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A constructive approach to nonstandard analysis☆

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Abstract

In the present paper we introduce a constructive theory of nonstandard arithmetic in higher types. The theory is intended as a framework for developing elementary nonstandard analysis constructively. More specifically, the theory introduced is a conservative extension of $\mathbf{HA}^{\omega} + \mathbf{AC}$. A predicate for distinguishing standard objects is added as in Nelson's internal set theory. Weak transfer and idealisation principles are proved from the axioms. Finally, the use of the theory is illustrated by extending Bishop's constructive analysis with infinitesimals.

1. Introduction

The constructive or intuitionistic approach to analysis [3, 8] still seems to be devoid of a full-fledged nonstandard aspect. As is well known, Robinson's classical nonstandard analysis from 1960 is based on highly nonconstructive notions. Schmieden and Laugwitz [27] introduced a nonstandard analysis which is far more constructive, albeit classical reasoning is still used. The ideas were further developed in subsequent papers by Laugwitz [11–13]. In their approach the strong transfer principle is lacking, so it cannot be related to standard analysis as easily as in that of Robinson. Earlier Chwistek [5, pp. 209–216] had suggested a similar interpretation of infinitesimals, but apparently did not develop this very far. Martin-Löf [17] makes a conceptual analysis of the notions of choice sequence [29], and of stream in computer science. From these he arrives at a definition of nonstandard objects in constructive type theory, which is quite similar to that of Laugwitz and Schmieden. The type theory is extended by infinite numbers, and its logic is given a nonstandard interpretation. Given this interpretation, the full transfer principle holds. Mycielski [21] develops a locally constructive theory of infinitesimals, where every proof can be interpreted in a finite model. There are other interesting approaches which are not of immediate concern to

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us here. As for Brouwer intuitionism [8] there is a first attempt by Vesley [31]. Moerdijk and Reyes [20] use topos theory to develop calculus with different kinds of infinitesimals. The logic used in the formal theories of their approach is intuitionistic, but the necessary properties of their models are not proved constructively. In Moerdijk [19] a constructive sheaf model of nonstandard arithmetic is given, and it is shown that it has a full transfer principle relative to the standard model. It remains to see whether it holds also for higher type arithmetic.

To give a quick but incomplete picture of the basic idea in Schmieden and Laugwitz' paper, we could say that they work with the reduced power of the reals, modulo the cofinite filter on the natural numbers, whereas Robinson's approach amounts to using instead a nonprincipal ultrafilter. Thus in the former approach two sequences of real numbers are identified if they eventually agree; this makes it possible to interpret infinitesimals as sequences converging to zero, and infinite numbers as sequences growing beyond all bounds. Because of the properties of the cofinite filter, the law of trichotomy fails, leaving us with numbers of indeterminate size, for instance those given by alternating sequences. This is in contrast to Robinson's nonstandard reals.

Another important, and related, idea is the *nonarchimedean extension* of arithmetical theories. This is an extension with one or many symbols for infinite numbers. The possibility of using the extended theories for developing elementary nonstandard analysis has been perceived by several authors: Jensen [10], Laugwitz [14], Liu [16], Martin-Löf [17] and Mycielski [21]. Laugwitz [14] gave a nonconstructive variant involving infinite proof rules.

The content of the paper is outlined as follows. In Section 2 we give some metamathematical results on nonarchimedean extensions, e.g. Martin-Löf's interpretation of infinity symbols. We also indicate how such theories might be used. Unfortunately, they have no useful external notions, such as being infinitesimal. In Section 3 we introduce a new theory, *internal* $\mathbf{HA}^{\omega}$, which remedies this limitation and where it is possible to distinguish standard and nonstandard objects. This theory is partly inspired by Nelson's [22] internal set theory. Internal $\mathbf{HA}^{\omega}$ is a conservative extension of $\mathbf{HA}^{\omega} + \mathbf{AC}$, that is of intuitionistic arithmetic in all finite types with an axiom of choice. In the internal theory, weak forms of the transfer and saturation principles can be proven. The theory formalises the main idea behind Schmieden and Laugwitz [27], that every nonstandard object is represented by an infinite sequence of standard objects. Indeed, its intended model is essentially a reduced power of a standard model of $\mathbf{HA}^{\omega} + \mathbf{AC}$. The model is given in Section 4. These features taken together make it possible, we believe, to extend Bishop constructivism with nonstandard methods. An investigation is started in Section 5.

2. Nonarchimedean extensions of arithmetic

Elementary nonarchimedean extensions of the real number structure R can be obtained in essentially two different ways, both nonconstructive: one is to use an ultrapower construction, the other is to use the compactness theorem. As a background for this section we briefly review the latter method. Let T^* be the theory

$$Th(R) \cup \{\underline{n} < \omega : n \in N\},$$

where $Th(R)$ is the theory of R and $\underline{n}$ is the numeral $S^n(0)$. Clearly every finite subset of T^* has a model, namely R with ω interpreted as a sufficiently large number. By compactness there is a model R^* of T^*. Since $R^* \models Th(R)$, R^* is an elementary extension of R. Thus a nonarchimedean extension of R, with the same true first-order formulas, has been constructed. Consequently we have the *transfer principle* for formulas A of $Th(R)$:

$$R \models A \text{ if and only if } R^* \models A.$$

As is well known, such a pair of models is sufficient for carrying out large parts of elementary nonstandard analysis (see for example [1] or [9]). A natural question is: can nonstandard analysis be done within a theory extended with just constants for infinite numbers? Such a theory is called *nonarchimedean.* In the introduction we mentioned that several persons have worked on this question. The most interesting for the present discussion are Martin-Löf [17] and Mycielski [21]. Through their work it became clear that there is nothing intrinsically nonconstructive in the notion of a nonarchimedean theory. Nonarchimedean extensions of intuitionistic arithmetic (**HA**, **HA**$^\omega$, etc.) with infinite numbers should be candidates for constructivising elementary nonstandard analysis. There are however limitations to this simple-minded approach (cf. Section 2.3 below).

2.1. Nonarchimedean extensions

Definition 2.1. Let T be a theory containing **HA**, and let α be a symbol not in the language of T. Define two different types of nonarchimedean extensions:

1. $T[\alpha] = T \cup \{\underline{n} < \alpha : n \in N\}$,
2. $T\langle\alpha\rangle = T \cup \{t < \alpha : t \text{ is a closed term in } T\}$

($<$ is the order relation on natural numbers, which is primitive recursive.)

When T is **HA** or **HA**$^\omega$ the extensions are equivalent. The interpretation of infinity symbols is given by the following simple theorem.

Theorem 2.2 (Martin-Löf [17]). *Let the theory T contain* **HA**, *and assume α is a symbol not in T. Then*

$$T[\alpha] \vdash A(\alpha) \Leftrightarrow (\exists n \in N)\, T \vdash A(\underline{n} + x),$$

where x is a fresh variable.

Proof. ($\Rightarrow$) Since proofs are finite, $A(\alpha)$ must be proved from T and finitely many axioms

$$\underline{n_1} < \alpha, \ldots, \underline{n_k} < \alpha.$$

Let $n = \max(n_1, \ldots, n_k) + 1$. We can then replace α by $\underline{n} + x$ throughout the proof, and it remains valid, provided the variable x is fresh w.r.t. the proof.

($\Leftarrow$) Let $\ominus$ denote cut-off subtraction. Since $\mathbf{HA} \vdash y > \underline{n} \to \underline{n} + (y \ominus \underline{n}) = y$, substituting $\alpha \ominus \underline{n}$ for x yields

$$T[\alpha] \vdash A(\alpha). \qquad \square$$

Theorem 2.3. *Let T be a theory containing* **HA**, *and suppose α is not in T. Then*

$$T\langle\alpha\rangle \vdash A(\alpha) \iff \textit{there exists a closed term } t \textit{ in } T \textit{ with } T \vdash A(t + x),$$

where x is a fresh variable.

Proof. Analogous to the above. $\square$

The theorems work for a wide variety of theories T (which need not be based on intuitionistic logic). In the case of $\mathbf{HA}^\omega$, α is introduced as a constant of type 0.

Corollary 2.4. *Both $T[\alpha]$ and $T\langle\alpha\rangle$ are conservative extensions of T.*

2.2. Successive extensions

The difference between the two types of extensions appears when considering successive extensions. We have

$$\mathbf{HA}[\omega][\omega'] \vdash A(\omega, \omega') \iff (\exists n \in N)\mathbf{HA} \vdash A(\underline{n} + x, \underline{n} + y). \tag{1}$$

Applying Theorem 2.3 twice it can be seen that

$$\mathbf{HA}\langle\omega\rangle\langle\omega'\rangle \vdash A(\omega, \omega')$$

if and only if there are closed terms t of **HA** and $t'(\omega)$ of $\mathbf{HA}\langle\omega\rangle$ s.t.

$$\mathbf{HA} \vdash A(t + x, t'(t + x) + y).$$

By definition $\omega < \omega'$ holds in $\mathbf{HA}\langle\omega\rangle\langle\omega'\rangle$, but (1) shows that this is impossible in $\mathbf{HA}[\omega][\omega']$.

Successive extensions according to the first type are proper.

Proposition 2.5. *Let T be a consistent theory containing* **HA**. *Then there are no terms $f(z)$ and $g(z)$ of T, such that for all formulas $A(u, v)$ of T:*

$$T[\omega] \vdash A(f(\omega), g(\omega)) \iff (\exists m \in N)\, T \vdash A(\underline{m} + x, \underline{m} + y).$$

Proof. Assume that $f(z)$ and $g(z)$ in fact are such. We have

$$T[\omega] \vdash \exists z\,[f(z) = f(\omega) \wedge g(z) = g(\omega)].$$

By the assumption, for some $m \in N$,

$$T \vdash \exists z\,[f(z) = \underline{m} + x \wedge g(z) = \underline{m} + y]. \tag{2}$$

Further, since $T \vdash \underline{m+1} + x > \underline{m}$, we have by the assumption

$$T[\omega] \vdash f(\omega) > \underline{m}.$$

Hence by Theorem 2.2,

$$T \vdash f(\underline{n} + u) > \underline{m} \tag{3}$$

for some n. Substituting 0 for x and $0, \dots, \underline{n}$ for y in (2), successively, yields

$$T \vdash \exists z_0 \cdots z_n\,[f(z_0) = \cdots = f(z_n) = \underline{m} \wedge g(z_0) = \underline{m} \wedge \cdots \wedge g(z_n) = \underline{m+n}].$$

Clearly all z_i are distinct. By the pigeonhole principle, one of the z_i's is greater than or equal to $\underline{n}$, contradicting (3). □

For the second type of extension we have a similar result for the first two levels above arithmetic.

Proposition 2.6. *There are no terms $f(z)$ and $g(z)$ such that for all formulas $A(u, v)$ of* **HA**: $\mathbf{HA}\langle\omega\rangle \vdash A(f(\omega), g(\omega))$ *if and only if for some terms $t, t'(u)$ of* **HA**,

$$\mathbf{HA} \vdash A(t + x, t'(t + x) + y).$$

Proof. Analogous to the previous proposition, noting that $\mathbf{HA}[\omega]$ and $\mathbf{HA}\langle\omega\rangle$ are equivalent. □

2.3. Nonstandard analysis in a nonarchimedean theory

Take an ordinary formulation of elementary analysis in an arithmetical theory T such as $\mathbf{HA}^\omega + \mathbf{AC}$. Consider the polynomial

$$\widetilde{\exp}(x) = \sum_{k=0}^{\omega} \frac{x^k}{k!},$$

in $T[\omega]$, where ω is a symbol for an infinite number. This symbol can be treated like a finite number inside the theory. Thus the derivative of $\widetilde{\exp}$ is immediately seen to be defined, and

$$D\,\widetilde{\exp}(x) = \sum_{k=0}^{\omega-1} \frac{x^k}{k!}.$$

In a nonarchimedean theory $\widetilde{\exp}(x)$ is of course not the same as $D\,\widetilde{\exp}(x)$, since ω can be interpreted as a (sufficiently large) finite number. But if x is not too big, the difference

$$\widetilde{\exp}(x) - D\,\widetilde{\exp}(x) = x^{\omega}/\omega!$$

should be small. In Robinson's nonstandard analysis this difference is indeed infinitesimal for finite x. In $T[\omega]$ we have nevertheless, for all numerals $\underline{m}, \underline{n}$

$$T[\omega] \vdash |x| < \underline{m} \to |\widetilde{\exp}(x) - D\,\widetilde{\exp}(x)| < 2^{-\underline{n}}.$$

But there is no way of expressing that two numbers have an infinitesimal difference, inside the theory. This is a severe limitation of nonarchimedean theories. It can be overcome to some extent by using a large stock of infinity symbols (cf. [21]).

2.4. Application to Mycielski's theories

Mycielski [21] treated a nonarchimedean extension of a fragment of Peano arithmetic. We consider his type of extension, but for a theory T containing all of Heyting arithmetic. Let $\{\omega_q : q \in Q\}$ be a set of distinct symbols, not in T, indexed by rational numbers. T^{M} is the union of all theories

$$T\langle\omega_{q_1}\rangle \cdots \langle\omega_{q_n}\rangle,$$

where $q_1 < \cdots < q_n$. This theory has a dense set of infinity symbols.

Proposition 2.7. *T^{M} is conservative over T.*

Proof. If A, expressed in the language of T, is proved in T^{M}, then for some sequence $q_1 < \cdots < q_n$ of rationals

$$T\langle\omega_{q_1}\rangle \cdots \langle\omega_{q_n}\rangle \vdash A.$$

Hence $T \vdash A$, using Corollary 2.4. $\square$

Mycielski calls a function f *rs-continuous* if $s < r$ and

$$|x - y| < 1/\omega_r \to |f(x) - f(y)| < 1/\omega_s.$$

Theorem 2.3 now gives an interpretation of this notion. For simplicity, suppose that f contains no infinity symbols. Then

$$T\langle\omega_s\rangle\langle\omega_r\rangle \vdash f \text{ is } rs\text{-continuous}$$

if and only if, for some terms t_r and t_s,

$$T \vdash |x - y| < \frac{1}{t_r(t_s + v) + u} \to |f(x) - f(y)| < \frac{1}{t_s + v}.$$

We see that this yields a continuity modulus for f.

2.5. *Representability of functions*

Let T be a theory including the symbols 0 and S, so that all numerals $\underline{n}$, $n \in N$, are available. A partial function $\varphi : N^k \to_{\mathrm{p}} N$ is *represented by the term* $t(x_1, \ldots, x_k)$ of T, if for all $n_1, \ldots, n_k, m \in N$:

$$\phi(n_1, \ldots, n_k)\downarrow = m \Leftrightarrow T \vdash t(\underline{n_1}, \ldots, \underline{n_k}) = \underline{m}.$$

($a\downarrow = b$ means that a is defined and equals b.) The following result shows that the term language of a nonarchimedean theory is very rich.

Theorem 2.8. *Let T be* **HA** *or* $\mathbf{HA}^\omega + \mathbf{AC}$. *Then all partial recursive functions are representable by terms in* $T[\omega]$.

Proof. Cf. [24]. We note that the restriction to **HA** in that paper is inessential. □

3. Internal $\mathbf{HA}^\omega$

Intuitionistic arithmetic in all finite types, $\mathbf{HA}^\omega$, suffices to formalize elementary parts of constructive analysis (cf. the discussion in [7]). The theory contains function types, such as the type of all functions on natural numbers, and the type of functions on this type, and so on. In particular, the real numbers and their functions can thereby be quantified over. However there is no concept of set in $\mathbf{HA}^\omega$, so, for instance, the general theory of metric spaces [4] has to be omitted.

In this section we shall extend $\mathbf{HA}^\omega$ with nonstandard objects. This new system, *internal* $\mathbf{HA}^\omega$, $\mathbf{iHA}^\omega$ for short, is partly inspired by Nelson's [22] extension of set theory. To each type σ we associate a predicate $St^\sigma(x)$, with the intuitive meaning that x is a standard object of type σ. The intended model of the theory can be described as the reduced power of the standard model of $\mathbf{HA}^\omega$ (w.r.t. the cofinite filter on the natural numbers). (For a general definition of reduced powers of many sorted structures, see [18].) In Section 4, we shall model $\mathbf{iHA}^\omega$ inside $\mathbf{HA}^\omega$ furnished with the axiom of choice (AC). By this procedure it will immediately be clear that the internal theory is constructive, and moreover conservative over $\mathbf{HA}^\omega + \mathbf{AC}$. This is in analogy with Nelson's internal set theory: it is a conservative extension of set theory.

3.1. *Arithmetic in all finite types*

We first present the system $\mathbf{HA}^\omega$ in some detail. (For a discussion the reader is referred to [28, 30].) This system is based on many-sorted intuitionistic logic, where the sorts are the *finite type symbols*, $\mathscr{T}$, defined inductively by:

$0 \in \mathscr{T}$	(the type of natural numbers)
$\sigma, \tau \in \mathscr{T} \Rightarrow \sigma \times \tau \in \mathscr{T}$	(product types)
$\sigma, \tau \in \mathscr{T} \Rightarrow \sigma \to \tau \in \mathscr{T}$	(function types)

The language of $\mathbf{HA}^\omega$ is defined as follows. We let $t : \sigma$ denote that t is of type σ.

- For each type $\sigma \in \mathscr{T}$ there are infinitely many variables $x^\sigma, y^\sigma, z^\sigma, \ldots$ of type σ.
- We have the following constants:

$0:0$	(zero)
$S:0 \to 0$	(successor)
$\mathbf{k}^{\sigma,\tau}:\sigma \to (\tau \to \sigma)$	(function space
$\mathbf{s}^{\rho,\sigma,\tau}:(\rho \to (\sigma \to \tau)) \to ((\rho \to \sigma) \to (\rho \to \tau))$	combinators)
$\mathbf{p}^{\sigma,\tau}:\sigma \to (\tau \to (\sigma \times \tau))$	(pairing)
$\mathbf{p}_0^{\sigma,\tau}:(\sigma \times \tau) \to \sigma,\ \mathbf{p}_1^{\sigma,\tau}:(\sigma \times \tau) \to \tau$	(projections)
$\mathbf{r}^\sigma:\sigma \to ((\sigma \to (0 \to \sigma)) \to (0 \to \sigma))$	(recursor)

- On each type σ there is a predicate symbol $(=_\sigma)$ for equality.
- For each pair of types σ, τ there is a function symbol for application, $Ap^{\sigma,\tau}$, with two arguments: the first being of type $\sigma \to \tau$ and the second of type σ, yielding a value of type τ.

The *terms* of $\mathbf{HA}^\omega$ are formed from applications, constants and variables, complying to the type discipline. The *formulas* are built from atomic formulas in the usual manner. An atomic formula is either $\perp$ (absurdity) or $t =_\sigma s$ where $t:\sigma$ and $s:\sigma$ are terms. Henceforth we use the convention that $A(x_1, \ldots, x_n)$ means that all free variables in A are among $x_1, \ldots, x_n$. It will be convenient to make the following abbreviations: $\sigma\tau$ for $\sigma \to \tau$ and ts for $Ap^{\sigma,\tau}(t,s)$. We moreover write $\sigma_1\sigma_2 \cdots \sigma_n$ for $\sigma_1(\sigma_2 \cdots (\sigma_{n-1}\sigma_n) \cdots)$ and $s_1s_2 \cdots s_n$ for $(\cdots (s_1s_2) \cdots s_n)$. The type information attached to subterms will frequently be omitted. We take the letters i, j, k, ℓ, m, n to be variables of type 0, unless otherwise indicated. If $\vec{x} = x_1, \ldots, x_n$, we write $\vec{x}y$ for $x_1y, \ldots, x_ny$.

Equality axioms:

$$x = x$$

$$x = y \to y = x$$

$$x = y \to y = z \to x = z$$

$$x = y \to u = v \to xu = yv$$

Defining axioms for combinators:

$$\mathbf{k}xy = x$$

$$\mathbf{s}xyz = xz(yz)$$

$$\mathbf{p}_0(\mathbf{p}xy) = x \qquad \mathbf{p}_1(\mathbf{p}xy) = y$$

$$\mathbf{p}(\mathbf{p}_0z)(\mathbf{p}_1z) = z$$

$$\mathbf{r}xy0 = x$$

$$\mathbf{r}xy(Sz) = y(\mathbf{r}xyz)z$$

Arithmetical axioms:

- The fourth Peano axiom: $\neg 0 = S(x)$.
- The induction schema: for every formula A assume

$$A(\vec{u},0) \wedge \forall n\,(A(\vec{u},n) \to A(\vec{u},S(n))) \to \forall n\, A(\vec{u},n).$$

The above axioms constitute $\mathbf{HA}^\omega$. We also consider an extension $\mathbf{HA}^\omega + \mathbf{AC}$ with the *axiom of choice*

(AC) $\quad \forall x^\sigma \exists y^\tau A(\vec{u},x,y) \to \exists z^{\sigma\tau} \forall x^\sigma A(\vec{u},x,zx)$,

for each formula A.

3.2. The internal theory

We expand the language of $\mathbf{HA}^\omega$ by adding a new constant for an infinite number $\infty : 0$, and for each type $\sigma \in T$ a new predicate

$St^\sigma(x)\quad$ for "x is standard".

The intended interpretation of $St^\sigma(x)$, in the reduced power model, is that x is eventually constant. The intended interpretation of ∞ is the simplest infinite number in this model, namely the identity $\lambda i.i$. The axioms directly taken over from $\mathbf{HA}^\omega$ are: (I) the equality axioms, (II) the defining axioms for combinators, (III) the fourth Peano axiom. A new equality axiom for the standard predicate is necessary:

(IV) $x = y \to St(x) \to St(y)$.

A term is said to be *internal* if it does not contain the infinity constant ∞. The internal constants c are standard:

(V) $St(c)$.

To formulate the remaining axioms we introduce the abbreviations $\forall^{\mathrm{st}} x^\sigma A(\vec{u},x)$ for $\forall x^\sigma (St^\sigma(x) \to A(\vec{u},x))$, and $\exists^{\mathrm{st}} x^\sigma A(\vec{u},x)$ for $\exists x^\sigma (St^\sigma(x) \wedge A(\vec{u},x))$. These are the relativisations of the respective quantifiers to standard objects. Standard functions applied to standard arguments give standard values:

(VI) $\forall^{\mathrm{st}} x^{\sigma\tau} \forall^{\mathrm{st}} y^\sigma St(xy)$.

Induction on standard numbers, so called *external induction*, is admissible:

(VII) $A(\vec{u},0) \wedge \forall^{\mathrm{st}} n\,(A(\vec{u},\underline{n}) \to A(\vec{u},Sn)) \to \forall^{\mathrm{st}} n\, A(\vec{u},n)$,

for arbitrary A. Furthermore we have the *external axiom of choice* for any A:

(VIII) $\forall^{\mathrm{st}} x^\sigma \exists^{\mathrm{st}} y^\tau A(\vec{u},x,y) \to \exists^{\mathrm{st}} z^{\sigma\tau} \forall^{\mathrm{st}} x^\sigma A(\vec{u},x,zx)$.

Remark. If A is a formula, let A^{st} be the result of relativising all quantifiers to St. It is not difficult to see that $A \mapsto A^{\mathrm{st}}$ defines an interpretation of $\mathbf{HA}^\omega + \mathbf{AC}$ into the present theory. More precisely, if $\mathbf{HA}^{\omega} + \mathbf{AC} \vdash A(x_1, \dots, x_n)$, then

$$\mathbf{iHA}^{\omega} \vdash St(x_1) \wedge \cdots \wedge St(x_n) \to A^{\mathrm{st}}(x_1, \dots, x_n).$$

The verification uses only the hitherto given axioms.

The following axiom, the *limit principle*, states that every nonstandard object is the "limit" of a sequence of standard objects:

(IX) $\forall x^{\sigma} \exists^{\mathrm{st}} y^{0\sigma} [x = y \infty]$.

Finally, there is the *limit equality axiom* which determines when two such limits are equal:

(X) $\forall^{\mathrm{st}} x^{0\sigma} y^{0\sigma} [x \infty = y \infty \leftrightarrow \exists^{\mathrm{st}} k \forall^{\mathrm{st}} n \geqslant k (xn = yn)]$.

We often refer to such a (standard) k as a *stage*. This concludes the axiomatisation of $\mathbf{iHA}^{\omega}$.

We define classes of formulas which will be of importance later on. A formula free from ∞-symbols A is called *internal* if it does not contain the standard predicate St; the formula is *almost internal* if the St-predicate occurs only in subformulas

$$\forall i^0 [St(i) \wedge i < t \to B] \tag{4}$$

of A, where the free variables of t are either free in A or bound by quantifiers where the range is restricted to standard objects. A variable occurring in such a t is called *confining*. The idea is that such subformulas are really conjunctions of variable finite length when t is standard.. A formula is *subgeometric* it is formed from atomic formulas using only $\wedge$ and $\exists$; the formula is *almost subgeometric* if it in addition can contain universal quantifications of the form (4), subject to the same conditions on t. The wider classes of formulas represent slight but useful improvements, in the Łos principle (see below), over the more natural. The class of *constructive Horn formulas* is the least class $\mathscr{CH}$ such that

- $\mathscr{CH}$ contains the atomic formulas,
- $\mathscr{CH}$ is closed under conjunction, existential and universal quantification,
- if A is subgeometric and $B \in \mathscr{CH}$, then $A \to B \in \mathscr{CH}$.

One can prove that every Horn formula is classically equivalent to a constructive Horn formula, and conversely. If we allow almost subgeometric A in the last clause, we call the resulting class of formulas *almost constructive Horn*. Almost internal formulas, which are almost subgeometric, are called *amenable*, those which are almost constructive Horn are called *light*.

Theorem 3.1 (The Łos principle). *Let $A(\vec{x}, m)$ be an amenable formula where m is not a confining variable. Then*

$$\forall^{\mathrm{st}} \vec{x} [A(\vec{x}, \infty) \leftrightarrow \exists^{\mathrm{st}} k \forall^{\mathrm{st}} n \geqslant k\, A^{\mathrm{st}}(\vec{x}, n)].$$

Proof. By induction on the complexity of A. The case $A \equiv \perp$ is trivial. Let

$$A(\vec{x}, m) \equiv (s_1(\vec{x}, m) = s_2(\vec{x}, m)).$$

Using abstraction on variables we find internal terms t_1, t_2 such that

$$t_i m = s_i(\vec{x}, m)$$

for $i = 1, 2$. The limit equality axiom applied to these terms gives

$$A(\vec{x}, \infty) \leftrightarrow \exists^{\mathrm{st}} k \forall^{\mathrm{st}} n \geqslant k\, A^{\mathrm{st}}(\vec{x}, n).$$

This proves the atomic cases. The conjunctive case is easy. Consider the $\rightarrow$-direction of the $\exists$-case: Suppose $A(\vec{x}, \infty) \equiv \exists z\, B(\vec{x}, z, \infty)$. By the limit principle, there is a standard w such that $B(\vec{x}, w\infty, \infty)$. z is not confining, so the induction hypothesis can be applied, and yields a standard k such that

$$(\forall^{\mathrm{st}} n \geqslant k)\, B^{\mathrm{st}}(\vec{x}, wn, n).$$

Hence for all standard $n \geqslant k$, $\exists^{\mathrm{st}} z\, B^{\mathrm{st}}(\vec{x}, z, n)$.

$\exists$-*case*, $\leftarrow$-*direction*: Suppose

$$\exists^{\mathrm{st}} k\, \forall^{\mathrm{st}} n \geqslant k\, \exists^{\mathrm{st}} z\, B^{\mathrm{st}}(\vec{x}, z, n).$$

Since $\geqslant$ is decidable, $\exists^{\mathrm{st}} z$ can be moved outside $n \geqslant k$. The external axiom of choice can thereby be applied:

$$\exists^{\mathrm{st}} w\, \exists^{\mathrm{st}} k \forall^{\mathrm{st}} n \geqslant k\, B^{\mathrm{st}}(\vec{x}, wn, n).$$

Since z is not confining, we have by the inductive hypothesis, $B(\vec{x}, w\infty, \infty)$.

Bounded $\forall^{\mathrm{st}}$-*case*: Assume that $A(\vec{x}, m) \equiv (\forall^{\mathrm{st}} i < t(\vec{x}))\, B(\vec{x}, i, m)$. We need only to note that if $\vec{x}$ is standard, then $t(\vec{x})$ is standard, and thus

$$(\forall^{\mathrm{st}} i < t(\vec{x}))\, \exists^{\mathrm{st}} k\, (\forall^{\mathrm{st}} n \geqslant k)\, B^{\mathrm{st}}(\vec{x}, i, n)$$

and

$$\exists^{\mathrm{st}} k\, (\forall^{\mathrm{st}} n \geqslant k)(\forall^{\mathrm{st}} i < t(\vec{x}))\, B^{\mathrm{st}}(\vec{x}, i, n)$$

are equivalent. The case now follows by the induction hypothesis. $\square$

Corollary 3.2. *Let $A(\vec{x}, \vec{v})$ be an amenable formula, where none of the variables $\vec{v}$ are confining. Then*

$$\forall^{\mathrm{st}} \vec{x}, \vec{y}\, [A(\vec{x}, \vec{y}\infty) \leftrightarrow \exists^{\mathrm{st}} k\, \forall^{\mathrm{st}} n \geqslant k\, A^{\mathrm{st}}(\vec{x}, \vec{y} n)].$$

Proof. Let $B(\vec{x}, \vec{y}, z) \equiv A(\vec{x}, \vec{y} z)$. By the assumption z is not confining in B and the result follows from the Łos principle. $\square$

Remarks. In Martin-Löf [17] it was pointed out that, in effect, the Łos principle holds for internal, subgeometric formulas, in the setting of type theory. The Łos principle implies that ∞ is infinite; take $A(x, y)$ to be $x < y$.

Theorem 3.3 (The lifting principle). *Let $A(\vec{x}, m)$ be a light formula, where m is not confining. Then*

$$\forall^{\mathrm{st}}\vec{x}\,[\exists^{\mathrm{st}}k(\forall^{\mathrm{st}}n \geqslant k)\,A^{\mathrm{st}}(\vec{x}, n) \to A(\vec{x},\ \infty)].$$

We say that A^{st} is lifted.

Proof. By induction on the complexity of A. The base cases have already been dealt with in Theorem 3.1. The $\wedge$-case is easy, and the $\exists$-case is analogous to that of Theorem 3.1.

$\to$-case: Suppose there is a standard k such that

$$\forall^{\mathrm{st}}n \geqslant k\,[B^{\mathrm{st}}(\vec{x}, n) \to C^{\mathrm{st}}(\vec{x}, n)].$$

The assumption $B(\vec{x},\ \infty)$ gives that for some stage $k' \geqslant k$, $(\forall^{\mathrm{st}}n \geqslant k')\,B^{\mathrm{st}}(\vec{x}, n)$. Thus $(\forall^{\mathrm{st}}n \geqslant k')\,C^{\mathrm{st}}(\vec{x}, n)$, and $C(\vec{x},\ \infty)$ follows from the inductive hypothesis.

$\forall$-case: The case where $A(\vec{x}, m) \equiv (\forall^{\mathrm{st}}i < t(\vec{x}))\,B(\vec{x}, i, m)$ is analogous to the corresponding case in Theorem 3.1. Say $A(\vec{x}, m) \equiv \forall z^{\sigma}\,B(\vec{x}, z, m)$, where B is a light formula. Suppose

$$\exists^{\mathrm{st}}k(\forall^{\mathrm{st}}n \geqslant k)\,\forall^{\mathrm{st}}z^{\sigma}\,B^{\mathrm{st}}(\vec{x}, z, n).$$

Let $w{:}o\sigma$ be standard. Hence for $(\forall^{\mathrm{st}}n \geqslant k)\,B^{\mathrm{st}}(\vec{x}, wn, n)$. Since z is not confining, we have by the inductive hypothesis

$$B(\vec{x}, w\infty,\ \infty).$$

The limit principle gives the result. $\square$

The lifting principle corresponds to the well-known result from classical model theory, that taking reduced powers preserves the truth of Horn formulas.

Remark. We point out a possible source of confusion. Let $P(i)$ be an internal atomic formula. Then both $A(x) \equiv (\forall^{\mathrm{st}}i < x)\,P(i)$ and $A'(x) \equiv (\forall i < x)\,P(i)$ are light, and they are not equivalent. However A^{st} and A'^{st} are equivalent. Thus certain formulas can be lifted in two different ways.

Theorem 3.4 (The transfer principle). *Let $A(\vec{x}) \equiv \forall\vec{y}\,[B(\vec{x}, \vec{y}) \to C(\vec{x}, \vec{y})]$ be a light formula with B and C almost subgeometric. Then*

$$\forall^{\mathrm{st}}\vec{x}\,[A^{\mathrm{st}}(\vec{x}) \leftrightarrow A(\vec{x})].$$

Proof. This follows from the Łos principle and the lifting principle. $\square$

Remark. The restriction in the lifting principle to Horn formulas is necessary in general. There are true formulas (a) $\forall^{\mathrm{st}} n(A \vee B)$ and (b) $\forall^{\mathrm{st}} n(\neg A \to B)$ that cannot be lifted. As for (a), let $A \equiv (n \bmod 2 = 0)$ and $B \equiv (n \bmod 2 = 1)$. The lifted formula would imply $A(\infty) \vee B(\infty)$, which is impossible. For (b) consider $A \equiv (n > 0)$, $B \equiv (n = 0)$. But $\neg(\infty \bmod 2 > 0)$ and $\infty \bmod 2 \neq 0$. See also Section 5.1.

3.3. Definitional extensions

It turns out that Theorem 3.1 and its consequences are sometimes too weak for application to constructive analysis. The equality relation for real numbers, e.g., is given by a logically complex formula. We wish to treat this and similar relations as if they were atomic in order to obtain useful transfer principles.

Let L be the language of $\mathbf{iHA}^\omega$. Let $R_1, \ldots, R_N, R_1^{\mathrm{st}}, \ldots, R_N^{\mathrm{st}}$ be new relation symbols, and denote L extended with these L'. Suppose we are given arbitrary formulas A_i $(i = 1, \ldots, N)$ of L with free variables among $x_1, \ldots, x_{n_i}$. We let $\mathbf{iHA}^\omega[\vec{R}, \vec{A}]$ be the extension of $\mathbf{iHA}^\omega$ obtained by adding the definitions

$$R_i^{\mathrm{st}}(x_1, \ldots, x_{n_i}) \leftrightarrow A_i(x_1, \ldots, x_{n_i}),$$

$$R_i(y_1\, \infty, \ldots, y_{n_i}\, \infty) \leftrightarrow \exists^{\mathrm{st}} k (\forall^{\mathrm{st}} m \geqslant k)\, R_i^{\mathrm{st}}(y_1 m, \ldots, y_{n_i} m).$$

An equality axiom for each new predicate is also added. The limit principle in L' guarantees that R_i is completely defined. We note that for standard $x_1, \ldots, x_{n_i}$:

$$R_i^{\mathrm{st}}(x_1, \ldots, x_{n_i}) \leftrightarrow R_i(x_1, \ldots, x_{n_i}).$$

R_i^{st} is the standard predicate, while R_i is said to be its *nonstandard version*. Moreover the axiom schemata of $\mathbf{iHA}^\omega$ are extended to cover also formulas of L'. It is immediate that $\mathbf{iHA}^\omega[\vec{R}, \vec{A}]$ is a conservative extension of $\mathbf{iHA}^\omega$. A formula in L' is (*almost*) *internal* if it is (almost) internal in the usual sense, and possibly contains R_i-predicates, but not R_i^{st}-predicates. We extend the $(\cdot)^{\mathrm{st}}$-translation so that R_i is translated by R_i^{st}.

Theorem 3.1, Corollary 3.2 and Theorem 3.3 also hold when taking internal formulas in the extended sense. This is seen by observing that the new base cases are trivially true by the very definition of the new predicates.

3.4. Idealisation principles

The idealisation principle of Nelson's internal set theory states that for internal formulas $B(x, y)$: *If for every finite standard set z, there exists a set x such that for all $y \in z$, $B(x, y)$, then there exists a set x such that for all standard y, $B(x, y)$.* In Robinson's nonstandard analysis this corresponds to the saturation property of enlargements. We prove weak versions of these principles inside $\mathbf{iHA}^\omega$, adapted to the restricted expressiveness of our theory. Examples are given in this and later sections which show that they are still useful.

Theorem 3.5 (Subgeometric saturation). *Let $A(\vec{u}, v, \vec{w}, i)$ be an amenable formula, where $\vec{u}$ are nonconfining, and which satisfies the chain condition*

$$\forall^{st} i\, \forall \vec{u}, v, \vec{w}\, [A(\vec{u}, v, \vec{w}, i+1) \rightarrow A(\vec{u}, v, \vec{w}, i)].$$

(i) *If v is also nonconfining in A,*

$$\forall \vec{u}\, \forall^{st} \vec{w}\, [\forall^{st} i\, \exists v\, A(\vec{u}, v, \vec{w}, i) \rightarrow \exists v\, \forall^{st} i\, A(\vec{u}, v, \vec{w}, i)].$$

(ii) *Let A' be the formula resulting by removing all restrictions to St in A. Then*

$$\forall \vec{u}\, \forall^{st} \vec{w}\, [\forall^{st} i\, \exists^{st} v\, A(\vec{u}, v, \vec{w}, i) \rightarrow \exists v\, \forall^{st} i\, A'(\vec{u}, v, \vec{w}, i)].$$

Proof. We first prove (i). Let $\vec{u} = \vec{x}\infty$, where $\vec{x}$ are standard; let $\vec{w}$ be standard. Suppose that for each standard i there is a $v_i = y_i \infty$ with y_i standard and $A(\vec{u}, v_i, \vec{w}, i)$. Thus by Corollary 3.2, we can successively choose an increasing sequence of stages $k_0 < k_1 < k_2 < \cdots$ such that

$$(\forall^{st} j \geqslant k_i)\, A^{st}(\vec{x}j, y_i j, \vec{w}, i). \tag{5}$$

Now define z as follows: for $j < k_0$, let $z(j) = k_0$; for $k_m \leqslant j < k_{m+1}$, let $z(j) = y_m(j)$. (To find the m we need only to search up to and including k_j.) We prove that $v = z\infty$ is the desired object. Let i be standard and suppose that $j \geqslant k_i$. Then for some l,

$$k_{i+l} \leqslant j < k_{i+l+1},$$

so $z(j) = y_{i+l}(j)$, and thus by (5),

$$A^{st}(\vec{x}j, zj, \vec{w}, i+l).$$

By transfer the chain condition holds also when applying the $(\cdot)^{st}$-translation. Using this and induction on l we get

$$A^{st}(\vec{x}j, zj, \vec{w}, i).$$

Hence by the Łos principle $A(\vec{u}, v, \vec{w}, i)$. (ii) is proved by observing that since v_i is standard, Corollary 3.2 is applicable without requirement on v. Now $y_i j$ can be replaced by v_i in (5) and in the sequel. The last step is modified: from $(\forall^{st} j \geqslant k_i)\, A^{st}(\vec{x}j, zj, \vec{w}, i)$, we conclude by *lifting* (with $(\forall j \geqslant k_i)\, A'$ as the light formula of Theorem 3.3):

$$(\forall j \geqslant k_i)\, A'(\vec{x}j, zj, \vec{w}, i).$$

Thus letting $j = \infty$ yields $A'(\vec{u}, v, \vec{w}, i)$. $\square$

The idea of this proof is present already in customary proofs of saturation for ultra products. A constructive proof of the restricted form above, for reduced products, is given in Palmgren [23].

Corollary 3.6 (Weak idealisation). *Let $A(\vec{u}, v, \vec{w}, i)$ be an amenable formula, where $\vec{u}$ and v are nonconfining. Then*

$$\forall \vec{u}\, \forall^{st} \vec{w}\, [\forall^{st} i\, \exists v\, (\forall^{st} j < i)\, A(\vec{u}, v, \vec{w}, j) \rightarrow \exists v\, \forall^{st} i\, A(\vec{u}, v, \vec{w}, i)].$$

Proof. Let $B(\vec{u}, v, \vec{w}, i) \equiv (\forall^{st} j < i)\, A(\vec{u}, v, \vec{w}, j)$. This formula is amenable and nonconfining in $\vec{u}$ and v. It also satisfies the chain condition of Theorem 3.5. The result is immediate from part (i) of the theorem. □

Laugwitz [13, p. 119] proves a similar result for enlargements based on cofinite filters. We now have weak analogues of the three nonstandard principles of Nelson's internal set theory:

idealisation	weak idealisation,
standardisation	external induction,
transfer	Theorem 3.4.

Corollary 3.7 (du Bois-Reymond's lemma (Laugwitz [13])). *Suppose (u_i) is a decresing sequence of numbers, where u_i is infinite for each standard i. Then there is an infinite v with $\forall^{st} i\, (u_i \geqslant v)$.*

Proof. We have by the assumption

$$\forall^{st} i\, \exists v\, (\forall^{st} j < i)\, [u_j \geqslant v \wedge v \geqslant j].$$

By Corollary 3.6 there exists a v such that

$$\forall^{st} i\, [u_i \geqslant v \wedge v \geqslant i].$$

Thus v is the desired number. □

The following is a refinement of a result due to Martin-Löf which he proved directly.

Corollary 3.8 (Overspill). *Let $A(\vec{u}, \vec{w}, n)$ be an amenable formula, where the variables $\vec{u}$ are nonconfining. Then*

$$\forall \vec{u}\, \forall^{st} \vec{w}\, [\forall^{st} n\, A(\vec{u}, \vec{w}, n) \rightarrow (\exists v)\, v \text{ infinite } \wedge\ \forall \mu \leqslant v\, A'(\vec{u}, \vec{w}, \mu)],$$

where A' is A when all restrictions to St have been removed.

Proof. Let $B(\vec{u}, n', \vec{w}, n) \equiv (\forall^{st} \mu \leqslant n')\, A(\vec{u}, \vec{w}, \mu) \wedge n' \geqslant n$. This formula satisfies the conditions of Theorem 3.5(ii). From $\forall^{st} n\, A(\vec{u}, \vec{w}, n)$ it follows that

$$\forall^{st} n\, \exists^{st} n'\, B(\vec{u}, n', \vec{w}, n),$$

choosing $n' = n$. The theorem yields a number v such that $\forall^{\mathrm{st}} n (v \geqslant n)$ and

$$(\forall \mu \leqslant v) A'(\vec{u}, \vec{w}, \mu),$$

as required. □

4. A model of internal $\mathbf{HA}^{\omega}$

We construct the model inside $\mathbf{HA}^{\omega} + \mathbf{AC}$. The construction is essentially a reduced power; a nonstandard object of type σ is interpreted as a sequence of ordinary objects of type σ, and two such sequences are considered equivalent if they eventually agree. The modelling proves conservativity over $\mathbf{HA}^{\omega} + \mathbf{AC}$. Moreover, everything that is true in the model is provable in $\mathbf{iHA}^{\omega}$. We let $(\cdot)^*$ denote the interpretation function from the expressions of $\mathbf{iHA}^{\omega}$ to the expressions of $\mathbf{HA}^{\omega}$. It will satisfy the conditions:

- If σ is a type, then σ^* is a type.
- If $a : \sigma$, then $a^* : \sigma^*$.
- If A is a formula, then A^* is a formula.

The interpretation is defined as follows for the different syntactic categories.

Types: For $\sigma \in \mathscr{T}$, let $\sigma^* \equiv 0 \to \sigma$.

Terms: Each variable x^{σ} is interpreted as a new variable $x^{*\sigma^*}$.

If $c : \sigma$ is one of the internal constants $0, S, \mathbf{s}, \mathbf{k}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1$ or $\mathbf{r}$, then

$$c^* \equiv \mathbf{k}^{\sigma,0} c.$$

Thus c^* is constantly c, and $c^* : \sigma^*$. The infinite number is however a nonstandard object, and is interpreted as

$$\infty^* \equiv \mathbf{I}^0$$

where $\mathbf{I}^{\sigma}$ is the identity function on σ. Thus $\infty^* : 0^*$. The application function is interpreted as acting pointwise along the sequence; if $t^* : (\sigma \to \tau)^*$, $s^* : \sigma^*$, then

$$Ap^{\sigma,\tau}(t, s)^* \equiv \mathbf{s}^{0,\sigma,\tau} t^* s^*.$$

This term clearly has type τ^*. Note that for $m : 0$, $(ts)^* m = t^* m(s^* m)$.

Formulas: For sequences $u, v : 0 \to \sigma$ we define two predicates

$$St^*_{\sigma}(u) \equiv \exists k \, \forall m \geqslant k \, [um =_{\sigma} u(Sm)],$$

$$u =^*_{\sigma} v \equiv \exists k \, \forall m \geqslant k \, [um =_{\sigma} vm].$$

Thus $St^*_{\sigma}(u)$ if and only if u is eventually constant. It is easily seen that the second predicate is an equivalence relation, which is respected by the former predicate. The interpretation of atomic formulas is as follows:

$$St^{\sigma}(t)^* \equiv St^*_{\sigma}(t^*)$$

and

$$(t =_\sigma s)^* \equiv (t^* =^*_\sigma s^*).$$

Absurdity is interpreted as itself $\bot^* \equiv \bot$. The logical constants are translated literally: $(A \circ B)^* \equiv A^* \circ B^*$ for $\circ \equiv \wedge, \vee, \rightarrow$ and $(Qx^\sigma A)^* \equiv Qx^{*\sigma^*} A^*$ for $Q = \forall, \exists$. This defines the interpretation.

In preparation for the soundness theorem we need a few results on substitution, and interpretation of standard quantifiers. It is easily shown that the interpretation commutes with substitution, i.e. $b^*[\vec{a}^*/\vec{x}^*] \equiv b[\vec{a}/\vec{x}]^*$ and $A^*[\vec{a}^*/\vec{x}^*] \equiv A[\vec{a}/\vec{x}]^*$.

Lemma 4.1. *Let $a:\tau$ be a term, and let A be a formula of* $\mathbf{iHA}^\omega$. *Suppose $\vec{u} \equiv u_1, \ldots, u_n$, $\vec{v} \equiv v_1, \ldots, v_n$, and $\vec{x}^* \equiv x_1^*, \ldots, x_n^*$ where u_i, v_i, x_i^* are of type $0\sigma_i$. Then:*

(i) $u_1 =^*_{\sigma_1} v_1 \wedge \cdots \wedge u_n =^*_{\sigma_n} v_n \Rightarrow a^*[\vec{u}/\vec{x}^*] =^*_\tau a^*[\vec{v}/\vec{x}^*]$.

(ii) $u_1 =^*_{\sigma_1} v_1 \wedge \cdots \wedge u_n =^*_{\sigma_n} v_n \Rightarrow A^*[\vec{u}/\vec{x}^*] \Leftrightarrow A^*[\vec{v}/\vec{x}^*]$.

(iii) $(\forall^{\mathrm{st}} y^\sigma A)^*[\vec{u}/\vec{x}^*] \Leftrightarrow \forall v^\sigma A^*[\vec{u}, \mathbf{k}v/\vec{x}^*, y^*]$.

(iv) $(\exists^{\mathrm{st}} y^\sigma A)^*[\vec{u}/\vec{x}^*] \Leftrightarrow \exists v^\sigma A^*[\vec{u}, \mathbf{k}v/\vec{x}^*, y^*]$.

Proof. (i) and (ii) are proved by a straightforward induction on the complexity of a and A respectively. Noting that $St^*_\sigma(v)$ if and only if $v =^*_\sigma \mathbf{k}u$, for some $u:\sigma$, the equivalences (iii) and (iv) easily follow from (ii). □

Lemma 4.2. *Let $t:\tau$ be an internal term with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$. Then for all $y_1 : \sigma_1^*, \ldots, y_n : \sigma_n^*$,*

$$t^*[y_1, \ldots, y_n/x_1^*, \ldots, x_n^*]m \ =_\tau t[y_1 m, \ldots, y_n m/x_1, \ldots, x_n].$$

Proof. By induction on the complexity of t. □

Lemma 4.3. *Let A be an internal formula with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$. Then for all $y_1 : \sigma_1, \ldots, y_n : \sigma_n$,*

$$\forall y_1^{\sigma_1} \cdots y_n^{\sigma_n} [A[\vec{y}/\vec{x}] \Leftrightarrow (A^{\mathrm{st}})^*[\mathbf{k}y_1, \ldots, \mathbf{k}y_n/x_1^*, \ldots, x_n^*]].$$

Proof. By induction on the complexity of A. The atomic case = follows from Lemma 4.2. The quantifier case follows from Lemma 4.1. □

Any closed internal formula A is thus equivalent to $(A^{\mathrm{st}})^*$. We are now in a position to prove the soundness theorem.

Theorem 4.4. *The interpretation $(\cdot)^*$ of* $\mathbf{iHA}^\omega$ *into* $\mathbf{HA}^\omega + \mathbf{AC}$ *is sound.*

Proof. The soundness for the logical axioms and rules follows since the interpretation is literal for logical constants, and since the interpretation function commutes with substitution. We need thus only to check the nonlogical axioms.

Equality axioms: Reflexivity, symmetry and transitivity follow because $=^*$ is an equivalence relation. The cases for application and St are consequences of Lemma 4.1.

Defining axioms for combinators: These are readily checked by noting that the combinators act pointwise. To give one example, consider the equation

$$\mathbf{r}xy(Sz) = y(\mathbf{r}xyz)z.$$

We have

$$\begin{aligned}(\mathbf{r}xy(Sz))^*m &= (\mathbf{r}^*m)(x^*m)(y^*m)((S^*m)(z^*m))\\ &= \mathbf{r}(x^*m)(y^*m)(S(z^*m))\\ &= (y^*m)(\mathbf{r}(x^*m)(y^*m)(z^*m))(z^*m)\\ &= (y(\mathbf{r}xyz)z)^*m.\end{aligned}$$

The second member follows since $c^*m = \mathbf{k}cm = c$ for internal constants c. By the definition of $=^*$, we now see the validity of the equation under the interpretation.

Arithmetical axioms: The fourth Peano axiom is easily seen to be valid under the interpretation. The validity of external induction follows by applying Lemma 4.1 to translate it into usual induction.

Axioms on internal constants and application: Since an internal constant c is interpreted as a constant sequence, the axiom $St(c)$ is obviously valid. The axiom $\forall^{st} x^{\sigma\tau} \forall^{st} y^{\sigma} St(xy)$ is also easily checked.

The external axiom of choice: Suppose $(\forall^{st} x^{\sigma} \exists^{st} y^{\tau} A(\vec{u}, x, y))^*$. Lemma 4.1 gives

$$\forall v^{\sigma} \exists w^{\tau} A^*[\mathbf{k}v, \mathbf{k}w/x^*, y^*].$$

The axiom of choice gives $t^{\sigma\tau}$ so that, for all v,

$$A^*[\mathbf{k}v, \mathbf{k}(tv)/x^*, y^*].$$

Now $\mathbf{k}(tv) =^*_{\tau} \mathbf{s}(\mathbf{k}t)(\mathbf{k}v)$, so by Lemma 4.1,

$$A^*[\mathbf{k}v, \mathbf{s}(\mathbf{k}t)(\mathbf{k}v)/x^*, y^*].$$

From this it follows that $(\exists^{st} z^{\sigma\tau} \forall^{st} x^{\sigma} A(\vec{u}, x, zx))^*$.

The limit principle: Let $x^* : \sigma^* \equiv 0\sigma$ be given. Thus

$$x^*m = \mathbf{s}(\mathbf{k}x^*)\mathbf{I}m = \mathbf{s}(\mathbf{k}x^*)(\infty^*)m.$$

Thus $x^* =^* \mathbf{s}(\mathbf{k}x^*)(\infty^*)$ and since $St^*_{0\sigma}(\mathbf{k}x^*)$,

$$(\exists^{st} y^{0\sigma}[x = y\infty])^*.$$

The limit equality axiom: The equivalence

$$\mathbf{s}(\mathbf{k}t)\,\infty^* =^* \mathbf{s}(\mathbf{k}v)\,\infty^* \leftrightarrow \exists l\, \forall m \geqslant l\, [\mathbf{s}(\mathbf{k}t)(\mathbf{k}m) =^* \mathbf{s}(\mathbf{k}v)(\mathbf{k}m)], \tag{6}$$

for $t, v : 0\sigma$, follows from the definition of $=^*$. Now $\geqslant$ has a primitive recursive characteristic function $f_{\geqslant}$ so that

$$m \geqslant l \;\Leftrightarrow_{\mathrm{def}}\; f_{\geqslant}(m,l) = 0 \;\Leftrightarrow\; f^*_{\geqslant}(\mathbf{k}m, \mathbf{k}l) \;=^*\; \mathbf{k}0.$$

Thus the right-hand side of (6) is equivalent to

$$(\exists^{\mathrm{st}} l\, \forall^{\mathrm{st}} m \geqslant l\, [xm = ym])^* [\mathbf{k}t, \mathbf{k}v / x^*, y^*].$$

This validates the axiom and concludes the soundness proof. □

Lemma 4.5. *For formulas A of* **iHA**$^{\omega}$ *with free variables among $\vec{x}$, the following holds in* **iHA**$^{\omega}$:

$$\forall^{\mathrm{st}} \vec{u}\, (A^{*\mathrm{st}}[\vec{u}/\vec{x}^*] \;\Leftrightarrow\; A[\vec{u}\infty/\vec{x}]).$$

Proof. By a straightforward induction on A, using the **iHA**$^{\omega}$-interpreted form of Lemma 4.2. □

Corollary 4.6. *For closed A in* **HA**$^{\omega}$, *and closed B in* **iHA**$^{\omega}$:
(i) **iHA**$^{\omega} \vdash A^{\mathrm{st}}$ *if and only if* **HA**$^{\omega}$ + **AC** $\vdash A$,
(ii) **HA**$^{\omega}$ + **AC** $\vdash B^*$ *if and only if* **iHA**$^{\omega} \vdash B$.

Proof. (i) follows from the soundness theorem and Lemma 4.3. (ii) follows from Lemma 4.5, and the soundness theorem. □

The first result says that **iHA**$^{\omega}$ is a conservative extension of **HA**$^{\omega}$ + **AC**. The second states that **iHA**$^{\omega}$ completely axiomatises the model given inside **HA**$^{\omega}$ + **AC**.

Corollary 4.7. *The theory* **iHA**$^{\omega}$ *has the explicit definability property in the following forms. Let $A(x)$ be a formula where x is the only free variable.*
(i) *If* **iHA**$^{\omega} \vdash \exists x\, A(x)$, *then for some closed $t : \sigma$,* **iHA**$^{\omega} \vdash A(t)$.
(ii) *If* **iHA**$^{\omega} \vdash \exists^{\mathrm{st}} x\, A(x)$, *then for some closed internal $t : \sigma$,* **iHA**$^{\omega} \vdash A(t)$.

Proof. This follows from Lemmas 4.3 and 4.5, using the explicit definability property of **HA**$^{\omega}$ + **AC**. □

Finally, we characterise provability in **HA**$^{\omega}$ + **AC**$[\infty]$.

Corollary 4.8. *Let $A(x)$ be an internal formula. Then*

$$\mathbf{HA}^{\omega} + \mathbf{AC}[\infty] \vdash A(\infty) \;\Leftrightarrow\; \mathbf{iHA}^{\omega} \vdash \exists^{\mathrm{st}} k\, \forall^{\mathrm{st}} n \geqslant k\, A^{\mathrm{st}}(n).$$

Proof. ($\Rightarrow$) Follows from Theorem 2.2 and the $(\cdot)^{\mathrm{st}}$-interpretation.
($\Leftarrow$) Suppose

$$\mathbf{iHA}^{\omega} \vdash \exists^{\mathrm{st}} n \forall^{\mathrm{st}} m \geqslant n\, A^{\mathrm{st}}(m).$$

By Corollary 4.7, there is a numeral $\underline{n}$ such that

$$\mathbf{iHA}^{\omega} \vdash \forall^{\mathrm{st}} k\, A^{\mathrm{st}}(\underline{n} + k)$$

and hence by Corollary 4.6(i) and Theorem 2.2, $\mathbf{HA}^{\omega} + \mathbf{AC}[\infty] \vdash A(\infty)$. □

Remark. An immediate consequence of this result, and Theorem 2.8, is that all partial recursive functions are representable by terms in $\mathbf{iHA}^{\omega}$.

5. Constructive nonstandard analysis in the internal theory

In Section 3 we remarked that elementary parts of Bishop's constructive analysis can be carried out in $\mathbf{HA}^{\omega}$. We now illustrate the possibilities given by $\mathbf{iHA}^{\omega}$ to handle nonstandard real numbers. Nonstandard characterisations of a few standard notions concerning sequences and functions are given. The fundamental theorem of calculus is proved using nonstandard methods.

5.1. Extending Bishop's constructive analysis with nonstandard notions

In Section 3 we saw that the syntactic translation $A \mapsto A^{\mathrm{st}}$ given by relativising all quantifiers to standard objects defines an interpretation of $\mathbf{HA}^{\omega} + \mathbf{AC}$ into $\mathbf{iHA}^{\omega}$. Thus any theorem A of constructive analysis proved in the former theory is valid in the latter as the relativised statement A^{st}. In $\mathbf{iHA}^{\omega}$ we have general methods for dealing with nonstandard objects of the kind considered by Laugwitz/Schmieden and Martin-Löf. The limit principle embodies the idea that every nonstandard object is essentially a sequence of standard objects. All operations defined on standard entities extend to nonstandard entities by applying them termwise to the representing sequences. Moreover every standard concept or predicate gives rise to a canonical nonstandard concept, so that it holds of a nonstandard object, if its representing sequence eventually falls under the standard concept. The latter is formalised by the procedure of definitional extension given in Section 3.3. We illustrate this with real numbers, where we use the definition of Bishop [3]:

$$R^{\mathrm{st}}(x) \Leftrightarrow \forall^{\mathrm{st}} n\, Q^{\mathrm{st}}(x_n) \wedge (\forall^{\mathrm{st}} m, n > 0) |x_m - x_n| \leqslant^{\mathrm{st}}_{Q} \frac{1}{m} + \frac{1}{n},$$

where $Q^{\mathrm{st}}(x)$ is the predicate for "the natural number x represents a rational number" and $\leqslant^{\mathrm{st}}_{Q}$ is the order relation on rationals. The second condition is that the sequence is *regular* (a strong form of Cauchy condition). The nonstandard real numbers are given by

$$R(x\infty) \Leftrightarrow \exists^{st}k(\forall^{st}l \geqslant k)R^{st}(xl).$$

Thus 2^∞, $(-1)^\infty$, $1/\infty$ and $\sin\infty$ are examples of nonstandard real numbers. We define standard predicates for equality and positivity of real numbers.

$$x =_R^{st} y \Leftrightarrow (\forall^{st}n > 0)|x_n - y_n| \leqslant_Q^{st} \frac{1}{2n},$$

$$P^{st}(x) \Leftrightarrow (\exists^{st}n > 0)\frac{1}{n} <_Q^{st} x_n.$$

Again we have the corresponding nonstandard predicates $=_R$ and P. While the standard and nonstandard predicates agree on standard objects, they mean completely different things for nonstandard objects.

Example. We have

$$P(\infty^{-1}) \Leftrightarrow \exists^{st}k(\forall^{st}m \geqslant k)(\exists^{st}n > 0)[n^{-1} < m^{-1}],$$

but

$$P^{st}(\infty^{-1}) \Leftrightarrow (\exists^{st}n > 0)\exists^{st}k(\forall^{st}m \geqslant k)[n^{-1} < m^{-1}].$$

The right-hand side of the latter is of course false.

The order relations $<_R^{st}$, $<_R$ are given by

$$x <_R^{st} y \Leftrightarrow P^{st}(y - x).$$

The apartness and the not-greater-than relation are defined through

$$x \neq^{st} y \Leftrightarrow x <_R^{st} y \vee y <_R^{st} x,$$

$$x \leqslant^{st} y \Leftrightarrow \neg y <_R^{st} x.$$

Almost all basic standard results on these relations, and the arithmetic of real numbers, can be lifted to nonstandard real numbers. We here refer to the results proved in Ch. 2.2 of Bishop and Bridges [4]. The reason is that they can be formulated as *constructive Horn formulas* in the language extended with R^{st}, $=_R^{st}$, $<_R^{st}$, $\leqslant_R^{st}$ and $\neq_R^{st}$. Consider the following statement:

$$\forall^{st}xyzt[R^{st}(x) \wedge R^{st}(y) \wedge R^{st}(z) \wedge R^{st}(t) \wedge x <_R^{st} z \wedge y <_R^{st} t \rightarrow x + y <_R^{st} z + t].$$

By the lifting theorem, the same holds with the superscripts st removed. A result which does *not* lift is e.g.: for all standard reals x, y, z,

$$x < y \rightarrow x < z \vee z < y.$$

This does not hold for nonstandard reals, take $x = 0$, $y = 1$ and $z = (-2)^\infty$.

Definition 5.1. Let a and b be real numbers. Define
(i) a is *finite*, if $\exists^{st} k(|a| <_R k)$,
(ii) a is *infinite*, if $\forall^{st} k(k <_R |a|)$,
(iii) a is *infinitesimal*, if $\forall^{st} k(|a| <_R 2^{-k})$,
(iv) a and b are *infinitely close*, if $a - b$ is infinitesimal; in this case we write $a \simeq b$.

Thus all standard reals are finite. ∞ and $(-2)^{\infty}$ are infinite. ∞^{-1} is infinitesimal. Note that the only standard infinitesimal is 0.

5.2. Nonstandard characterisations: sequences

We give first the precise standard definitions in the internal theory.

Definition 5.2. Let (x_n) be a standard sequence of real numbers.
(i) (x_n) *converges to the standard real number* L, $(x_n) \to L$, if

$$(\forall^{st} k > 0)\exists^{st} l(\forall^{st} n \geqslant l)|x_n - L| <^{st} 2^{-k}.$$

(ii) L is a *limit point* of (x_n), if $\forall^{st} k\, \forall^{st} l(\exists^{st} n \geqslant l)|x_n - L| <^{st} 2^{-k}$.
(iii) (x_n) is *bounded* if $\exists^{st} k\, \forall^{st} n(|x_n| <^{st} k)$.

The following simple but important results give nonstandard characterisations of these notions.

Theorem 5.3. *Let (x_n) be a standard sequence of real numbers. Let L be a standard real number. Then:*
(i) $(x_n) \to L$ if and only if $x_\infty \simeq L$.
(ii) L is a limit point of (x_n) if and only if there is an infinite natural number η such that $x_\eta \simeq L$.

Proof. (i) $(x_n) \to L$ is by the Łos principle equivalent to

$$\forall^{st} k\, |x_\infty - L| < 2^{-k},$$

i.e. $x_\infty \simeq L$.

(ii) Suppose that $\forall^{st} kl(\exists^{st} n \geqslant l)|x_n - L| <^{st} 2^{-k}$. Applying the axiom of choice twice, we find a standard function f such that

$$\forall^{st} kl[f(k,l) \geqslant l \wedge |x_{f(k,l)} - L| <^{st} 2^{-k}].$$

By first lifting this formula, and then letting $k = l = \infty$ we get $f(\infty, \infty) \geqslant \infty$ and

$$|x_{f(\infty,\infty)} - L| < 2^{-\infty}.$$

Thus let $\eta = f(\infty, \infty)$.

Conversely, let $\eta = g(\infty)$ be infinite with $x_\eta \simeq L$. Let k and l be fixed standard numbers. The Łos principle applied to $|x_\eta - L| < 2^{-k}$ yields a stage p such that

$$(\forall^{st} i \geqslant p)\,|x_{g(i)} - L| < 2^{-k}.$$

Since $g(\infty)$ is infinite, we have $n = g(i) \geqslant l$ for some $i \geqslant p$, and thus $|x_n - L| < 2^{-k}$. □

Proposition 5.4. *Let (x_n) be a standard sequence of real numbers. Then: (x_n) is bounded if and only if x_∞ is finite.*

Proof. ($\Rightarrow$) Let k be standard and suppose $\forall^{st} n(|x_n| <^{st} k)$. By lifting $\forall n(|x_n| < k)$. In particular, $|x_\infty| < k$.

($\Leftarrow$) Suppose $|x_\infty| < k$ for some standard k. By the Łos principle,

$$\exists^{st} l(\forall^{st} n \geqslant l)(|x_n| < k).$$

Now let k' be a standard upper bound of $|x_0|, \dots, |x_{l-1}|, k$. Thus (x_n) is bounded by k'. □

The number ∞ acts as a generic infinity in certain formulas.

Proposition 5.5. *Let $A(\vec{x}, n)$ be an amenable formula where n is nonconfining. Then*

$$\forall^{st} \vec{x}\,[A(\vec{x}, \infty) \to (\forall \textit{ infinite } \eta)\, A(\vec{x}, \eta)].$$

Proof. Suppose $A(\vec{x}, \infty)$ for fixed standard $\vec{x}$. Thus by the Łos principle, there is a stage k such that $A^{st}(\vec{x}, n)$ for $n \geqslant k$. Let $\eta = f(\infty)$ be infinite, where f is standard. Thus for some standard l, $(\forall^{st} m \geqslant l)\, f(m) \geqslant k$. Hence

$$A^{st}(\vec{x}, f(m))$$

for $m \geqslant l$. Hence, by the Łos principle, $A(\vec{x}, \eta)$. □

Remark. It can easily be shown that the proposition holds also for formulas $A(\vec{x}, y) \equiv \forall^{st} \vec{z}\; B(\vec{x}, y, \vec{z})$ or $\exists^{st} \vec{z}\; B(\vec{x}, y, \vec{z})$, where B is amenable and not confining in y. We thus have, for example, $a_\infty \simeq b_\infty$ iff for all infinite η, $a_\eta \simeq b_\eta$. Moreover a_∞ is finite iff for all infinite η, a_η is finite.

Proposition 5.6. (Robinson's sequential lemma (Laugwitz [13])). *Assume (s_n) to be a sequence of real numbers. If for every standard n, $s_n \simeq 0$, then for some infinite ν, $(\forall \mu \leqslant \nu)\, s_\mu \simeq 0$.*

Proof. We use overspill. Let $A(s,n)$ be the formula $(\forall^{st} k \leqslant n)|s_k| \leqslant 2^{-n}$. Clearly $\forall^{st} n\, A(s,n)$, so there is an infinite ν with

$$(\forall \mu \leqslant \nu)\,|s_\mu| \leqslant 2^{-\nu},$$

and we are done. □

5.3. Nonstandard characterisations: functions

We recall the usual continuity notion from Bishop [3].

Definition 5.7. A standard function $f: I \to R$ where I is an interval, is *uniformly continuous* if there exists a standard function m (which does not have to respect equality on reals) such that $(\forall^{st} \varepsilon > 0)\, m(\varepsilon) > 0$ and

$$(\forall^{st} \varepsilon > 0)(\forall^{st} x, y \in I)[|x - y| \leqslant m(\varepsilon) \Rightarrow |f(x) - f(y)| \leqslant \varepsilon].$$

m is called a continuity modulus. A function defined on a general interval J is locally uniformaly continuous if it is uniformly continuous on each compact subinterval of J.

The axiom of choice gives several convenient standard characterisations of uniform continuity. Here it is important that the continuity modulus is not required to respect equality on real numbers.

Proposition 5.8. *The following are equivalent for a standard function* $f: I \to R$:
(i) *f is uniformly continuous,*
(ii) $(\forall^{st} \varepsilon > 0)(\exists^{st} \delta > 0)(\forall^{st} x, y \in I)[|x - y| \leqslant \delta \Rightarrow |f(x) - f(y)| \leqslant \varepsilon]$,
(iii) $(\forall^{st} k)(\exists^{st} n)(\forall^{st} x, y \in I)[|x - y| \leqslant 2^{-n} \Rightarrow |f(x) - f(y)/ \leqslant 2^{-k}]$.

Proof. The proof is standard in both senses. □

Definition 5.9. A function $f: J \to R$ defined on a standard interval J is *monad preserving* if

$$\forall x, y \in J\,[x \simeq y \Rightarrow f(x) \simeq f(y)].$$

f is locally monad preserving if it is monad preserving on every compact standard subinterval of J.

Theorem 5.10. *Let $f: J \to R$ be a standard function on the standard interval J.*
(i) *If f is uniformly continuous, then f is monad preserving.*
(ii) *If f is locally uniformly continuous, then f is locally monad preserving.*

Proof. (ii) follows easily from (i). We prove (i). By lifting

$$(\forall^{\mathrm{st}}\varepsilon > 0)(\forall x, y \in J)[|x - y| \leqslant m(\varepsilon) \Rightarrow |f(x) - f(y)| \leqslant \varepsilon].$$

Suppose $x \simeq y$, $x, y \in J$. For a standard $\varepsilon > 0$, $m(\varepsilon)$ is standard and positive, so $|x - y| \leqslant m(\varepsilon)$. Hence $|f(x) - f(y)| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $f(x) \simeq f(y)$. □

The converse does not seem to hold (constructively). The reason for this conclusion is that monad preservation can be characterised in terms of sequential continuity.

Definition 5.11. Let $f: J \to R$ be a standard function on the standard interval J. Then f is *sequentially uniformly continuous* if for all standard sequences (x_n) and (y_n) in J,

$$x_n - y_n \to 0 \;\Rightarrow\; f(x_n) - f(y_n) \to 0.$$

Proposition 5.12. *Let $f: J \to R$ be a standard function on the standard interval J. Then: f is sequentially uniformly continuous if and only if it is monad preserving.*

Proof. ($\Leftarrow$) This follows easily by the nonstandard characterisation of limits.

($\Rightarrow$) Let $x \simeq y$, where $x, y \in J$. Thus $x = u\infty$ and $y = v\infty$ for some standard u and v. Thus there exists a stage k such that for all standard $n \geqslant k$, $un, vn \in J$. We let u' and v' be modifications of u and v, respectively, leaving them unchanged for arguments $\geqslant k$, but such that $u'n, v'n \in J$ for all n. Thus $x = u'\infty$ and $y = v'\infty$ and $u'\infty \simeq v'\infty$. By the characterisation of limits, $u'n - v'n \to 0$. The assumption gives $f(u'n) - f(v'n) \to 0$. Hence again by the characterisation $f(u'\infty) \simeq f(v'\infty)$, i.e. $f(x) \simeq f(y)$. □

Nevertheless, uniform continuity can be given a nonstandard characterisation. Define a standard predicate,

$$C^{\mathrm{st}}(f, I, \delta, \varepsilon) \leftrightarrow (\forall^{\mathrm{st}} x, y \in I)[|x - y| \leqslant \delta \to |f(x) - f(y)| \leqslant \varepsilon].$$

Let C be its nonstandard version.

Proposition 5.13. *Let $f: I \to R$, and let I be standard. Then the following are equivalent:*

(i) *f is uniformly continuous,*

(ii) $(\forall \varepsilon > 0)(\exists \delta > 0)\, C(f, I, \delta, \varepsilon)$,

(iii) *there are $\delta, \varepsilon > 0$ with ε infinitesimal such that $C(f, I, \delta, \varepsilon)$.*

Proof. (i) $\Rightarrow$ (ii) follows by lifting the alternate characterisation of uniform continuity given in Proposition 5.8(ii). (ii) $\Rightarrow$ (iii) is immediate.

We prove (iii) $\Rightarrow$ (i). Let $\delta = \alpha_\infty$ and $\varepsilon = \beta_\infty$ where α and β are standard. Thus there is a stage k such that for all standard $n \geqslant k$, we have $\alpha_n > 0$, $\beta_n > 0$ and

$$(\forall^{\mathrm{st}} x, y \in I)[|x - y| \leqslant \alpha_n \to |f(x) - f(y)| \leqslant \beta_n].$$

ε is infinitesimal, so for each k, there is an $n \geqslant k$ such that $0 < \beta_n < 2^{-k}$ and thus

$$(\forall^{st} x, y \in I)[|x - y| \leqslant \alpha_n \to |f(x) - f(y)| \leqslant 2^{-k}].$$

It follows that f is uniformly continuous. □

Example. We prove that $f(x) = x^2$ is uniformly continuous on $I = [-M, M]$ where M is standard. We have for standard $x, y \in I$, $\delta > 0$ and $|x - y| \leqslant \delta$, that

$$|y^2 - x^2| = |2(y - x)x + (y - x)^2| \leqslant 2\delta|x| + \delta^2 \leqslant 2\delta M + \delta^2.$$

By lifting, we thus have for arbitrary $\delta > 0$, $C(f, I, \delta, 2\delta M + \delta^2)$. If δ is infinitesimal, $2\delta M + \delta^2$ is also infinitesimal, since M is standard. Thus f is uniformly continuous on I, by (iii) in the above proposition. This seems to capture a familiar mode of reasoning which begins: "let $\delta > 0$ be small $\cdots$".

The characterisation in Proposition 5.13 can be generalised to notions that can be defined by a similar quantifier combination, such as differentiability. Consider a formula $(\forall^{st}\varepsilon > 0)(\exists^{st}\delta > 0)A(\vec{x}, \delta, \varepsilon)$. Defining $C^{st} \equiv A$, the following is a sufficient condition on the formula $A(\vec{x}, \delta, \varepsilon)$ with real parameters δ, ε: for standard $\vec{x}$ and $\delta, \delta', \varepsilon, \varepsilon' > 0$:

$$A(\vec{x}, \delta, \varepsilon) \wedge \delta' < \delta \wedge \varepsilon < \varepsilon' \;\Rightarrow\; A(\vec{x}, \delta', \varepsilon').$$

Wattenberg [32] notes that the proof of the intermediate value theorem (IVT) in classical nonstandard analysis is constructive up to the point of applying the standardisation map. Here we can reason thus. Let $U^{st}(f, I)$ be the standard predicate for "$f: I \to R$ is a uniformly continuous function". A constructive version of IVT now reads:

$$\forall^{st} f \forall^{st} I (\forall^{st} a, b \in I)(\forall^{st}\varepsilon > 0)$$

$$[U^{st}(f, I) \wedge a < b \wedge f(a) < 0 < f(b) \to \exists^{st} x [a \leqslant x \leqslant b \wedge |f(x)| < \varepsilon]].$$

Clearly this is a constructive Horn formula, so it can be lifted. Taking $\varepsilon \simeq 0$, we thus find x with $f(x) \simeq 0$. But in our approach the representing sequence of x could even contain subsequences converging to two different zeros of f.

Definition 5.14. Let $f, g: I \to R$ be standard, uniformly continuous functions, where I is a standard interval. Let d be a standard function on reals (which do not have to respect equality on reals) with $(\forall^{st}\varepsilon > 0)\, d(\varepsilon) > 0$. Then f is *uniformly differentiable*, with derivative g, if

$$(\forall^{st}\varepsilon > 0)(\forall^{st} x, y \in I)[|y - x| \leqslant d(\varepsilon) \to |f(y) - f(x) - g(x)(y - x)| \leqslant \varepsilon|y - x|].$$

We then write $Df = g$.

Proposition 5.15. *Let $f: I \to R$ be standard and uniformly differentiable. For all $x \in I$ and $|\xi| > 0$ infinitesimal with $x + \xi \in I$,*

$$Df(x) \simeq \frac{f(x+\xi) - f(x)}{\xi}.$$

Proof. We have by lifting, that $\forall^{st} n (\exists^{st} \delta > 0)$:

$$(\forall x \in I)(\forall \xi)\left[|\xi| > 0 \wedge x + \xi \in I \wedge |\xi| \leqslant \delta \to \left| \frac{f(x+\xi) - f(x)}{\xi} - Df(x) \right| \leqslant 2^{-n} \right].$$

Since $|\xi| \leqslant \delta$ for every standard $\delta > 0$, we have for all standard n:

$$\left| \frac{f(x+\xi) - f(x)}{\xi} - Df(x) \right| \leqslant 2^{-n}.$$

This yields the result. □

Finally, we prove the fundamental theorem of calculus.

Theorem 5.16. *Let $F: I \to R$ be a standard, uniformly differentiable function with dervative f. Then for $a < b$ in I,*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Proof. We have for all infinite ω,

$$\int_a^b f(x)\,\mathrm{d}x \simeq \Delta \sum_{i=0}^{\omega - 1} f(a + i\Delta),$$

where $\Delta = (b - a)/\omega$, by the nonstandard characterisation of limits. By Proposition 5.15,

$$f(a + i\Delta) = \Delta^{-1}(F(a + i\Delta + \Delta) - F(a + i\Delta)) + \eta_i,$$

where η_i is infinitesimal. Telescoping the sum, we get

$$\Delta \sum_{i=0}^{\omega-1} f(a + i\Delta) = F(b) - F(a) + \Delta \sum_{i=0}^{\omega-1} \eta_i.$$

We can make the estimates

$$\Delta \sum_{i=0}^{\omega-1} |\eta_i| \leqslant \Delta\omega \max_{0 \leqslant i < \omega} |\eta_i| \leqslant \Delta\omega(|\eta_j| + \omega^{-1}) = (b - a)|\eta_j| + \Delta \simeq 0,$$

where j is some index, $0 \leqslant j < \omega$. Thus

$$\int_a^b f(x)\,\mathrm{d}x \simeq F(b) - F(a).$$

Since both sides are standard, they are actually equal. $\square$

6. Concluding remarks

In this paper we introduced a nonstandard arithmetical theory which we believe is suitable for the formalisation of elementary constructive nonstandard analysis. Further investigations into the elementary parts would be desirable; by, for example, constructivising the work of Laugwitz [13]. For some applications to differential equations, see Palmgren [25]. The present theory has the obvious limitation that it cannot handle general sets. We could instead have built the nonstandard theory on $\mathbf{HA}^\omega$ with predicative set quantification. The nonstandard theory would then have two kinds of set variables, standard and general. The intended interpretation of general set is a sequence of standard sets. Membership of such a set is eventual standard membership (cf. the internal sets in [23]). With this interpretation it is possible to motivate a standardisation principle, similar to Nelson's [22].

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