has pointed out that the first part of Theorem 1.1, when  $\mathcal{C}$  does not contain an infinite simple group, and the first part of Corollary 1.2 follow from Lemma 2.2 (2) (below) and the result of Baudisch [1, Theorem 1.1], as follows. By [1, Corollary 1.9],  $G$  contains a unique maximal normal solvable subgroup  $rad(G)$  (which is equationally-definable). Hence, by Lemma 2.2(2),  $G/rad(G) = G^*$  is a residually- $\mathcal{C}$  group. Assume  $G^*$  infinite. By [1, Theorem 1.1],  $G^*$  contains an infinite definable subnormal subgroup  $H$  that is either abelian or simple modulo a finite center. A subnormal abelian definable subgroup would be contained in a definable normal nilpotent subgroup. This is impossible. In the other case  $G^*$  is not a residually- $\mathcal{C}$  group. Hence  $G^*$  is finite as desired.

The proof of Theorem 1.1 is based on the use of different notions of *components* related to  $\mathcal{C}$  (see Definition 2.1 and 2.6), the use of equationally-definable subgroups, and induction on the  $U$ -rank. It is also based on the use of the indecomposability theorem for superstable groups, and for that purpose we prove Proposition 2.4 which states that a superstable group has either a big normal  $\alpha$ -connected definable subgroup or a big normal nilpotent definable subgroup.

We will be interested in finitely generated  $\omega$ -stable groups. This arises naturally in the context of Cherlin-Zil'ber Conjecture. In that context, one can ask about the existence of an infinite finitely generated group of finite Morley rank. This is motivated by the fact that the existence of such a group implies the existence of a simple group of finite Morley rank which can not be algebraic over an algebraically closed field and gives a counterexample to the Cherlin-Zil'ber Conjecture (Proposition 3.2). But one must ask first about the existence of an infinite finitely generated  $\omega$ -stable group. By Corollary 1.2, we may conclude that a finitely generated residually finite  $\omega$ -stable group is finite. In the superstable case we have the following characterization.

**Corollary 1.3** *A finitely generated residually finite group is superstable if and only if it is abelian-by-finite.*

**Proof**

A finitely generated solvable non(abelian-by-finite) group interprets arithmetic and thus it can not be stable. Therefore a superstable finitely generated

residually finite group is abelian-by-finite. For the other direction, the conclusion follows from [4].  $\square$

It is a theorem of Mal'cev [9, Theorem 4.2] that a finitely generated linear group is residually finite. Combining this theorem with the above results, we may conclude that a finitely generated linear superstable group is abelian-by-finite and if it is  $\omega$ -stable then it is finite.

The paper is organized as follows. In the next section we prove some propositions and lemmas needed in the sequel and we end by the proof of Theorem 1.1. The goal of Section 3 is to give some observations and remarks in the finitely generated case.

## 2 Preliminaries and proofs

All our notations are standard and concerning conventions about  $U$ -rank and superstability we refer the reader to [2]. Before proving Theorem 1.1, we need some preparatory lemmas and definitions.

**Definition 2.1** Let  $G$  be a group and let  $\mathcal{C}$  be a class of groups.

- (1) We define the  $\mathcal{C}$ -component of  $G$ , denoted  $G^{\mathcal{C}}$ , to be the intersection of all normal equationally-definable subgroups  $K$  of  $G$  for which  $G/K \in \mathcal{C}$ .
- (2) We say that  $G$  is  $\mathcal{C}$ -connected if  $G = G^{\mathcal{C}}$ .

If  $G$  is  $\omega$ -stable, then by the DCC,  $G^{\mathcal{C}}$  is equationally-definable. It is known, when  $\mathcal{C}$  is closed by subgroups and finite direct products, that  $G/A, G/B \in \mathcal{C}$  imply  $(G/A \cap B) \in \mathcal{C}$ . This last property will be used freely without explicit reference. It follows that if  $\mathcal{C}$  is a pseudovariety, then  $G/G^{\mathcal{C}} \in \mathcal{C}$ .

It is more natural to take in the definition of the  $\mathcal{C}$ -component, definable subgroups instead of equationally-definable subgroups. But for our purpose, we want to conserve the property of being residually- $\mathcal{C}$  by quotients and thus we need the equational definability, as shown by the property (2) of the following lemma.

**Lemma 2.2** Let  $\mathcal{C}$  be a pseudovariety and  $G$  a group.

- (1)  $G^{\mathcal{C}}$  is normal in  $G$ . If  $G$  is stable and if  $K$  is an equationally-definable subgroup which contains a normal subgroup  $L$  such that  $G/L \in \mathcal{C}$ , then  $G^{\mathcal{C}} \leq K$ .
- (2) If  $G$  is residually- $\mathcal{C}$ , then  $G/K$  is residually- $\mathcal{C}$  for any equationally-definable normal subgroup  $K$  of  $G$ .
- (3) Suppose that  $G$  is  $\mathcal{C}$ -connected and  $\omega$ -stable. If  $H$  is an equationally-definable normal subgroup of  $G$ , then  $G/H$  is  $\mathcal{C}$ -connected.

**Proof**

- (1) Clearly  $G^{\mathcal{C}}$  is normal. Let  $K$  be an equationally-definable subgroup of  $G$  containing a normal subgroup  $L$  such that  $G/L \in \mathcal{C}$ . Then  $M = \bigcap_{g \in G} K^g$  is normal, and equationally-definable by the Baldwin-Saxl lemma. Moreover,  $M$  contains also  $L$ . Therefore  $G/M \in \mathcal{C}$ . Since  $M \leq K$  and  $G^{\mathcal{C}} \leq M$  we get  $G^{\mathcal{C}} \leq K$  as desired.

(2) Let  $K$  be definable by  $\bigwedge_{1 \leq i \leq n} w_i(\bar{a}_i, x) = 1$ . Let us denote  $\pi : G \rightarrow G/K$  the canonical morphism.

Let  $g \in G$  such that  $\pi(g) \neq 1$ ; that is  $g \notin K$ . Then there is some  $1 \leq p \leq n$  such that  $w_p(\bar{a}_p, g) \neq 1$ . Since  $G$  is residually- $\mathcal{C}$ , there exists a surjective morphism  $\phi : G \rightarrow L$ , where  $L \in \mathcal{C}$ , such that  $\phi(w_p(\bar{a}_p, g)) \neq 1$ .

We claim that  $\phi(g) \notin \phi(K)$ . If  $\phi(g) \in \phi(K)$ , then there exists an element  $g' \in K$  such that  $\phi(g) = \phi(g')$ . Since  $g' \in K$  we get  $\phi(w_p(\bar{a}_p, g')) = w_p(\phi(\bar{a}_p), \phi(g')) = 1$  and thus  $\phi(w_p(\bar{a}_p, g)) = 1$ ; a contradiction.

Since  $\phi$  is surjective,  $\phi(K)$  is a normal subgroup of  $L$ . Let  $\pi' : L \rightarrow H = L/\phi(K)$  be the canonical morphism. We define  $f : G/K \rightarrow H$  by  $f(\pi(x)) = \pi'(\phi(x))$ . Then  $f$  is a morphism. Now if  $f(\pi(g)) = 1$ , then  $\pi'(\phi(g)) = 1$  and thus  $\phi(g) \in \phi(K)$ ; a contradiction. Thus  $G/K$  is residually- $\mathcal{C}$  as desired.

(3) Let us denote by  $\pi_1 : G \rightarrow G/H$  and by  $\pi_2 : G/H \rightarrow M = (G/H)/(G/H)^{\mathcal{C}}$  the canonical morphisms. Since  $H$  is equationally-definable in  $G$  and  $(G/H)^{\mathcal{C}}$  is equationally-definable in  $(G/H)$  it follows that  $L = \ker(\pi_2 \circ \pi_1)$  is an equationally-definable normal subgroup of  $G$ . Since  $G/L \cong M$  and  $M \in \mathcal{C}$  we find  $G/L \in \mathcal{C}$  and since  $G$  is  $\mathcal{C}$ -connected we get  $G = L$  and thus  $M = 1$ .  $\square$

We need the next lemma which is a refinement of some statements in [8]. Let  $G$  be a subgroup of a stable group  $\mathcal{G}$  and let  $H \leq G$ . We say that  $H$  is *relatively equationally-definable* if there is an equationally-definable subgroup  $N \leq \mathcal{G}$ , with parameters only from  $G$ , such that  $H = G \cap N$ . Notice that in that case,  $H$  is equationally-definable in  $G$ . We notice also that if  $H$  is relatively equationally-definable, then  $H$  is relatively definable in the sense of [8, Definition 1.0.2].

**Lemma 2.3** *A subnormal solvable subgroup of a substable group lies in a normal solvable and relatively equationally-definable subgroup.*

**Proof**

Let  $G$  be a substable group in the ambient group  $\mathcal{G}$ . We first show the lemma for solvable normal subgroups. Let  $H$  be a normal solvable subgroup of  $G$ . The proof is by induction on the derived length of  $H$ . If  $H$  has derived length 0, the result is clear.

Suppose  $H$  is of derived length  $n + 1$ . Then  $H^{(n)}$  is abelian and normal. Let  $K$  be the center of  $C_G(H^{(n)})$  in  $G$ . Then  $H^{(n)} \leq K \trianglelefteq G$ . Since  $K$  can be written as an intersection of centralizers of elements of  $G$ ,  $K$  is the intersection with  $G$  of an equationally-definable (with parameters only from  $G$ ) subgroup  $\bar{K}$  of  $\mathcal{G}$ .

Let  $N = \bigcap_{g \in G} \bar{K}^g$  and  $B = N_G(N)$ . Then, as in [8, Lemma 1.1.6],  $G \leq B$ ,  $K \leq N$ , the group  $G_1 = G/K$  is substable and it is a subgroup of the stable group  $\mathcal{G}_1 = B/N$ .

Let  $f : B \rightarrow B/N$  be the canonical morphism. Then  $f(H)$  is a normal and solvable subgroup of derived length  $n$  in  $G_1$ . Thus, by induction,  $f(H) \leq K_1$  and  $K_1$  is a solvable normal and relatively equationally-definable subgroup of  $G_1$ . Let  $\bar{K}_1$  be an equationally-definable (with parameters only from  $G_1$ ) subgroup of  $\mathcal{G}_1$  such that  $G_1 \cap \bar{K}_1 = K_1$ . Then  $H \leq f^{-1}(K_1) = f^{-1}(\bar{K}_1) \cap G$ . Now since

$\bar{K}_1$  (resp.  $N$ ) is equationally-definable with parameters only from  $G_1$  (resp. from  $G$ ),  $f^{-1}(\bar{K}_1)$  is equationally-definable (with parameters only from  $G$ ) and the result follows.

Let  $H$  be a subnormal solvable subgroup of  $G$ . The proof is, as above, by induction on the derived length of  $H$ . If  $H$  is of derived length 0, then the result is clear. Suppose  $H$  is of derived length  $n + 1$ . Then  $H^{(n)}$  is abelian and subnormal. By [8, Corollary 1.2.12],  $H^{(n)}$  lies in a normal nilpotent subgroup; which by the first case above, lies in a normal solvable relatively equationally-definable subgroup  $K$ . We do the same construction as above:  $f : B \rightarrow B/N$ ,  $f(H)$  is solvable and of derived length  $n$ , and  $f^{-1}(K_1)$  is a solvable and normal relatively equationally-definable subgroup containing  $H$ .  $\square$

**Proposition 2.4** *Let  $G$  be a superstable group of  $U$ -rank  $\omega^\alpha n + \beta$  where  $\alpha$  and  $\beta$  are ordinals,  $n \in \mathbb{N}^*$  and  $\beta < \omega^\alpha$ . Then either the  $\alpha$ -connected component  $\Gamma$  of  $G$  contains a definable  $\alpha$ -connected normal subgroup  $K$  of  $G$  such that  $U(K) \geq \omega^\alpha$ , or  $\Gamma$  lies in a normal definable nilpotent subgroup  $K$  such that  $U(K) \geq \omega^\alpha n$ .*

**Proof**

Write  $\Gamma = \bigwedge_{i \in \lambda} \phi_i(G)$  where each  $\phi_i(x)$  is a formula defining a subgroup of  $G$ . Let  $G \prec \mathbb{G}$  be an elementary saturated extension of  $G$  of cardinality greater than  $\max\{\lambda^+, |\text{Th}(G)|^+\}$ , and let  $\Gamma_1 = \bigwedge_{i \in \lambda} \phi_i(\mathbb{G})$ .

We have  $U(\Gamma_1) = \omega^\alpha n$ ,  $\Gamma_1$  is  $\bigwedge$ -definable and  $\alpha$ -connected and thus  $\alpha$ -indecomposable. We notice also that  $\Gamma_1$  is a normal subgroup of  $\mathbb{G}$ .

Suppose first that  $\Gamma_1$  is not nilpotent. By the Indecomposability Theorem [2, V, Theorem 3.1] (or by [2, VI.2., Proposition 2.4]), for any  $s \in \mathbb{N}$ ,  $\Gamma_1^s$  is  $\bigwedge$ -definable and  $\alpha$ -connected. Since  $\Gamma_1^s$  is nontrivial,  $U(\Gamma_1^s) = \omega^\alpha p$  for some  $p \leq n$ . Therefore the descending central series stabilizes and thus there exists  $m \in \mathbb{N}$  such that  $\Gamma_1^m = [\Gamma_1^m, \Gamma_1]$ .

Let us show that  $\Gamma_1^m$  is definable by using only parameters from  $G$ . By [2, VI, Lemma 2.3], for every nontrivial  $g \in \Gamma_1^m$ ,  $g^{-1}g^{\Gamma_1}$  is  $\alpha$ -indecomposable. By the Indecomposability Theorem [2, V, Theorem 3.1], there exist  $q \in \mathbb{N}^*$  and nontrivial elements  $g_1, \dots, g_q$  in  $\Gamma_1^m$  such that  $[\Gamma_1^m, \Gamma_1] = g_1^{-1}g_1^{\Gamma_1} \cdots g_q^{-1}g_q^{\Gamma_1}$ .

Since  $\Gamma_1^m = [\Gamma_1^m, \Gamma_1]$  and, since  $\Gamma_1^m$  is normal in  $G$ ,  $\Gamma_1^m$  is clearly defined by the formula  $\exists y_1 \cdots \exists y_m (x = g_1^{-1}g_1^{y_1} \cdots g_q^{-1}g_q^{y_q})$ . Now since  $\Gamma_1$  is definable using only parameters from  $G$  and since  $\Gamma_1^m$  is a characteristic subgroup of  $\Gamma_1$  we find that  $\Gamma_1^m$  is fixed setwise by any automorphism of  $\mathbb{G}$  which fixes  $G$  pointwise. Therefore  $\Gamma_1^m$  is definable using only parameters from  $G$  (see for instance [6, Proposition 4.3. 25]). Thus by taking  $H = \phi(G)$ , where  $\phi$  is formula defining  $\Gamma_1^m$  by using only parameters from  $G$ , we find  $U(H) \geq \omega^\alpha$ .

We conclude that  $G$  has a definable normal subgroup  $K$  which is  $\alpha$ -connected and  $U(K) \geq \omega^\alpha$ .

Suppose now that  $\Gamma_1$  is nilpotent. Then, by [8, Theorem 1.1.10],  $\Gamma_1 \leq K$ , where  $K$  is a definable nilpotent normal subgroup of  $\mathbb{G}$ .

Since  $\mathbb{G}$  is  $\lambda$ -saturated we find that  $\mathbb{G} \models \forall x (\bigwedge_{1 \leq i \leq n} \phi_i(x) \Rightarrow \phi(x))$ , where  $\phi$  is a formula defining  $K$ . We also have  $U(K) \geq U(\Gamma_1) = \omega^\alpha n$ . Therefore, by

elementary equivalence and since  $U(\phi_i(\mathbb{G})) \geq U(\Gamma)$ ,  $G$  has a definable nilpotent normal subgroup with  $U$ -rank greater than  $\omega^\alpha n$ .  $\square$

**Proposition 2.5** *Let  $\mathcal{C}$  be a pseudovariety and  $G$  a superstable residually- $\mathcal{C}$  group of  $U$ -rank  $\omega^\alpha n + \beta$  where  $\alpha$  and  $\beta$  are ordinals,  $n \in \mathbb{N}^*$  and  $\beta < \omega^\alpha$ . If  $K$  is a definable  $\alpha$ -connected normal subgroup of  $G$ , then  $G/C_G(K) \in \mathcal{C}$ .*

**Proof**

Let  $(H_i)_{i \in \lambda}$  be the list of all normal subgroups of  $G$  such that  $G/H_i \in \mathcal{C}$ . Since  $G$  is residually- $\mathcal{C}$ , we obtain  $\bigcap_i H_i = 1$ . Now since  $H_i$  is normal in  $G$  we have  $[K, H_i] \leq H_i$ . Therefore  $\bigcap_i [K, H_i] = 1$ .

Since  $K$  is  $\alpha$ -connected, for every  $h \in H_i$ ,  $h^{-1}h^K$  is  $\alpha$ -indecomposable and definable. By the Indecomposability Theorem [2, V, Theorem 3.1],  $[K, H_i]$  is a definable and connected subgroup of  $G$  and

$$[K, H_i] = h_{i,1}^{-1}h_{i,2}^K \cdots h_{i,m_i}^{-1}h_{i,m_i}^K, \text{ for some } m_i \leq 2n \text{ and } h_{i,j} \in H_i \text{ for } j = 1, m_i.$$

Let

$$\phi(x, y_1, y_2, \dots, y_{2n}) := \exists z_1 \cdots \exists z_{2n} (x = y_1^{-1}y_1^{z_1} \cdots y_{2n}^{-1}y_{2n}^{z_{2n}} \wedge \bigwedge_{1 \leq i \leq 2n} \psi(z_i)),$$

$$\bar{h}_i = (h_{i,1}, \dots, h_{i,m_i}, 1, \dots, 1),$$

where  $\psi(x)$  is a formula defining  $K$ . Then  $\{[K, H_i] : i \in \lambda\} = \{\phi(G, \bar{h}_i) : i \in \lambda\}$  is uniformly definable by the formula  $\phi$  above. Therefore, by the Baldwin-Saxl lemma, we get

$$1 = \bigcap_{i \in \lambda} [K, H_i] = [K, H_1] \cap \cdots \cap [K, H_p].$$

Therefore  $[H_1 \cap \cdots \cap H_p, K] = 1$  and thus  $L = H_1 \cap \cdots \cap H_p \leq C_G(K)$ . Since  $G/L \in \mathcal{C}$ , we obtain  $G/C_G(K) \in \mathcal{C}$ .  $\square$

**Definition 2.6** Let  $G$  be a group and  $\mathcal{C}$  be a pseudovariety.

(1) If  $G$  is  $\omega$ -stable, by the DCC on definable subgroups, we find that  $G$  has a finite series  $G_0 \trianglelefteq G_1 \cdots \trianglelefteq G_n = G$  such that  $G_{n-1} = G^{\mathcal{C}}$  and  $G_i = G_{i+1}^{\mathcal{C}}$  and  $G_0 = G_0^{\mathcal{C}}$ . We call  $G_0$  the *smallest  $\mathcal{C}$ -component* of  $G$  and we denote it  ${}^{\mathcal{C}}G$ . Notice that  ${}^{\mathcal{C}}G$  is  $\mathcal{C}$ -connected and that  $G_{i+1}/G_i \in \mathcal{C}$ .

(2) Suppose that  $G$  is stable. We define the  $\mathcal{C}$ -centralizer component of  $G$ , denoted  $G_{\mathcal{C}}$ , to be the intersection of all normal subgroups  $K$  which are definable by intersection of centralizers and such that  $G/K \in \mathcal{C}$ . Since  $G$  is stable,  $G_{\mathcal{C}}$  is definable and is an intersection of centralizers. Notice also that  $G/G_{\mathcal{C}} \in \mathcal{C}$ . By the DCC on centralizers, we find that  $G$  has a finite series  $G_0 \trianglelefteq G_1 \cdots \trianglelefteq G_n = G$  such that  $G_{n-1} = G_{\mathcal{C}}$  and  $G_i = G_{i+1}^{\mathcal{C}}$  and  $G_0 = G_{0\mathcal{C}}$ . We call  $G_0$  the *smallest  $\mathcal{C}$ -centralizer component* of  $G$  and we denote it  ${}_{\mathcal{C}}G$ . Notice that  $G_{i+1}/G_i \in \mathcal{C}$ .

**Proof of Theorem 1.1**

We treat first the  $\omega$ -stable case. If  ${}^cG$  is nilpotent, then  $G$  satisfies the conclusion of the theorem. Thus we may show the theorem for  $\mathcal{C}$ -connected groups.

Let  $G$  be a  $\mathcal{C}$ -connected group of minimal Lascar rank for which the theorem does not hold. Let  $U(G) = \omega^\alpha n + \beta$ , where  $n \in \mathbb{N}^*$ ,  $\alpha, \beta$  are ordinals and  $\beta < \omega^\alpha$ . Let  $K$  be the  $\alpha$ -connected component of  $G$ . Since  $G$  is  $\omega$ -stable,  $K$  is definable. Therefore, by Proposition 2.5,  $G/C_G(K) \in \mathcal{C}$ .

Since  $G$  is  $\mathcal{C}$ -connected, we find  $C_G(K) = G$ . Therefore  $K \leq Z(G)$  and thus  $\omega^\alpha n = U(K) \leq U(Z(G))$ . Thus  $U(G/Z(G)) < U(G)$ . By Lemma 2.2 (2),  $G/Z(G)$  is residually- $\mathcal{C}$  and, by Lemma 2.2 (3),  $G/Z(G)$  is  $\mathcal{C}$ -connected. By the minimality of the rank of  $G$ ,  $G/Z(G)$  is nilpotent. It follows that  $G$  is also nilpotent; a final contradiction.

If every  $\omega$ -stable residually- $\mathcal{C}$  abelian group is of bounded exponent, then the conclusion for the bounded exponent follows as above, by induction on the rank and by seeing that  $Z(G)$  is of bounded exponent.

We treat the superstable case. Let  $G$  be a superstable residually- $\mathcal{C}$  group of minimal Lascar rank for which the theorem does not hold. Let  $U(G) = \omega^\alpha n + \beta$ , where  $n \in \mathbb{N}^*$ ,  $\alpha, \beta$  are ordinals and  $\beta < \omega^\alpha$ .

Suppose that the theorem holds for the smallest  $\mathcal{C}$ -centralizer connected component  ${}^cG$  of  $G$ . Since  ${}^cG = G_0 \trianglelefteq G_1 \cdots \trianglelefteq G_n = G$ ,  $G_{i+1}/G_i \in \mathcal{C}$ , and any equationally-definable subgroup of  ${}^cG$  is also an equationally-definable subgroup of  $G$ , we may conclude that the theorem holds also for  $G$ . Therefore  $U({}^cG) = U(G)$  and hence we may assume without loss of generality that  $G = {}^cG$ ; that is if  $K$  is a normal subgroup which is an intersection of centralizers such that  $G/K \in \mathcal{C}$ , then  $K = G$ .

By Proposition 2.4, either  $G$  has a normal definable  $\alpha$ -connected subgroup  $K$  such that  $U(K) \geq \omega^\alpha$  or  $G$  has a normal nilpotent subgroup  $K$  such that  $U(K) \geq \omega^\alpha n$ .

We treat the first case; that is  $G$  has a normal definable  $\alpha$ -connected subgroup  $K$  such that  $U(K) \geq \omega^\alpha$ . Then, by Proposition 2.5,  $G/C_G(K) \in \mathcal{C}$ . Since  $C_G(K)$  is an intersection of centralizers, and by our assumption we find that  $G = C_G(K)$  and thus we obtain  $K \leq Z(G)$ .

Therefore  $K \leq Z(G)$  and thus  $\omega^\alpha \leq U(K) \leq U(Z(G))$ . Thus  $U(G/Z(G)) < U(G)$ . By Lemma 2.2 (2),  $G/Z(G)$  is residually- $\mathcal{C}$ . By the minimality of the rank of  $G$ ,  $G/Z(G)$  satisfies the conclusion of the theorem and so does  $G$ ; a contradiction.

Now we suppose that  $G$  has a normal nilpotent subgroup  $K$  such that  $U(K) \geq \omega^\alpha n$ . By Lemma 2.3,  $K$  lies in a solvable equationally-definable subgroup  $L$ . Therefore  $U(L) \geq \omega^\alpha$  and thus  $U(G/L) < U(G)$ . By Lemma 2.2 (2),  $G/L$  is residually- $\mathcal{C}$ . By the minimality of the rank of  $G$ ,  $G/L$  satisfies the conclusion of the theorem and so does  $G$ ; a contradiction.  $\square$

### 3 The finitely generated case and final remarks

We are interested in finitely generated  $\omega$ -stable groups. As noticed in the introduction this is motivated by the Cherlin-Zil'ber conjecture and by the possible existence of an infinite finitely generated  $\omega$ -stable group.

In several situations, when one works with ranks, one realizes that the propositions which one wants to show depend only on certain properties of the classes of groups considered. For that reason and to avoid repetitions we introduce the following definitions. Let  $\mathcal{C}$  be a class of groups. We say that  $\mathcal{C}$  is *closed under definability* (abbreviated as CD-class) if whenever  $G$  is in  $\mathcal{C}$  and  $K$  is a normal definable subgroup of  $G$ ,  $K$  and  $G/K$  are in  $\mathcal{C}$ . For instance, the following classes are a CD-classes: any variety of groups, solvable groups, finite groups, pseudofinite groups,  $\omega$ -stable groups, groups of finite Morley rank, affine algebraic groups.

We say that  $\mathcal{C}$  is *closed under definability inversely* (abbreviated as CDI-class) if whenever  $G$  is a group having a normal definable subgroup  $K$  such that  $K$  and  $G/K$  are in  $\mathcal{C}$ , then  $G$  is in  $\mathcal{C}$ . For instance, the class of solvable groups and the class of finite groups are CDI-classes.

Let  $(\mathcal{C}_1, \mathcal{C}_2)$  be a pair of classes of groups. We say that  $(\mathcal{C}_1, \mathcal{C}_2)$  is a *good pair of classes of groups* if the following conditions are satisfied:

- (i)  $\mathcal{C}_2$  is a CD-class and a CDI-class,
- (ii) if  $G \in \mathcal{C}_1$  and  $K$  is a normal definable subgroup of  $G$ , then  $G/K \in \mathcal{C}_1$ ,
- (iii) if  $G \in \mathcal{C}_1$  and  $K$  is a normal definable subgroup of  $G$  of finite index, then  $K \in \mathcal{C}_1$ ,
- (iv) if  $G \in \mathcal{C}_1$  and  $K$  is a normal definable subgroup of  $G$  such that  $G/K \in \mathcal{C}_2$ , then  $K \in \mathcal{C}_1$ ,
- (v) if  $G \in \mathcal{C}_1$  and  $G/Z(G) \in \mathcal{C}_2$ , where  $Z(G)$  is finite, then  $G \in \mathcal{C}_2$ .

For instance, if  $\mathcal{C}_1$  is a CD-class and  $\mathcal{C}_2$  is a CD-class and a CDI-class satisfying (v), then  $(\mathcal{C}_1, \mathcal{C}_2)$  is a good pair of classes of groups.

**Theorem 3.1** *Let  $(\mathcal{C}_1, \mathcal{C}_2)$  be a good pair of classes of groups. If  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains an infinite superstable group (resp.  $\omega$ -stable, resp. of finite MR), then  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains also an infinite superstable (resp.  $\omega$ -stable, resp. of finite MR) group which is either  $\mathcal{C}_2$ -by-finite or simple or nilpotent-by-finite (resp. abelian-by-finite).*

#### Proof

We shall treat first the superstable case. Let  $G \in \mathcal{C}_1 \setminus \mathcal{C}_2$  be infinite of minimal Lascar rank. Let  $U(G) = \omega^\alpha n + \beta$ , where  $n \in \mathbb{N}^*$  and  $\alpha, \beta$  are ordinals with  $\beta < \omega^\alpha$ .

Suppose that  $G$  has a normal definable subgroup  $K$  satisfying  $U(K) \geq \omega^\alpha$  and of infinite index. By our supposition,  $U(K) < U(G)$  and also  $U(G/K) < U(G)$ . By (ii),  $G/K \in \mathcal{C}_1$ . Since  $G/K$  is infinite, by minimality of the rank of  $G$  we get  $G/K \in \mathcal{C}_2$  and therefore, by (iv),  $K \in \mathcal{C}_1$ . Since  $U(K) < U(G)$  we get  $K \in \mathcal{C}_2$ . Since  $\mathcal{C}_2$  is a CDI-classe we get  $G \in \mathcal{C}_2$ ; a contradiction.



Thus any definable normal subgroup  $K$  of  $G$  is either of finite index or satisfies  $U(K) < \omega^\alpha$ . In particular  $\beta = 0$  and  $U(G) = \omega^\alpha n$ .

By Proposition 2.4, either  $G$  has a normal definable  $\alpha$ -connected subgroup  $H$  such that  $U(H) \geq \omega^\alpha$  or  $G$  has a normal nilpotent subgroup  $K$  such that  $U(K) \geq \omega^\alpha n$ .

If  $G$  has a definable  $\alpha$ -connected normal subgroup  $H$  such that  $U(H) \geq \omega^\alpha$ , then  $H$  has a finite index in  $G$ . By (iii),  $H \in \mathcal{C}_1$ . If  $H \in \mathcal{C}_2$ , then  $G$  is  $\mathcal{C}_2$ -by-finite and we are done. Otherwise  $H \in \mathcal{C}_1 \setminus \mathcal{C}_2$ . In that case, we replace  $G$  by  $H$  and thus we may assume that  $G$  is  $\alpha$ -connected. Therefore, every definable normal proper subgroup  $K$  of  $G$  satisfies  $U(K) < \omega^\alpha$ . If  $U(Z(G)) \geq \omega^\alpha$ , then  $G$  is abelian and we are done. If is not the case, by [2, Proposition 6],  $G/Z(G)$  is simple and it is clearly infinite.

Clearly  $G/Z(G) \in \mathcal{C}_1$ . Suppose that  $G/Z(G) \in \mathcal{C}_2$ . By (iv),  $Z(G) \in \mathcal{C}_1$ .

If  $Z(G)$  is infinite, then since  $U(Z(G)) < U(G)$  we get  $Z(G) \in \mathcal{C}_2$ . Therefore  $G \in \mathcal{C}_2$ , a contradiction.

If  $Z(G)$  is finite, then by (v), we find  $G \in \mathcal{C}_2$  which is also a contradiction.

Therefore  $G/Z(G) \in \mathcal{C}_1 \setminus \mathcal{C}_2$  and thus  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains an infinite simple group.

If  $G$  has a normal nilpotent subgroup  $K$  such that  $U(K) \geq \omega^\alpha n$ , then  $K$  has a finite index and thus  $G$  is nilpotent-by-finite.

This ends the proof in the superstable case.

In the  $\omega$ -stable case, by the DCC we may assume  $G$  connected. As in the previous case,  $G$  is either abelian or  $G/Z(G)$  is infinite simple.  $\square$

**Proposition 3.2** *If there exists an infinite finitely generated  $\omega$ -stable (resp. of finite MR) group, then there exists an infinite simple finitely generated  $\omega$ -stable (resp. of finite MR) group (which is not linear, and in particular is not algebraic over an algebraically closed field).*

**Proof**

Let  $\mathcal{C}_1$  be the class of finitely generated groups and let  $\mathcal{C}_2$  be the class of finite groups. Then it is easily checked that  $(\mathcal{C}_1, \mathcal{C}_2)$  is a good pair of classes of groups. Using Macintyre's Theorem on the structure of  $\omega$ -stable abelian groups and by Theorem 3.1, there exists an infinite simple  $\omega$ -stable (resp. of finite MR) finitely generated group  $G$ . Now if  $G$  is linear, then  $G$  is residually finite and thus finite as it is simple.  $\square$

The previous proposition shows in particular that if the Cherlin-Zil'ber Conjecture is true, then every finitely generated group of finite Morley rank is finite. This in particular suggests the following problem.

**Question 1.** Is there an infinite finitely generated group of finite Morley rank ?

We believe that every finitely generated  $\omega$ -stable group is finite. What can be said about finitely generated groups of small Morley rank ? By [3, Corollary 6.6, Theorem 9.19] a group of finite Morley rank  $\leq 2$  is solvable-by-finite. The following proposition shows that in that case such groups are finite.

**Proposition 3.3** *A finitely generated solvable-by-finite or residually-solvable  $\omega$ -stable group is finite.*

**Proof**

By Theorem 1.1, a residually-solvable  $\omega$ -stable group is solvable and thus we may show the proposition for solvable-by-finite groups. Let  $\mathcal{C}_1$  be the class of finitely generated solvable-by-finite groups and let  $\mathcal{C}_2$  be the class of finite groups. Then it is easily checked that  $(\mathcal{C}_1, \mathcal{C}_2)$  is a good pair of classes of groups. By Theorem 3.1, if  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains an infinite  $\omega$ -stable group, then  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains an infinite abelian-by-finite or simple  $\omega$ -stable group  $G$ . Since  $G$  is finitely generated, by Macintyre's Theorem on the structure of abelian  $\omega$ -stable groups, the first case is impossible. The second case is clearly impossible.  $\square$

We notice that the above proposition can also be deduced from that fact that an infinite finitely generated solvable-by-finite  $\omega$ -stable group  $G$  interprets arithmetic and thus it must be abelian-by-finite (and we conclude by Macintyre's Theorem on the structure of abelian  $\omega$ -stable groups).

**Proposition 3.4** *If there exists a finitely generated (resp. finitely presented) group of Morley rank 3, then there exists a finitely generated (resp. finitely presented) simple bad group of Morley rank 3.*

**Proof**

Let  $G$  be a finitely generated group of Morley rank 3. We may assume  $G$  connected. Suppose that there exists an infinite normal definable subgroup  $K$  of  $G$ . Then  $G/K$  is of rank  $\leq 2$ . Therefore  $G/K$  is nilpotent and finitely generated, and thus finite by Proposition 3.3. Since  $K$  is of finite index in  $G$ ,  $K$  is finitely generated and thus, since it is of rank  $\leq 2$ , it is finite. Therefore  $G$  is finite, a contradiction. Therefore every definable normal subgroup of  $G$  is finite and thus  $M = G/Z(G)$  is a simple finitely generated group of rank 3. By [3, Lemma 13.8]  $M$  is a simple bad group or isomorphic to  $PSL_2(K)$  for some algebraically closed field  $K$ . But it is not difficult to see that  $PSL_2(K)$  is never finitely generated and therefore  $M$  is a simple bad group. Now if  $G$  is finitely presented, then  $M$  is finitely presented since  $Z(G)$  is finite.  $\square$

We give another application of Theorem 3.1.

**Proposition 3.5** *A superstable pseudofinite group is solvable-by-finite.*

**Proof**

By taking  $\mathcal{C}_1$  to be the class of pseudofinite groups and by  $\mathcal{C}_2$  to be the class of solvable-by-finite groups, it is easily checked that  $(\mathcal{C}_1, \mathcal{C}_2)$  is a good pair of classes of groups. By Theorem 3.1, if  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains an infinite superstable group, then  $\mathcal{C}_1 \setminus \mathcal{C}_2$  contains an infinite simple group  $G$ . So there exists an infinite simple pseudofinite superstable group  $G$ . By Wilson's Theorem [10],  $G$  is elementary equivalent to a Chevalley group over a pseudofinite field. But such a group is not stable by [5, Fact 2.2].  $\square$

The precedent proposition can also be deduced from the result of Baudisch previously cited [1, Theorem 1.1] and by using Wilson's Theorem [10]. The

above result is a special case of a more general result by Macpherson and Tent which states that a pseudofinite stable group is solvable-by-finite [5]. Notice that Khelif has shown that any pseudofinite group of finite Morley rank is abelian-by-finite. Notice also that nilpotent-by-finite pseudofinite  $\omega$ -stable groups which are not abelian-by-finite exist and some examples are known to Sabbagh and Zilb'er and there is an example in [5] (see the discussion of this in [5]). In fact in the  $\omega$ -stable case we observe the following.

**Proposition 3.6** *An  $\omega$ -stable pseudofinite group is nilpotent-by-abelian-by-finite.*

**Proof**

The connected component of  $G$  is solvable and therefore nilpotent-by-abelian, by [7] (or see [3], p. 171).  $\square$

Notice that it is known that a finitely generated pseudofinite solvable-by-finite group is finite. By the above mentioned result of Macpherson and Tent, we conclude that a stable finitely generated pseudofinite group is finite.

The present paper leaves the following question open.

**Question 2.** What can be said about finitely generated residually finite stable groups ?

An announced recent result of Z. Sela claims that the free group of rank 2 is stable. Therefore one can hope to get, certainly under additional hypothesis, an alternative which look likes the Tits alternative for finitely generated linear groups.

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Abderezak OULD HOUCINE,  
Institut Camille Jordan,  
Université Claude Bernard Lyon-1,  
Bâtiment Braconnier, 21 Avenue Claude Bernard,  
69622 Villeurbanne Cedex, France.  
ould@math.univ-lyon1.fr