The solution to Alhazen problem

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Abstract

Alhazen problem of reflection at a concave spherical surface is one of the most discussed problems in optics. It was solved by Alhazen, The number of solution points vary from zero to a maximum of four. However, his solution is known to be prolix. Huygens solved the problem and identified the solution points to be points of intersection of the given circle and a hyperbola. Many other lines of attack were attempted with their solutions. New solutions are offered from time to time. In this paper we offer a solution based on the criterion to be satisfied by reflection of a ray of light at a concave spherical surface. Based on that criterion we show that it is impossible for a circle containing both end points of the path of the ray inside it, to reflect rays from one point to the other. However, for every point on the given circle, we can construct two orthogonal circles (orthogonal conics in general) which reflect rays from one given point to the other, while the given circle is a non-reflector.

Key words

Alhazen, Alhazen problem, Optics, Light, Reflection, Spherical surface, Apollonius circle, Harmonic division, Orthogonal circles

Introduction

The centuries old Alhazen's problem was originally formulated by Ptolemy (ca 100-170) and was later taken up by Alhazen[1] (ca 965-1040) who is also known as Ibn al–Haytham. That his fame as a mathematician rested on his treatment of this problem goes to show his contribution to its solution. The problem appears deceptively simple, but yet it took centuries for mathematicians to solve it. Alhazen's solution was "horribly prolix"[2].Notable among others who solved this problem was Huygens. Mihas gives a detailed version of Huygens' solution[3]. Elkin gave a method[4] to show how to calculate the position of the required points in the general case with respect to the relative location of the points and the circle. Smith gave a reconstruction of Alhazen's solution[5]. He credits Alhazen's solution to be remarkably ingenious and elegant in its conceptual simplicity. He also gave Ptolemy's solutions for different cases of location of the two points within the circle[6]. We demonstrate in this paper, using a new geometrical approach, that it is impossible for a concave spherical reflector to reflect light rays from a given point A, to another given point B, when both points lie within the circle (cross section of the spherical reflector). If it is simply a question of the points merely be located on the circle at which reflections occur - while the reflector is another circle (spherical reflector) which intersects the given circle at the point of incidence - then all points on the given circle can be solution points of the problem. Our solution is based on the construction of Apollonius circle that intersects the line segment formed from the two given points harmonically in a ratio k : 1, ($k \neq 1$).

Statement of the Alhazen problem

Alhazen's problem may be stated thus: From any two points opposite a reflecting surface – which may be plane, spherical concave (or convex) – to find a point (or points) on the surface at which a ray of light from one of the two points will be reflected to the other.

Solution to Alhazen problem

This solution is applies for concave spherical reflectors Let the two given points be A, B and the given circle (section of the spherical reflector) centered at O enclose the two points A and B (see Fig. 1). The problem is to find the points on the circle at which a ray of light from A is reflected to pass through B.

Apollonius circle[7]

Let P be an arbitrarily chosen point on the given circle (see Fig. 1). Then the ratio of the distances AP, BP of the point P from A and B defines a ratio, k.

$$AP:BP = k:1 \qquad (k > 1, here) \qquad (1)$$

Let us bisect the angle APB. Let the internal bisector intersect the line through A, B at C. Let the external bisector PD intersect that line at D.

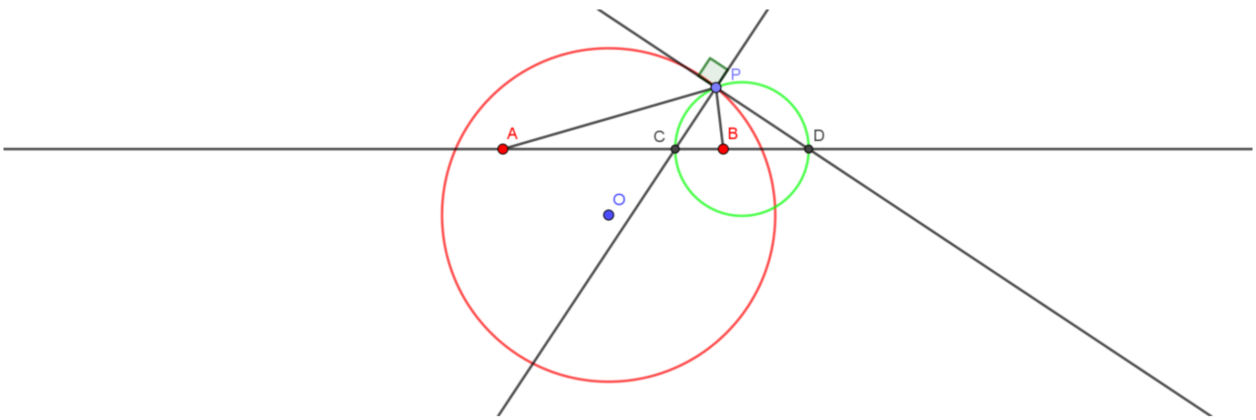

Fig. 1 A, B are the given points inside the given circle (red) centered at O. P is an arbitrary point on the circle. Angle APB is bisected by PC internally and PD externally. The circle through C, P, D is Apollonius circle determined by the ratio AP : PB = k : 1.

Let us draw the circle with CD as diameter (green). It passes through the point P, since angle CPD is a right angle. (the internal and the external bisectors PC and PD are mutually perpendicular). The circle is called Apollonius circle defined by the value of the constant k[7]. Every point P_i on the Apollonius circle follows Eq. (2) below.

$$AP_i = k.BP_i \qquad (2)$$

Therefore P_iC bisects the angle AP_iB for all P_i.

The family of α circles

Since P is an arbitrarily chosen point on the given circle the above arguments are valid for any choice of P. For different choices we get different values of the ratio of distances AP : PB and different values of k.

For different values of k we get different Apollonius circles (See Fig. 2.). They form a family of coaxial, non intersecting circles. Let us call it the family of α circles[7].

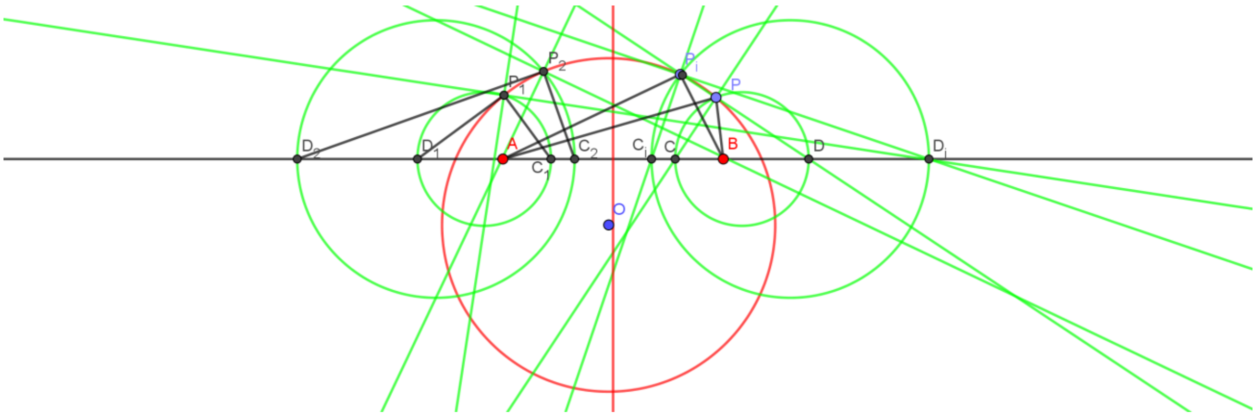

Fig. 2. Figure shows a few members of the family of the α circles (coaxial, non intersecting circles).

If we construct the perpendicular bisector of AB, called the radical axis (the red vertical line), the Apollonius circles on the left of it have values of k < 1. That is, for circles on left of the radical axis, k < 1 and for those on the right k > 1.

The family of β circles

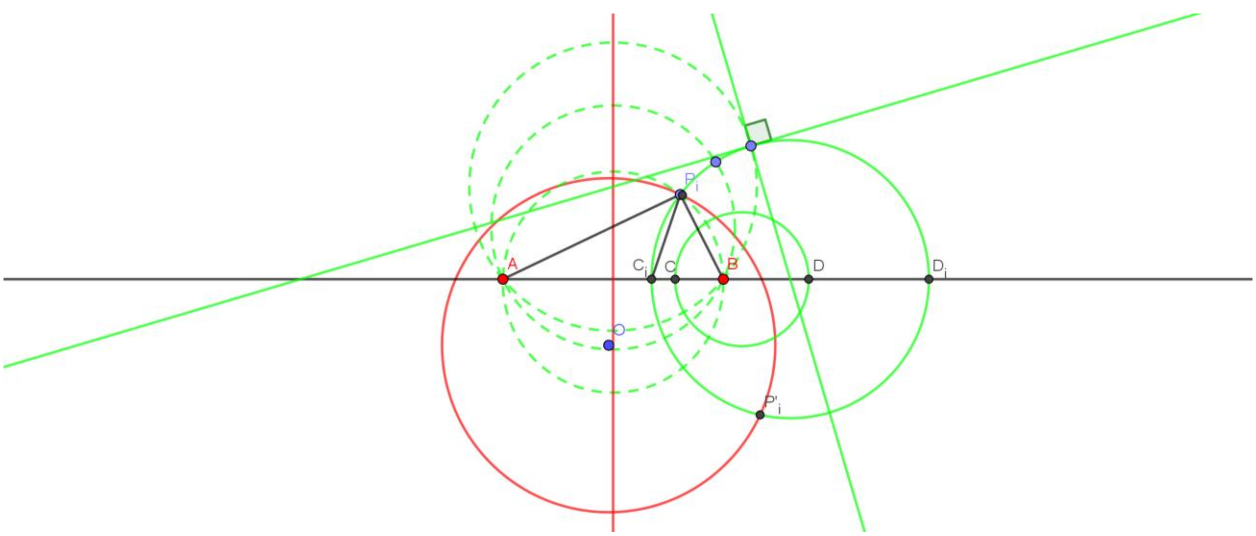

Fig. 3, Figure shows a few members of the β family of circles. They intersect the members of the α family orthogonally.

Let us draw the circles passing through points A, B and P_i. These circles form a family of coaxial intersecting circles (see Fig. 3). Let us call this, the family of β circles. Each of the β circles intersect every member of the α circles orthogonally[7].

Reflection of light at points on the given circle

Let a ray of light from A be incident at point P_i on the given circle. P_i is the point of intersection of three circles. (see Fig. 4) - a pair of orthogonal α and β circles and the given circle.

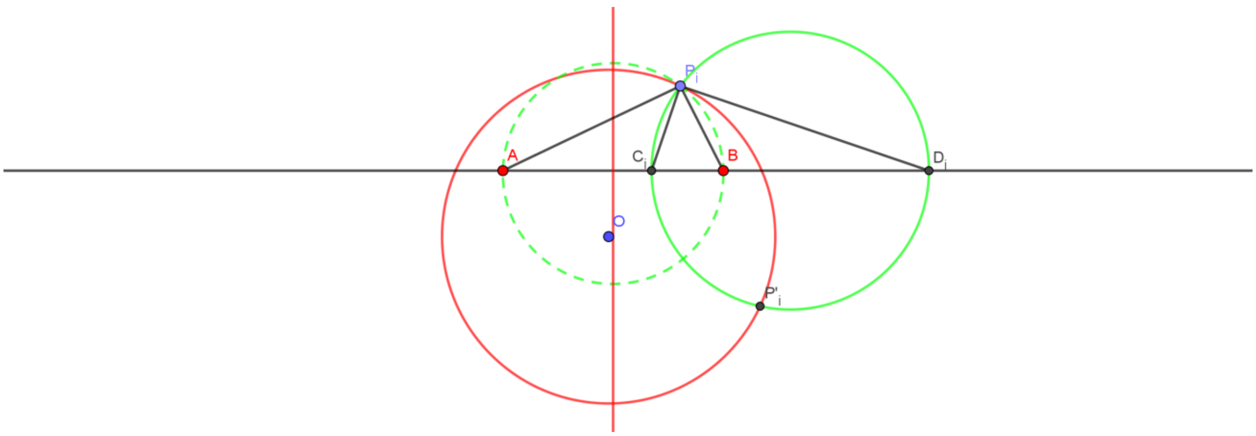

Fig. 4, Figure shows three circles intersecting at P_i. The Apollonius circle, the circle orthogonal to it at P_i and the given circle (red). AP_i is an incident ray of light at P_i. P_i B is the reflected ray.

The ray AP_i gets reflected to go along P_i B making equal angles with P_i C.

The question arises: Which of the three circles reflects the ray AP_i to go through B? Do all the three circles reflect it? Or, are there many more circles that reflect rays from A to go through B? Is there any criterion to decide which one, if only one of them reflects?

Criterion to decide if a given circle reflects a ray of light from a given point to go through another given point

If the diameter CD of a given circle is divided harmonically by points A and B, then one point lies inside and the other point lies outside the circle. Any point P on that circle (we may call it the mirror circle) reflects rays from A to pass through B or rays from B to pass through A. The reflector circle is the Apollonius circle defined by the ratio of the distances PA and PB. The circle orthogonal to it at P and, passing through points C and D acts as the normal circle (we may call it the mirror circle). Thus, it is to be noted that it is the pair of orthogonal circles that reflect the rays from one point to another point. It is not one circle that decides the reflection of a ray of light incident at a point on it. (see the Appendix).

Solution to Alhazen problem

From the criterion of harmonic division given above, we see that the given circle is not a member of the family of α circles. It is not a member of the family of β circles either. Therefore, it is clear that it is impossible for the given circle to reflect rays from one given point to another, since both the given points lye inside it.

Looking at it from a different angle, if the condition required to be satisfied is only that, that the point of reflection merely lies on the circle, then every point on the given circle satisfies this condition. As such, all points on the given circle reflect rays from one given point to another in it. But note that the circle containing the points is not the circle that reflects the rays from one given point to another point in it.

Discussion

Ptolemy's solutions to the problem correspond to points of intersection of the given circle and the circle passing through the three points – the center of the given circle and the two given points[6]. But the two circles are not orthogonal at the solution points. Therefore, those points cannot be considered to be the end points of path of reflection from the given circle.

Huygens solutions correspond to points of intersection of the given circle and a hyperbola[3]. Obviously, the two conic sections cannot be orthogonal at the solution points. Therefore, those points cannot be considered to be the points of reflection from the given circle. However, it is possible that the ellipse orthogonal to the hyperbola at that point could act as the reflector.

Fujimura et. al[1]. call this problem as 'Ptolemy - Alhazen interior Problem'. Their solutions correspond to the solutions of a fourth degree equation and include all points of common tangency of an ellipse and a unit circle. But it could be the ellipse and not the circle that acts as the reflector. In their geometric approach to the problem their solutions correspond to points of intersection of the given circle (a unit circle) and a conic section. For the exterior problem they show the points of intersection of the unit circle and a hyperbola as the solution points. Here again, it cannot be ruled out that the ellipse which is orthogonal to the hyperbola at the solution points that acts as the reflector.

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Appendix

It is not only in the case of reflection of light rays at spherical surfaces that *two* circles (orthogonal) are involved. In general, reflection at curved surfaces involves two orthogonal conics, for example, two coaxial, confocal central conics such as an ellipse and hyperbola; or two coaxial, confocal, intersecting parabolas. The condition to be satisfied being that that the point of incidence must be a point of intersection of two orthogonal conics.

The internal and external bisectors (which are orthogonal) of the angle subtended by a reflected ray couple, define two points of intersection with the line through the feet of the ray couple and, a circle – the Apollonius circle. The feet of the ray couple (the object and the eye as they are usually referred to) define two pints that intersect the diameter of the Apollonius circle harmonically.

Thus, basically it is the harmonic division of a line segment in a given ratio that underlies the solution of Alhazen Problem, through the construction of Apollonius circle and reflection of light from it.

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