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Constructive Sheaf Semantics

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Abstract. Sheaf semantics is developed within a constructive and predicative framework, Martin-Löf's type theory. We prove strong completeness of many sorted, first order intuitionistic logic with respect to this semantics, by using sites of provably functional relations.

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1 Introduction

Topos theory may to a large extent be developed within a constructive higher order logic (see BELL [1]). However the very definition of an elementary topos relies on a nonpredicativity: the axiom for the subobject classifier. Fortunately, the more restricted class of Grothendieck topoi (see [4]), i.e. sheaves over sites (generalised topological spaces), are amenable to a predicative treatment. In this paper we focus on defining sheaf semantics in a predicative way, and prove, constructively, the completeness of the semantics for first order logic. Somewhat surprisingly, we can use the same construction principle of the sites as for so called geometric theories (cf. [1]). There are of course other approaches to semantics with constructive completeness proofs: H. C. M. DE SWART with Beth models, W. VELDMAN with Kripke models, and more recently DRAGALIN [2], and SAMBIN [10] with formal spaces. Sheaf semantics is a generalisation of all of these kinds of semantics, and gives a richer class of models. As examples, we have the constructive nonstandard models of arithmetic [6, 7] and analysis [9].

2 Categories and presheaves

The type theory we use as a metatheory is essentially the one presented in MARTIN-LÖF [5], with the exception that we do not assume extensional equality. The reader is referred to NORDSTRÖM, PETERSSON and SMITH [8] for a precise "modern" for-

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mulation of the type theory. We reformulate some basic notions of category theory, using sets (types) with equivalence relations.

A *small category* C consists of a set C, the set of *objects* of the category, and for each pair of objects $a, b \in C$ a set of morphisms $\mathrm{Hom}_{\mathsf{C}}(a,b)$ with an equivalence relation $\sim_{(\mathsf{C},a,b)}$, moreover there is a *composition function*

$$\mathrm{comp} \in \mathrm{Hom}_{\mathsf{C}}(b,c) \to \mathrm{Hom}_{\mathsf{C}}(a,b) \to \mathrm{Hom}_{\mathsf{C}}(a,c)$$

(as usual we write $f \circ g$ for $\mathrm{comp}(f,g)$), and an *identity morphism* $1_a \in \mathrm{Hom}_{\mathsf{C}}(a,a)$, for each $a, b, c \in C$. The composition $\circ$ should respect $\sim$, and satisfy the usual monoid laws up to $\sim$-equality.

It will not be necessary to consider equalities between objects of a category. The category of sets, with equalities, belonging to a universe U is a small category. See HOFMANN [3 Chapter 6.3] for categorical properties. A set in this category is called *finite* if it is isomorphic to a canonical finite set $N_k = \{0, \ldots, k-1\}$. Another simple example of a small category is a preorder on a set.

Definition 2.1. Let C be a small category. A *presheaf F over* C is a triple $(F, \approx, M)$, where $F(a)$ is a set, equipped with an equivalence relation $\approx_a$, for each $a \in C$, and where, for any $a, b \in C$, $M_{a,b}(f) \in F(b) \to F(a)$ is a function such that $M_{a,b}(f)(u) \approx_a M_{a,b}(g)(v)$, whenever $f \sim_{(\mathsf{C},a,b)} g$, $u \approx_b v$, and $f, g \in \mathrm{Hom}_{\mathsf{C}}(a,b)$. We write $F(f)$ for $M_{a,b}(f)$. Moreover, the contravariant functorial properties are required of F up to $\approx$-equality.

Natural transformations are now defined in the obvious way, as a family of morphisms depending on a category. It can be shown straightforwardly that the (large) category of presheaves over C has finite products. Let $\mathbf{1}$ denote the terminal presheaf. A presheaf F is *inhabited* if there is a natural transformation from $\mathbf{1}$ to F.

3 Grothendieck topologies and sheaves

A *Grothendieck topology*, or a *site*, is a generalised topology on a category, which is given by specifying what families of morphisms, with a common codomain, are to be counted as basic covers of the codomain (see MAC LANE and MOERDIJK [4]).

We shall only consider sites on small categories C with pullbacks. Let $\mathrm{Co}(d) = (\Sigma c \in C)\mathrm{Hom}_{\mathsf{C}}(c,d)$, the set of morphisms with codomain d. We assume there is a function that produces the pullback, given two objects from this set. Let $\Delta(d)$ be the set of small families of morphisms with codomain d, i.e. $(\Sigma I \in U)[I \to \mathrm{Co}(d)]$. We write elements of this set as $\langle f_i : c_i \to d\rangle_{i \in I}$.

Definition 3.1. A *basis for a site* on C is a relation $K(d,w)$ ($d \in C, w \in \Delta(d)$), read as *w covers d*, satisfying the following conditions

(a) if $f \in \mathrm{Hom}_{\mathsf{C}}(c,d)$ is an isomorphism, then the unit family $\langle f : c \to d\rangle_{i \in N_1}$ covers d;

(b) if $\langle f_i : c_i \to d\rangle_{i \in I}$ covers d and $g \in \mathrm{Hom}_{\mathsf{C}}(c,d)$, then the pullbacks of the family along g, $\langle \pi_2 : c_i \times_d c \to c\rangle_{i \in I}$, covers c;

(c) if $\langle f_i : c_i \to d\rangle_{i \in I}$ covers d, and $\langle g_{ij} : b_{ij} \to c_i\rangle_{j \in J_i}$ covers c_i for each $i \in I$, then the composite family $\langle f_i \circ g_{ij} : b_{ij} \to d\rangle_{(i,j) \in (\Sigma i \in I) J_i}$ covers d.

The notion of sheaf is now defined with respect to covers as usual (see [4]).

Definition 3.2. Let P be a presheaf over the category C and let K be a basis on C. A *(C, K)-relation R on P* is a relation $R(c, x)$ ($c \in C$, $x \in P(c)$) respecting the equality of P in the second argument, and satisfying the monotonicity and cover conditions (writing $x \dot\in R(c)$ for $R(c, x)$ below):

(a) if $x \dot\in R(c)$ and $f \in \mathrm{Hom}_{\mathsf{C}}(d, c)$, then $P(f)(x) \dot\in R(d)$;

(b) if $\langle f_i : c_i \to c\rangle_{i \in I}$ covers c, and $x \in P(c)$ is such that $P(f_i)(x) \dot\in R(c_i)$ for each $i \in I$, then $x \dot\in R(c)$.

We exemplify this notion by the objectwise equality on a sheaf. First note that the product of finitely many sheaves is a sheaf.

Proposition 3.3. *Let P be a sheaf over (C, K), and let $\approx_c$ be the equality on $P(c)$. Then*

$$\pi_1(z) \approx_c \pi_2(z) \qquad (c \in C,\ z \in P(c) \times P(c))$$

is a (C, K)-relation on the product sheaf $P^2 = P \times P$.

Proof. The first two conditions in the definition of a (C, K)-relation are immediate. To prove the covering property, assume $z \in P(c) \times P(c)$ and that $\langle f_i : c_i \to c\rangle_{i \in I}$ is a cover of c such that $\pi_1(P^2(f_i)(z)) \approx_{c_i} \pi_2(P^2(f_i)(z))$, $(i \in I)$. By naturality it follows that $P(f_i)(\pi_1(z)) \approx_{c_i} P(f_i)(\pi_2(z))$. Since $\langle f_i : c_i \to c\rangle_{i \in I}$ is a cover and P is a sheaf, it follows that $\pi_1(z) \approx_c \pi_2(z)$. □

4 Presheaf semantics

Definition 4.1. Let L be a many sorted first order language. Let C be a category with pullbacks and let K be a basis for a site on this category. A *presheaf interpretation of L* over (C, K) is an assignment of objects to the sorts and to the constant-, function- and relation-symbols of L as follows. To every sort τ, there is a presheaf $[\![\tau]\!]$ over (C, K). To every constant symbol c of sort τ there is an element $[\![c]\!] : \mathbf{1} \dot\to [\![\tau]\!]$, where $\mathbf{1}$ is the terminal presheaf. To every function symbol f of sort $\tau_1 \times \cdots \times \tau_n \to \tau$ there is a natural transformation $[\![f]\!] : [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \dot\to [\![\tau]\!]$. To every relation symbol R of sort $\tau_1 \times \cdots \times \tau_n$, there is a (C, K)-relation $[\![R]\!]$ on the presheaf $[\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$. We denote such an interpretation by $\mathcal{M} = (\mathsf{C}, K, [\![\cdot]\!])$.

An interpretation of an L-term t of sort τ with variables among $\overline{x} = x_1^{\tau_1}, \ldots, x_n^{\tau_n}$ is a natural transformation $[\![t]\!]_{\overline{x}} : [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \dot\to [\![\tau]\!]$. The product is $\mathbf{1}$ if the list $\overline{x}$ is empty. The interpretation is defined by induction on t in the obvious way. A variable $t = x_i^{\tau_i}$ is interpreted as the projection, $[\![t]\!]_{\overline{x}} = \pi_i^n : [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!] \dot\to [\![\tau_i]\!]$. A constant $t = c$ is interpreted as the composition of $[\![c]\!]$ with the unique terminal morphism from $[\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!]$. If $t = f(t_1, \ldots, t_n)$, then $[\![t]\!]_{\overline{x}} = [\![f]\!] \circ \langle [\![t_1]\!]_{\overline{x}}, \ldots, [\![t_n]\!]_{\overline{x}}\rangle$.

Definition 4.2. Let there be given an interpretation $[\![\cdot]\!]$ of L over (C, K). Define for each L-formula φ with free variables among $\overline{x} = x_1^{\tau_1}, \ldots, x_n^{\tau_n}$, each $c \in C$, and each sequence $\overline{\alpha} = \alpha_1, \ldots, \alpha_n$ with $\alpha_i \in [\![\tau_i]\!](c)$, the *forcing relation over* (C, K), $c \Vdash_{\overline{x}} \varphi[\overline{\alpha}]$. To simplify notation we write τ instead of $[\![\tau]\!]$. The forcing relation is defined inductively:

1. $c \Vdash_{\overline{x}} \bot[\overline{\alpha}]$ iff c is covered by an empty family;
2. $c \Vdash_{\overline{x}} R(t_1, \ldots, t_m)[\overline{\alpha}]$ iff $(([\![t_1]\!]_{\overline{x}})_c(\overline{\alpha}), \ldots, ([\![t_m]\!]_{\overline{x}})_c(\overline{\alpha})) \dot\in [\![R]\!](c)$;

3. $c \Vdash_{\overline{x}} (\varphi \wedge \psi)[\overline{\alpha}]$ iff $c \Vdash_{\overline{x}} \varphi[\overline{\alpha}]$ and $c \Vdash_{\overline{x}} \psi[\overline{\alpha}]$;
4. $c \Vdash_{\overline{x}} (\varphi \vee \psi)[\overline{\alpha}]$ iff there is some cover $\langle f_i : c_i \to c\rangle_{i\in I}$ such that for all $i \in I$, $c_i \Vdash_{\overline{x}} \varphi[\tau_1(f_i)(\alpha_1), \ldots, \tau_n(f_i)(\alpha_n)]$ or $c_i \Vdash_{\overline{x}} \psi[\tau_1(f_i)(\alpha_1), \ldots, \tau_n(f_i)(\alpha_n)]$;
5. $c \Vdash_{\overline{x}} (\varphi \to \psi)[\overline{\alpha}]$ iff for all $d \in C$ and $f \in \mathrm{Hom}_{\mathsf{C}}(d, c)$, $d \Vdash_{\overline{x}} \varphi[\tau_1(f)(\alpha_1), \ldots, \tau_n(f)(\alpha_n)]$ implies $d \Vdash_{\overline{x}} \psi[\tau_1(f)(\alpha_1), \ldots, \tau_n(f)(\alpha_n)]$;
6. $c \Vdash_{\overline{x}} (\exists y^\tau \varphi)[\overline{\alpha}]$ iff there is some cover $\langle f_i : c_i \to c\rangle_{i\in I}$, and some $\gamma_i \in \tau(c_i)$ $(i \in I)$, such that for all $i \in I$, $c_i \Vdash_{\overline{x},u^\tau} \varphi'[\tau_1(f_i)(\alpha_1), \ldots, \tau_n(f_i)(\alpha_n), \gamma_i]$, where $\varphi' = \varphi(u^\tau/y^\tau)$ and u^τ is the first variable (in some fixed ordering) not in $\overline{x}$ and not bound in φ;
7. $c \Vdash_{\overline{x}} (\forall y^\tau \varphi)[\overline{\alpha}]$ iff for all $d \in C$, every $f \in \mathrm{Hom}_{\mathsf{C}}(d, c)$, and for all $\gamma \in \tau(d)$, $d \Vdash_{\overline{x},u^\tau} \varphi'[\tau_1(f)(\alpha_1), \ldots, \tau_n(f)(\alpha_n), \gamma]$, where $\varphi' = \varphi(u^\tau/y^\tau)$ and u^τ is chosen as above.

The forcing relation naturally extends to sequences of formulas $\Gamma = \langle\varphi_i\rangle_{i\in I}$ with free variables among $\overline{x}$:

$$c \Vdash_{\overline{x}} \Gamma[\overline{\alpha}] \text{ iff for each } i \in I,\ c \Vdash_{\overline{x}} \varphi_i[\overline{\alpha}].$$

Remark. Note that we do not need to consider covering sieves as in [11]. The variable trick in the $\exists$- and $\forall$-cases is to ensure that the forcing relation is welldefined. The forcing relation depends only on the variables actually free in the forced formula. We can show in the usual manner that this semantics satisfies the monotonicity and covering property (see [4] or [11]).

Definition 4.3. Let $\mathcal{M} = (\mathsf{C}, K, [\![\cdot]\!])$ be a presheaf interpretation of L. A formula φ is *valid in $\mathcal{M}$ under the assumptions* Γ, if whenever $\overline{x} = x_1^{\tau_1}, \ldots, x_n^{\tau_n} \supseteq \mathrm{FV}(\Gamma, \varphi)$, $c \in C$, $\alpha_1 \in [\![\tau_1]\!](c), \ldots, \alpha_n \in [\![\tau_n]\!](c)$, and $c \Vdash_{\overline{x}} \Gamma[\alpha_1, \ldots, \alpha_n]$, then $c \Vdash_{\overline{x}} \varphi[\alpha_1, \ldots, \alpha_n]$. In this case we write $\Gamma \Vdash_{\mathcal{M}} \varphi$.

Let $\vdash$ denote derivability in first order, many-sorted intuitionistic logic without axioms for equality.

Theorem 4.4 (Soundness). *For every presheaf interpretation $\mathcal{M}$ of L, where each sort is interpreted by an inhabited presheaf, and for all sequences of L-formulas Γ and L-formulas φ,*

$$\Gamma \vdash \varphi \ \textit{implies}\ \Gamma \Vdash_{\mathcal{M}} \varphi.$$

Proof. By induction on derivations. The assumption that the presheaves are inhabited is used in the introduction rule for $\exists$. □

Remark. For sheaf semantics, Proposition 3.3 gives a simple interpretation of equality. Since the natural transformations and the (C, K)-relations respect this equality objectwise, it is easy to check the validity of the usual equality axioms for a language L.

5 Completeness

For the completeness theorem we use a syntactic site, similar to the one introduced in BELL [1, p. 249], for proving a completeness theorem for so called geometric theories (see also [4]). However we will allow arbitrary formulas in the site and not just geometric formulas. Our own contribution is the proof that this topology indeed yields a *constructive* completeness theorem for *full* first order intuitionistic logic.

Throughout this section L is a first order, many sorted language. We let $\varphi \subseteq \overline{x}$ denote that $\overline{x}$ is a string of distinct variables, which includes the free variables of φ. We abbreviate $x_1^{\tau_1} =_{\tau_1} y_1^{\tau_1} \wedge \cdots \wedge x_n^{\tau_n} =_{\tau_n} y_n^{\tau_n}$ by $\overline{x} = \overline{y}$. Two strings of variables $x_1^{\sigma_1}, \ldots, x_m^{\sigma_m}$ and $y_1^{\tau_1}, \ldots, y_n^{\tau_n}$ are *compatible* if $m = n$ and $\sigma_i = \tau_i$ for $i = 1, \ldots, m$.

Definition 5.1. Let Σ be a theory with closed axioms in the language L including the equality theory for L. The objects of the *category* $\mathsf{Syn}(\Sigma)$ *of Σ-provably functional relations* are pairs $(\varphi, \overline{x})$, where φ is an L-formula with $\varphi \subseteq \overline{x}$; the morphisms from $(\varphi, \overline{x})$ to $(\psi, \overline{y})$ are triples $(\overline{u}; \theta; \overline{v})$, where θ is an L-formula with $\theta \subseteq \overline{u}, \overline{v}$ such that $\overline{u}$ is compatible with $\overline{x}$, and $\overline{v}$ is compatible with $\overline{y}$, and such that θ is *Σ-provably functional*, i.e.

(a) $\Sigma \vdash \forall \overline{s}\,\overline{w}\,[\theta(\overline{s}, \overline{w}/\overline{u}, \overline{v}) \rightarrow \varphi(\overline{s}/\overline{x}) \wedge \psi(\overline{w}/\overline{y})]$,
(b) $\Sigma \vdash \forall \overline{s}\,[\varphi(\overline{s}/\overline{x}) \rightarrow \exists \overline{w}\,\theta(\overline{s}, \overline{w}/\overline{u}, \overline{v})]$,
(c) $\Sigma \vdash \forall \overline{s}\,\overline{w}\,\overline{z}\,[\theta(\overline{s}, \overline{w}/\overline{u}, \overline{v}) \wedge \theta(\overline{s}, \overline{z}/\overline{u}, \overline{v}) \rightarrow \overline{w} = \overline{z}]$.

Two such arrows $(\overline{u}_1; \theta_1; \overline{v}_1)$ and $(\overline{u}_2; \theta_2; \overline{v}_2)$ are equal ($\sim$) if

$$\Sigma \vdash \forall \overline{s}\,[\varphi(\overline{s}/\overline{x}) \rightarrow \forall \overline{w}\,(\theta_1(\overline{s}, \overline{w}/\overline{u}_1, \overline{v}_1) \leftrightarrow \theta_2(\overline{s}, \overline{w}/\overline{u}_2, \overline{v}_2))].$$

The identity morphism on $(\varphi, \overline{x})$ is given by $(\overline{x}; \overline{x} = \overline{y} \wedge \varphi; \overline{y})$, where $\overline{x}$ and $\overline{y}$ are compatible but have no common variables. The composition of $(\overline{u}_1; \theta; \overline{v})$ and $(\overline{w}; \varrho; \overline{s}_1)$ is $(\overline{u}_2; \exists \overline{z}\,\theta(\overline{u}_2, \overline{z}/\overline{u}_1, \overline{v}) \wedge \varrho(\overline{z}, \overline{s}_2/\overline{w}, \overline{s}_1); \overline{s}_2)$. It is assumed that the substitutions are always done with fresh variables.

Lemma 5.2. *The category* $\mathsf{Syn}(\Sigma)$ *has pullbacks and finite products.*

Proof. Straightforward (cf. BELL [1]). E.g. the terminal object is $(\top, \varepsilon)$, where ε denotes the empty string of variables. The construction of the pullback object of the arrows $(\overline{x}; \theta; \overline{y})$ and $(\overline{u}; \rho; \overline{v})$ is $(\exists \overline{z}\,\theta(\overline{s}, \overline{z}/\overline{x}, \overline{y}) \wedge \varrho(\overline{w}, \overline{z}/\overline{u}, \overline{v}), \overline{s}, \overline{w})$. □

Definition 5.3. The *syntactic basis* $(\mathsf{Syn}(\Sigma), K_\Sigma)$ is the following. Let the cover relation K_Σ be given by saying that $\langle(\overline{x}_i; \theta_i; \overline{y}_i) : (\varphi_i, \overline{u}_i) \rightarrow (\psi, \overline{y})\rangle_{i \in I} \in \Delta((\psi, \overline{y}))$ is a cover iff I is finite and $\Sigma \vdash \forall \overline{u}\,[\psi(\overline{u}/\overline{y}) \rightarrow \bigvee_{i \in I} \exists \overline{x}_i\,\theta_i(\overline{u}/\overline{y}_i)]$.

It is straightforward to check that this relation actually satisfies Definition 3.1.

Lemma 5.4. *Every representable presheaf* $\mathrm{Hom}(-, (\psi, \overline{y}))$ *over* $(\mathsf{Syn}(\Sigma), K_\Sigma)$ *is a sheaf.* □

Let Σ be a theory with closed axioms in the language L, including the equality theory for L. The *generic sheaf model* $\mathcal{M}(\Sigma, L)$ of this theory is defined as follows. The basis of the site is $(\mathsf{Syn}(\Sigma), K_\Sigma)$.

- A sort τ is interpreted as the sheaf $[\![\tau]\!] = \mathrm{Hom}_{\mathsf{Syn}(\Sigma)}(-, (\top, z^\tau))$.
- A constant symbol c of sort τ is interpreted by the natural transformation given by $[\![c]\!]_{(\varphi, \overline{x})} = (\overline{x}; z^\tau = c; z^\tau)$, where $z^\tau \notin \overline{x}$.
- A function symbol f of sort $\tau_1 \times \cdots \times \tau_n \rightarrow \tau$ is interpreted by the natural transformation defined by

$$[\![f]\!]_{(\varphi, \overline{x})}((\overline{x}_1; \theta_1; z_1), \ldots, (\overline{x}_n; \theta_n; z_n)) = (\overline{u}; \exists \overline{v}\,\textstyle\bigwedge_{i=1}^n \theta_i(\overline{u}, v_i/\overline{x}_i, z_i) \wedge f(\overline{v}) = z; z),$$

where $\overline{u}$, z and $\overline{v} = v_1, \ldots, v_n$ are freshly chosen variables.

- The interpretation of a relation symbol R of sort $\tau_1 \times \cdots \times \tau_n$ is given by letting $[\![R]\!]_{(\varphi,\overline{x})}$ be the set of all tuples $(\alpha_1, \ldots, \alpha_n) \in [\![\tau_1]\!](\varphi, \overline{x}) \times \cdots \times [\![\tau_n]\!](\varphi, \overline{x})$ such that $\Sigma \vdash \varphi(\overline{u}/\overline{x}) \wedge \bigwedge_{i=1}^{n} \theta_i(\overline{u}, v_i/\overline{u}_i, z_i) \to R(v_1, \ldots, v_n)$, where $\alpha_i = (\overline{u}_i; \theta_i; z_i)$ and the variables $\overline{u}, \overline{v}$ are freshly chosen.

It is readily checked that this constitutes a model.

Theorem 5.5. *Let Σ be a theory with closed axioms in the language L which includes the equality theory of L. Then for any $\alpha_i \in [\![\tau_i]\!](\psi, \overline{x})$ $(i = 1, \ldots, n)$ and any $\varphi \subseteq \overline{y} = y_1^{\tau_1}, \ldots, y_n^{\tau_n}$ we have*

$$(\psi, \overline{x}) \Vdash_{y_1,\ldots,y_n} \varphi[\alpha_1, \ldots, \alpha_n] \quad \textit{iff} \quad \Sigma \vdash \psi(\overline{u}/\overline{x}) \wedge \bigwedge_{i=1}^{n} \theta_i(\overline{u}, y_i/\overline{x}_i, z_i) \to \varphi,$$

where $\alpha_i = (\overline{x}_i; \theta_i; z_i^{\tau_i})$, and $\Vdash$ is the forcing relation associated with $\mathcal{M}(\Sigma, L)$.

Proof. By induction on the complexity of φ. We do three illustrative cases.

$\varphi = \bot$: By definition $(\psi, \overline{x}) \Vdash_{y_1,\ldots,y_n} \bot[\alpha_1, \ldots, \alpha_n]$ if and only if $(\psi, \overline{x})$ has an empty cover, i.e. $\Sigma \vdash \psi(\overline{u}/\overline{x}) \to \bot$. Since each θ_i is functional on ψ, this is equivalent to $\Sigma \vdash \psi(\overline{u}/\overline{x}) \wedge \overline{\theta} \to \bot$, where $\overline{\theta} = \bigwedge_{i=1}^{n} \theta_i(\overline{u}, y_i/\overline{x}_i, z_i)$.

$\varphi = \exists z^\tau \sigma$: We prove only the $(\Leftarrow)$-direction. Suppose

$$(1) \qquad \Sigma \vdash \psi(\overline{u}/\overline{x}) \wedge \overline{\theta} \to \varphi.$$

Let $\psi' = \psi(\overline{u}/\overline{x}) \wedge \exists \overline{y}(\overline{\theta} \wedge \sigma)$. We have $\psi' \subseteq \overline{u}, z$. By (1), projecting in all but the last coordinates yields a covering morphism,

$$\varrho = (\overline{u}, z; \psi' \wedge \overline{u} = \overline{v}; \overline{v}) : (\psi', \overline{u}, z) \to (\psi, \overline{x}).$$

Define a morphism $\eta : (\psi', \overline{u}, z) \to (\top, w^\tau)$ by $\eta = (\overline{u}, z; \psi' \wedge z = w; w)$, projection in the last coordinate. It follows that

$$\Sigma \vdash \psi' \wedge (\psi' \wedge \overline{u} = \overline{v}) \wedge \overline{\theta}(\overline{v}/\overline{u}) \wedge (\psi' \wedge z = w) \to \sigma.$$

So by the inductive hypothesis,

$$(\psi', \overline{u}, z) \Vdash_{y_1,\ldots,y_n,z} \sigma[\alpha_1 \circ \varrho, \ldots, \alpha_n \circ \varrho, \eta].$$

Since ϱ is a covering map, we have by definition of the $\exists$-case

$$(\psi, \overline{u}) \Vdash_{y_1,\ldots,y_n} \varphi[\alpha_1, \ldots, \alpha_n].$$

$\varphi = \varphi_1 \to \varphi_2$: We prove only the $(\Rightarrow)$ direction. Suppose $(\psi, \overline{x}) \Vdash_{\overline{y}} \varphi[\overline{\alpha}]$, where $\overline{\alpha} = \alpha_1, \ldots, \alpha_n$. Let $\mu \equiv \exists \overline{y}(\psi(\overline{u}/\overline{x}) \wedge \overline{\theta} \wedge \varphi_1)$. Then $\mu \subseteq \overline{u}$. Define an inclusion morphism

$$\beta = (\overline{u}; \nu; \overline{z}) : (\mu, \overline{u}) \to (\psi, \overline{x})$$

by $\nu \equiv \mu \wedge \overline{u} = \overline{z}$. Since $\theta_1, \ldots, \theta_n$ are functional, we have $\Sigma \vdash \mu \wedge \overline{\theta} \to \varphi_1$. By the inductive hypothesis, $(\mu, \overline{u}) \Vdash_{\overline{y}} \varphi_1[\alpha_1 \circ \beta, \ldots, \alpha_n \circ \beta]$. Hence by the assumption $(\mu, \overline{u}) \Vdash_{\overline{y}} \varphi_2[\alpha_1 \circ \beta, \ldots, \alpha_n \circ \beta]$. By the inductive hypothesis, $\Sigma \vdash \mu \wedge \overline{\theta} \to \varphi_2$, but this implies $\Sigma \vdash \psi(\overline{u}/\overline{x}) \wedge \overline{\theta} \to \varphi_1 \to \varphi_2$. □

Remark. Note that the generic model gives a sound interpretation of intuitionistic logic eventhough its sorts need not be inhabited. In the proof of the soundness theorem we just need to replace the covering identity map in the $\exists$-case by a suitable projection.

Corollary 5.6 (Strong Completeness Theorem). *Let Σ be a closed theory in the language L which includes the equality theory of L. Then for all closed L-formulas φ,*

$$\Vdash^{\mathcal{M}(\Sigma,L)} \varphi \quad \textit{iff} \quad \Sigma \vdash \varphi.$$

Proof. This follows from the previous theorem, by letting $(\psi, \overline{x})$ be the terminal object. □

Example. We may give a simple model-theoretic proof of the conservativity of adding function symbols. (There is of course a direct proof-theoretic argument too.) Let Σ be an L-theory, as in the completeness theorem, such that $\Sigma \vdash \forall x^{\sigma} \exists! y^{\tau}\, \varphi(x, y)$ and $f \notin L$. Then we can show that $\Sigma + \{\forall xy\, (f(x) = y \leftrightarrow \varphi(x, y))\}$ is a conservative extension of Σ, by expanding the generic model $\mathcal{M}(\Sigma, L)$ with an interpretation of f:

$$[\![f]\!]_{(\psi,\overline{x})}((\overline{u}; \theta; v)) = (\overline{u}; \exists v\, \theta \wedge \varphi(v, z/x, y); z).$$

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