Abstraction and Epistemic Economy

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March 13, 2015

1 Introduction

Most[1] of the arguments usually appealed to in order to support the view that some abstraction principles are analytic depend on ascribing to them some sort of existential parsimony or ontological neutrality[2], whereas the opposite arguments, aiming to deny this view, contend this ascription. As a result, other virtues that these principles might have are often overlooked. Among them, there is an epistemic virtue which I take these principles to have, when regarded in the appropriate settings, and which I suggest to call 'epistemic economy'. My present purpose is to isolate and clarify this notion. I shall also try to make clear that complying with this virtue is essentially independent of complying with existential parsimony or ontological neutrality.

The intimate connection between the analycity of an abstraction principle and its existential parsimony or ontological neutrality can be questioned and, in my view, the analyticity of such a principle can be made to depend on its epistemic economy instead. Hence, distinguishing epistemic economy from existential parsimony and/or ontological neutrality would allow one which denies that an abstraction principle is existentially parsimonious or ontologically neutral, or keeps an agnostic view on the matter, to nevertheless maintain that it is analytic, according to some plausible construal of analyticity.

In my view, an abstraction principle (like any other axiom or definition) is epistemically economic not because of its logical nature, nor because of some of its other intrinsic features, nor

[1]I thank Andrew Arana, Jeremy Avigad, Francesca Boccuni, Annalisa Coliva, Sorin Costreie, Michael Detlefsen, Sébastien Gandon, Guido Gherardi, Emmylou Haffner, Bob Hale, Brice Halimi, Greeg Landini, Paolo Mancosu, Ken Manders, Daniele Molinini, Julien Ross, Andrea Sereni, Stewart Shapiro, Giorgio Venturi, Sean Walsh, and David Waszek for useful comments and/or suggestions.

[2]I am not interested, here, in discussing the relation between existential parsimony and ontological neutrality. All I shall say of these virtues is independent of this matter.

even because of its being immersed in certain logical systems, but rather because of its context of use, that is, the setting and purpose in and for which it is used, when immersed in such logical systems.

In particular, I shall limit my attention to the use of abstraction principles in formal definitional contexts, namely to their being involved in different formal definitions of natural and real numbers (and, incidentally, of integer and rational ones). These definitions are all complex in the sense that they do not merely depend on a single stipulation, but rather depend on a system of stipulations of different sorts. The abstraction principles I shall take into account are among such stipulations, and I will take their being epistemically economic (or not) as being the same as their being involved in definitions which are themselves epistemically economic (or not). Insofar as all these complex definitions also include explicit definitions, I will then regard the relevant abstraction principles being or not being epistemically economic on par with these explicit definitions being or not being so. Broadly speaking, I take a definition, and the abstraction principles possibly involved in it, to be epistemically economic if its understanding involves less and/or more basic intellectual resources than other relevant definitions of the same items, or, in case the required resources are (nearly) the same, if its understanding is more progressive than that of other relevant definitions of the same items[3].

This explanation is quite rough. In § 2, I shall try to elucidate the notion of epistemic economy of a formal definition in general. This is a hard task, however. To achieve it, one should explain, in general, intricate notions such as those of understanding and of intellectual resources to be deployed to obtain understanding, as well as provide a way to compare the amounts of the intellectual resources involved in understanding different definitions of the same items, and their being more or less basic. It is therefore not surprising that I won't be able to get a general, clear-cut, unequivocal and compact characterisation of this virtue. I merely hope to make my general idea reasonably clear, so as to open the road for further enquiries.

In the following §§, I will look at the matter more *in concreto*, as it were, by considering different examples and dealing with them in comparative terms. In § 3, I will briefly take stock of (a part of) the discussion about the analyticity of Hume's principle (HP, in what follows), by mainly discussing a neo-logicist argument according to which HP contrasts with Peano axioms insofar as the latter provide an arrogant implicit definition of natural numbers, whereas the definition provided by the former avoids arrogance. This is intended to clarify that and how this discussion essentially focuses on existential issues. In reaction to this, in § 4, I will then suggest another way to contrast the same definitions, by comparatively assessing their epistemic costs, and I shall argue that Frege Arithmetic (FA, in what follows)—the neo-logicist version of second-order arithmetic, involving HP as an axiom—supplies an epistemically economic definition of natural numbers. In § 5, I will compare four definitions of real numbers, three of which involve

[3]Plausibly, epistemic economy does not only apply to definitions or parts of them. However, apart from some short remarks on the epistemic cost of proofs in § 2, I shall not investigate this matter further here.

abstraction principles, and I will argue that one of the latter is epistemically economic. In § 6, I will finally offer some concluding remarks.

2 Epistemic Economy

Speaking of epistemic economy might bring to mind Mach's idea of economy of though ([28], §V.4, pp. 452-466, esp. §V.4.6, pp. 460-46. E. C. Banks has distinguished two doctrines concerned with this idea ([1], p. 24).The first one pertains to the way in which "science structures its laws under one another to maximise desirable features", by grouping "the greatest number of particular experiences under the least number of super categories and principles". This doctrine is "descriptive", since, for Mach, "nature [. . .][is] lawlike", that is, "objective temporal and spatial patterns exist [. . .] in nature ready to be arranged under one another". The second doctrine is normative, instead, since it pertains to "the role of economy in the framing of basic laws". But normativity here essentially depends on our cognitive faculties, mainly memory, since "the emergence of general concepts and laws" is explained by Mach in terms of "memory's operation over traces" (*ibid.*, p. 31). This second doctrine was the target of Husserl's allegation of psychologism ([25], ch. 9). According to Husserl, when speaking of economy of though, Mach was "ultimately" bearing on "a branch of the theory of evolution", with the result that his "attempts to found epistemology on an economy of thought ultimately reduce[d] to attempts to found it on psychology" ([26], vol. I., p. 128). To these attempts, Husserl opposed a different program. He focused on the "thought-economy which occurs in the purely mathematical discipline, when genuine thought is replaced by surrogative, signitive thinking", from which "almost without specially directed mental labour, deductive disciplines arise having an infinitely enlarged horizon", and envisioned undertaking a detailed investigation of the different methods allowing this "economic achievement" (*ibid.*, pp. 127-128).

What I mean by epistemic economy stands between Mach's descriptive doctrine and Husserl's program. On the one hand, I do not endorse the claim that objective patterns (whatever they might be) already exist in nature "ready to be arranged under one another", but I agree that a scientific theory, as well as a mathematical one, results from a certain way of structuring some appropriate prior material. On the other hand, I do not endorse the ideas that, in pure mathematics, "genuine thought is replaced by surrogative signitive thinking", and that "deductive disciplines" require almost no "specially directed mental labor", but I agree that formal mathematical theories use signs, or, better, appropriate formal languages, to render a previous informal thinking.

To be more precise, I take mathematics to result from our intellectual activity and consequently consider that achieving a mathematical task depends on deploying some intellectual resources. I view formal theories as convenient means for expressing and controlling abstract thinking. More precisely, I consider that the purpose of a formal theory is to re-cast a certain piece of our informal knowledge in such a way that the different ingredients involved in this theory are so arranged that it becomes transparent what rests on what. Hence understanding a formal theory, or any component of it (for example a definition, a theorem, or a proof), consists, in my view, in recognising it as re-casting the relevant pieces of some informal knowledge in such a way that its different ingredients—which are, in turn, to be recognised as re-casting some pieces of our informal knowledge —obey a certain arrangement. Moreover, I take the intellectual resources involved in this understanding to be those that are required to achieve this recognition. Thus, a formal theory or any of its components are, in my view, epistemically economic if they are so shaped that achieving such a recognition in the relevant case calls for less and/or more basic intellectual resources than in the case of other theories or comparable components of them, or, if these resources are (nearly) the same in both cases, when they come about in the former more progressively than in the latter[4]. This means that the former theory (or component) re-casts the relevant piece of our informal knowledge by also re-casting for this purpose less and/or more basic other pieces of our informal knowledge than the latter theory, or do it by re-casting more progressively (nearly) the same pieces of our informal knowledge.

One could go further and maintain that a formal theory, or any of its components are as epistemically economic as possible, or utterly epistemically economic if achieving such a recognition relatively to them calls for as few and/or as basic intellectual resources as possible, that is, if they re-cast the relevant piece of our informal knowledge by also re-casting for this purpose as few and/or as basic other pieces of our informal knowledge as possible. Determining whether this last condition occurs would require a modal appraisal that could be difficult. I therefore conform to the former, weaker, account.

As an example (to which I shall come back in §§ 3 and 4), consider FA. It is intended to re-cast informal arithmetic, and it appeals, for this purpose, to an appropriate re-casting of several informal notions, like those of an object and of a first-level concept[5], that of identity for objects, that of the falling of an object under a (first-level) concept, etc. (cf. § 4, for a more comprehensive inventory). For short, we could say—and, indeed, I shall use this way of speaking, in what follows—that understanding FA calls for these notions. This means that understanding FA depends on recognising it as re-casting informal arithmetic by also re-casting all these notions according to a certain arrangement. Arguing that FA is epistemically economic is the same as arguing that these notions are fewer and/or more basic than those on the re-casting of which the understanding of other current formal theories (especially second-order ones), which are also intended to re-cast informal arithmetic, depends on. This means that FA achieves this task by also

[4]By this, I mean that the relevant resources, or a significant part of them, come about in the former case in agreement with an order in which some conceptually depend on others but not vice versa, whereas they come about in the latter case all together at once.

[5]Here, I adopt the current neo-logicist interpretation of monadic predicate variables as ranging over first-level concepts. *Mutatis mutandis*, one could go for another option and interpret such variables as ranging over properties of objects. If first-level concepts are conceived as they are by neo-logicists, then it does not seem to me that this would make any significants difference.

re-casting for this purpose fewer and/or more basic informal notions than these other theories.

It appears to me that, thus conceived, the notion of epistemic economy comes close, although in a different setting, to one of Frege's crucial concerns.

Famously, Frege considered a truth to be analytic if, in its proof, "one only runs into logical laws and definitions" ([14], § 3; I slightly modify the translation offered in [16]). By 'definitions' Frege clearly meant explicit definitions, and, for him, an explicit definition could certainly not be used to introduce new items and, *a fortiori*, for positing their existence. Together with the presently common view that a logical law can have no existential import, this could lead to the conclusion that Frege's pursuit of analyticity is a pursuit of existential parsimony. However, his conception of logic[6] suggests another view. He distinguished second-order logic as such, neither from first-order, nor from propositional logic, thus failing to grasp (or avoiding to emphasise) the essential difference (mainly in matters of ontology) we see between them. Moreover, he did not conceive of comprehension as a specific condition to be met through the admission of specific axioms, but rather came to results analogous to the ones we obtain by comprehension through the apparently innocent adoption of unrestricted rules of substitution in language. Finally, he candidly embraced the notion of a logical object. He took his logical language to be meaningful, and included under his (quite generic) notion of a law of logic both statements written in what is for us a purely logical language, and others involving what is for us a non-logical vocabulary. He was then perfectly comfortable with the idea that fixing the laws of logic comes together with ensuring (or revealing) the existence of some logical objects. Hence, restricting a proof to rely only on logical laws and (explicit) definitions was, for Frege, less a way of pursuing existential parsimony, than a way of limiting the tools to be used in conducting a proof. A similar concern also appears both in his notion of aprioriness—according to which a truth is *a priori* if it admits a proof that "proceeds as a whole from general laws which neither need nor admit a proof"—and in his claim that the question of whether a truth is *a priori* or *a posteriori*, analytic or synthetic is settled by "finding [...][its] proof and tracking it back up to the original truth" (*ibid.*). Frege's main purpose was, then, that of tracking arithmetical truths back to the minimal tools required to prove them.

This was for him a way to identify the place of these truths within the objective general order of truths, which he was aiming to reconstruct. Once the idea that there exists such an objective order is dismissed, and epistemology is no longer understood to be concerned with the reconstruction of such a putative order, but rather with the activity that human subjects perform in order to constitute a body of knowledge, and with the resources to be deployed to this end, probative tools appear as part of these resources, and minimising the former results in an effort to economise the latter. Moreover, once epistemology is so conceived, proofs appear less as procedures for discovering actual truths than as arguments for obtaining conclusions starting from some assumptions or from other previously established conclusions. Hence, assessing the

[6] Regarding my appraisal of Frege's conception of logic, I refer the reader to [6].

tools required to conduct a proof appears less as a way to estimate the objective role of a certain truth than as a way to evaluate the epistemic cost of reaching a certain conclusion within the relevant theory, namely the intellectual resources that are to be deployed within this theory to fix a certain thought. When replaced in such a framework, his concern for whether a truth is *a priori* or *a posteriori*, analytic or synthetic comes close to a concern for epistemic economy.

Still, whereas Frege's concern seems to point to the epistemic cost of the proofs involved in the relevant theories, that is, to the resources to be deployed to understand these proofs, I rather focus on the epistemic cost of the definitions that make these proofs possible, that is, on the resources to be deployed to understand these definitions, independently of the proofs they give rise to.

3 Arrogant vs. Non-Arrogant Implicit Definitions

I begin my enquiry by considering the neo-logicist definition of natural numbers.

Even though most discussions of this definition focus on HP, there is more in the former than the mere stipulation of the latter. For the definition to work, a previous definition of objects and first-level concepts is required. Immersing HP in a suitable system of second-order logic simply makes this definition available. Whether explicitly mentioned or not, objects and first-level concepts are taken in the former definition to be implicitly defined by this system, as the items which its individual and monadic predicate variables are supposed to respectively range over, by admitting that the existence of the latter is ensured by comprehension. Once objects and first-level concepts are thus defined, HP acts as the implicit definition of a function taking the latter and giving a particular kind of the former, namely numbers of first-level concepts, that is, cardinal numbers. To go from these numbers to natural ones, one has to extend comprehension to formulas including the functional constant designating this function, and to rely on this extension to state some explicit definitions, which allows to single out the natural numbers from the cardinal ones[7].

In this context, the neo-logicist thesis that HP is analytic admits two different readings. According to one of them, that which is claimed to be analytic is HP as such, merely conceived as providing an implicit definition of a function inputting first-level concepts and outputting cardinal numbers. According to the other reading, that which is claimed to be analytic is the whole system of stipulations providing the definition of natural numbers, *i.e.* the whole FA,

[7]Taking the relevant system of second-order logic to implicitly define first-level concepts (or properties of objects: cf footnote 5, above) is not mandatory. One might merely take HP as an implicit definition of a functional constant inputting monadic predicates and outputting terms. But in this case, adding this principle to the axioms of such a system of logic would merely result in introducing appropriate numerals, rather than in defining cardinal numbers. These numerals being given, defining natural numbers would then require much more than merely singling the latter out among the former, since this selection would at most provide a family of terms.

including the explicit definitions allowing to single out natural numbers among cardinal ones. For brevity, let us term this thesis 'weak neo-logicist analyticity thesis' if it is taken under the former reading, and 'strong neo-logicist analyticity thesis', if it is taken under the latter one. Insofar as FA involves comprehension extended to formulas including the functional constant introduced by HP, the weak thesis appears, at least at first glance, more plausible than the strong one. Still, it is only by endorsing the strong one that it can be argued that the neo-logicist definition of natural numbers results in a vindication of Frege's logicist program, as neo-logicists claim. It therefore seems that whatever the arguments for or against the analyticity of HP might be, for them to be fully relevant for our appreciation of this program, and, more generally, for the philosophy of arithmetic, they have to be for or against the strong thesis.

This is the case of the argument against the analyticity of HP which is most often repeated and considered as convincing. Shortly, its point is that HP cannot be analytic since for it to hold there must be infinitely many objects ([4], p. 306)[8]. But, as remarked by Boolos himself, there is something biased in this argument, since, in Boolos's words, "one person's *tollens* is another's *ponens*, and Wright happily regards the existence of infinitely many objects, and indeed, that of a Dedekind infinite concept, as analytic, since they are logical consequences of what he takes to be an analytic truth" (*ibid.*). This is reminiscent of Frege's view that natural numbers are logical objects since their existence is required for some proper names to refer, and these names actually refer since this is in turn required for some logical truths to be true.

But, one might retort, if natural numbers are logical objects or their existence analytic, then (second-order) Peano axioms, when conceived as categorical implicit definitions of these numbers, should be logical or analytic truths. And, if Peano axioms are so, why should appealing to HP to define natural numbers be preferable to merely stating these axioms? The obvious answer is that defining natural numbers though HP allows us to recognise that natural numbers are logical objects or their existence analytic, since it is the possibility of this very definition that makes it is so, whereas this is not the case for the definition of natural numbers appealing to Peano axioms[9].

This could lead one to think that the crucial point under discussion does not concern existential issues but rather an alleged intrinsic difference between HP and Peano axioms—namely, the former having a virtue (apparently an epistemic one) that the latter do not have—, which does not depend on their respective truth requiring or not requiring the existence of some objects, that is, on their having or not having an ontological import for objects[10]. However, when one looks at

[8]This argument questions the strong neo-logicist thesis since it is only in presence of second-order logic with comprehension appropriately extended, and of appropriate explicit definitions, that the existence of infinitely many objects follows from HP (through the admission of appropriate explicit definitions).

[9]Of course, the existence of natural numbers being analytic is strictly not the same as their being logical objects, as well as a truth being logical is not the same as its being analytic. These important distinctions are not relevant for the issue under discussion and it is then not necessary to insist on them here.

[10]For short, I say that a stipulation, or a system of stipulations, has an ontological import for objects if the truth of this stipulation, or these stipulations, requires the existence of some objects.

the situation more carefully, this appearance dissipates.

To see this, consider how Hale and Wright account for the relevant difference they see between HP and Peano axioms. Their main point is that the definition of natural numbers provided by the latter is arrogant, while the one provided by the former is not. The point is made on different occasions, but it receives particular emphasis in [20] and [22].

The former paper mainly focuses on implicit definitions. These are taken to include two parts: an unsaturated "matrix" formed by "previously understood vocabulary" (or possibly only by logical-constants and variables or schematic letters), and the *defininedum* or *definienda* to be inserted in this matrix, so as to form a well formed sentence or system of sentences (*ibid.* pp. 285 and 289)[11]. It is then arrogant if "the antecedent meaning" of the matrix and "the syntactic type" of the *definiendum* or *definienda* are such that the truth of the relevant sentence or sentences "cannot justifiably be affirmed without a collateral (*a posteriori*) epistemic work" (*ibid.*, p. 297).

To this general characterisation, a sufficient and a necessary condition for an implicit definition not to be arrogant are added. If I understand it correctly, the former (generally stated at pp. 314-315) goes as follows: let 'f' designate a *definiendum* and let '$\boldsymbol{S}(f)$', '$\boldsymbol{S}_I$' and '$\boldsymbol{S}_E$' be three appropriate sentences or schemes of sentences, the first of which includes one or more occurrences of 'f', while the two others include no occurrence of 'f' and of any constant designating another *definiendum*; and the third does not "introduce [any] fresh commitments"; then, for an implicit definition of f not to be arrogant, it suffices that it results from stipulating the truth either of '$\boldsymbol{S}_I \Rightarrow \boldsymbol{S}(f)$', or of '$\boldsymbol{S}(f) \Rightarrow \boldsymbol{S}_E$', or of both (*ibid.*, pp. 299 and 302). Hale and Wright seem, here, to imply that the mere conditional form of these sentences or schemes of sentences is enough to ensure that their truth can "justifiably be affirmed without a collateral (*a posteriori*) epistemic work". But why this is so? The necessary condition suggests a response, at least for the case to which it applies. According to it, in order to avoid arrogance "the stipulation of the relevant sentence as true ought not require reference for any of its ingredient terms in any way that cannot be ensured just by their possessing a sense" (*ibid.*, p. 314). It seems, then, that, according to Hale and Wright, in the cases where '$\boldsymbol{S}(f)$' includes some constant terms involving 'f', possibly schematic ones (that is, 'f' is either an individual or a functional constant), the conditional form of '$\boldsymbol{S}_I \Rightarrow \boldsymbol{S}(f)$' and '$\boldsymbol{S}(f) \Rightarrow \boldsymbol{S}_E$' ensures that the truth of these very sentences or of their relevant instances does not require (as a necessary condition) that these terms or their corresponding instances refer in a way that were not possibly ensured by their acquiring a sense thanks to the very stipulation of such a truth. This means that it is not a necessary condition for these sentences or their relevant instances to be true that some objects exist. Hence, the relevant "collateral (*a posteriori*) epistemic work" is that which would be required to ensure that these objects exist.

[11]As a matter of fact, Hale and Wright only consider the case of definitions given by a single sentence including a single *definiendum*, but the generalisation to the case of systems (or conjunctions) of sentences including more *definienda* is as natural as it is necessary to adapt the account to the case of Peano axioms.

This point well applies to HP and Peano axioms (perhaps too well to not generate the suspicion that it is taken *ad hoc*). Suppose that 'f' stands for the functional constant '#', '$\boldsymbol{S}\,(-)$' stands for '$-P = -Q$', and '$\boldsymbol{S}_I$' and '$\boldsymbol{S}_E$' both stand for '$P \approx Q$' (where 'P' and 'Q' are schematic monadic predicates). From the sufficient condition, it follows that stipulating the truth of HP—namely, '$\#P = \#Q \Leftrightarrow P \approx Q$' (where '$P \approx Q$' abbreviates a formula of second-order logic asserting that the objects falling under P and those falling under Q are in bijection)—provides a non-arrogant implicit definition of the function #, since, for '$\#P = \#Q \Leftrightarrow P \approx Q$' to be true, it is not necessary that '$\#P = \#Q$' be true, in turn, and, then, that the relevant instances of '$\#P$' and '$\#Q$' refer in the required way. On the other hand, for any Peano axiom—for example '$Suc(n) \neq 0$' (where 'n' is a schematic individual constant)—to be true, it is necessary that the constant terms included in it, or their relevant instances—namely '0' and the relevant instances of '$Suc(n)$'—refer in this way. From the necessary condition, it follows, then, that stipulating the truth of Peano axioms results in an arrogant implicit definition.

An easy reply to this argument has been suggested by MacFarlane ([27], pp. 454-455). Let PA be the conjunction of all Peano axioms (in some appropriate form), and CPA the double implication 'PA $\Leftrightarrow \forall x\,(x = x)$'. Clearly, CPA is as conditional as HP is, but stipulating its truth has the very same consequences as stipulating PA's truth. One could retort that CPA "makes the existence of [natural] numbers conditional on a logical truth" and is then conditional in a "Pickwickian sense". Surely. But "HP, too, makes the existence of [natural] numbers conditional on logical truths[, and][...] that is precisely why it can serve as the basis of a kind of logicism". Moreover, both the left-hand sides of CPA and of HP have "ontological commitments" of which their right-hand sides are "innocent". Hence if defining natural numbers through HP avoids arrogance because of the conditional form of this principle, then defining natural numbers through CPA should avoid arrogance also.

The answer might be too easy, since the right-hand side of HP is not, as such, a logical truth; only some of its relevant instances are so. Hence, the truth-conditions of HP do not reduce to the truth-conditions of its left-hand side, as it is the case for CPA. This is precisely what Hale and Wright retort to MacFarlane's objection in [22]: any instance of HP whose right-hand side is a logical truth "is to be viewed as part of a package of stipulations whose role is to fix the truth-conditions of statements of numerical identity"; what matters, then, are these conditions, namely the fact that they are "as feasible as any other purely meaning-conferring stipulations", which entails that "there is no need—indeed no room—for any associated stipulation of numerical existence" ([22], p. 476).

Here, as in the argument of [20], the essential point is that HP has, a such, no ontological import for objects, since its truth does not require that the relevant instances of '$\#P$' and '$\#Q$' refer in a way that is not ensured by their acquiring a sense, thanks to the stipulation of this truth, and, then, that cardinal numbers exist, whereas Peano axioms, as well as their conditionalisation CPA have such an import, since their truth requires that '0' and the relevant instance of '$Suc(n)$'

refer in this way, and then that natural numbers exist. To complete the argument, it is then enough to add that, once the truth of HP is stipulated, the existence of cardinal numbers is revealed by the ascertainment of the truth of appropriate instances of '$P \approx Q$', and that this truth is nothing but a logical truth (the existence of 0 is, for example, revealed by defining it as the cardinal number $\#[x : x \neq x]$ and by deriving the truth of '$0 = 0$' from HP and '$[x : x \neq x] \approx [x : x \neq x]$', which is a logical truth).

MacFarlane's objection does not come alone, however. It is part of a more general argument intended to show that PA fares as well as HP in all the requirements, other than non-arrogance, that Hale and Wright advance in [20] for an implicit definition to be acceptable as a meaning-conferring stipulation, namely consistency, conservativeness, generality, and harmony (supposing that the non-arrogance constraint is independent of these, which MacFarlane questions, at least insofar as non-arrogance reduces to conditionality, and Hale and Wright claim, instead: cf. [27], p. 455, and [22], pp. 467-468) [12]. In their reply, Hale and Wright question whether this is the case for the two last constraints ([22], pp. 466-467), but they do not insist on this point much. In [22], they mainly insist on the non-arrogance requirement to underline the difference between Peano axioms and HP, as alleged implicit definitions of natural numbers. They nonetheless seize the opportunity provided by this new paper to spread more light on this requirement.

They begin by suggesting both a new general characterisation of arrogance, and a new sufficient condition for it. Their characterisation is the following: arrogance is "the situation where the truth of the vehicle of the stipulation is hostage to the obtaining of conditions of which it's reasonable to demand an independent assurance, so that the stipulation cannot justifiably be made in a spirit of confidence, 'for free'" ([22], pp. 465). And the sufficient condition is this: "a stipulation is arrogant just if there are extant considerations to mandate doubt, or agnosticism, about whether we are *capable* of bringing about truth merely by stipulation in the relevant case" (*ibid.*, p. 468). By conversion, this implies that non-arrogant implicit definitions "are ones where there is no condition to which we commit ourselves in taking the vehicle to be true which we are not justified—either entitled or in possession of sufficient evidence—to take to obtain" (*ibid.*). The point is then concerned with the nature of the "independent assurance" that arrogant and non-arrogant explicit definitions do and do not require, respectively, and/or of the reasons that can "mandate doubt, or agnosticism" about our capability of "bringing about truth", by merely making the relevant stipulations.

A new argument advanced by Hale and Wright might suggest that this nature does not only pertain to existential considerations. Leaving their motivations apart, the claim is the following: "The stipulation of Hume['s principle] serves to communicate a singular-thought-enabling

[12]As a matter of fact, MacFarlane's paper is also concerned with Hale and Wright's conception of numerical definite descriptions as singular terms in relation with the sort of logic that FA actually requires (namely whether this logic is classical or free). In [22], Hale and Wright also reply to this point, but, though somehow connected with the question I'm discussing, this matter can be left aside here.

conception of the sort of objects the natural numbers are and explains their essential connection with the measure of cardinality. The stipulation of [...] Peano [axioms] communicates no such conception, and actually adds no real conceptual information to what would be conveyed by a stipulation of their collective Ramsey sentence" (*ibid.*, pp. 471-472). This is highly questionable, however, for reasons that do not directly depend on the reading of HP and of the consequent definition of natural numbers. What Hale and Wright base this conclusion on is their view that Peano axioms "convey no more than the collective structure of the finite cardinals—something which since it entails those axioms, Hume['s principle] also implicitly conveys"; a view that they specify by claiming that these axioms convey "no conception of the sort of thing that zero and its suite are" (*ibid.*, p. 471). Still, consenting to this requires admitting that there is something that natural numbers actually are, besides their forming a progression. In other words, what Hale and Wright take here for granted is not that there is room for coding these numbers with appropriate items having a determinate particular essence, but rather that these numbers have a determinate particular essence. It is only after having consented to this that the discussion can begin on whether HP conveys this essence, while Peano axioms do not. But consenting to this is all but anodyne, and no argument for the definitional advantage of HP on Peano axioms can require this without being biased from the very beginning.

This might be the reason for Hale and Wright's rapid shift back to existential considerations. They remark that there is no concern over a possible replacement of HP with its ramsification (whatever it might be), since "there is no need to (attempt to) stipulate that a suitable function [*i. e.* a function satisfying HP] exists", provided that "the existence of such a function is [...] a consequence of something known as an effect of the stipulation, viz. Hume's Principle itself" (*ibid*, p. 473). The point here is that HP merely fixes "the truth-conditions of the canonical statements of numerical identity in which [the operator '#'][...] occurs" (*ibid.*). In other terms, what really matters is that HP, as well as any abstraction principle suitable for working as an implicit definition, is "tantamount to legitimate schematic stipulations of truth conditions [...][, whereas] to lay down [...] Peano [axioms] as true is to stipulate, not truth-*conditions*, but *truth* itself" (*ibid.*, pp. 474). Hence, Hale and Wright go ahead, "as a stipulation, Hume['s principle] is considerably more modest than [...] Peano [axioms]: the attempted stipulation of the truth of [...] Peano [axioms] is effectively a stipulation of countable infinity; whereas whether or not Hume['s principle] carries that consequence is a function of the character of the logic in which it is embedded" (*ibid.*, pp. 475).

Even though it is enriched by a number of collateral considerations, the crucial argument seems to be the same one already advanced in [20]: what matters, concerning the definitional advantages of HP over Peano axioms, is that the latter have an ontological import for objects, better they entail the existence of an infinity of objects, whereas the former has no such import (and it perfectly works, as it is required to work in FA, without requiring that the existence of any function be admitted)[13].

For short, call this argument 'the existential argument'. Taken as such, it seems at most able to support the weak neo-logicist analyticity thesis, since it merely concerns HP, or better yet, nothing but the logical form of HP. Moreover, it is just part of the argument that HP acquires an ontological import for objects when it is embedded in an appropriate logical setting (and, I add, it is coupled with appropriate explicit definitions). Hence, to support the strong thesis, one should either advance an independent argument, or admit that the existential argument secures the weak thesis and infer the strong thesis from it. For this last purpose, one could admit: *i*) that a system of stipulations is analytic, even though it has an ontological import for objects, if the objects whose existence is required by the truth of these stipulations (taken together) are such that their existence is to be regarded as analytic, in turn; *ii*) that the existence of some objects is to be regarded as analytic if it follows from logic, plus some analytic principles, an appropriate extension of comprehension, and explicit definitions[14]. It would then be easy to conclude that the whole FA is analytic, though having an ontological import for objects, since HP, taken as such, is analytic. But both of these suppositions can be questioned. For example, one could argue that, for (*ii*) to be admissible, it should involve some conditions to be met by the relevant system of logic, so to avoid that one be licensed to take as analytic a complex definition involving a system of logic whose innocence might be questioned. Once this is admitted, one could argue, in agreement with Shapiro and Weir ([32], §§ II and III), that the system of second-order logic involved in FA does not meet these conditions. This would block the derivation of the strong neo-logicist thesis from the weak one, with the result that the existential argument could not be taken as part of a larger argument in favour of the former thesis, even if it were regarded as suitable for supporting the latter.

This is not all. The suitability of the existential argument for supporting the weak thesis can be questioned, too. One could contend, for example, that from the fact that the truth of HP does not require the truth of '$\#P = \#Q$'—nor the existence of references for the the relevant instances of '$\#P$' and '$\#Q$', *i.e.* of cardinal numbers—it does not follows that HP has no ontological import

[13] Neo-logicists have come back to this last point in different ways and in the context of different forms of argumentation. A very compact way to make the same point is found, for example, in [37], § II.2.

[14] Apart for the mention of the extension of comprehension (which neo-logicists seems to consider as existentially anodyne), condition (*ii*) is suggested by Wright's following remarks ([37], pp. 307 and 310):

> Analyticity, whatever exactly it is, is presumably transmissible across logical consequence. So if second-order consequence is indeed a species of logical consequence, the analyticity of Hume's Principle would ensure the analyticity of arithmetic.
>
> [...] on the classical account of analyticity the analytical truths are those which follow from logic and definitions. So if the existence of zero, one, etc. follows from logic plus Hume's Principle, then provided the latter has a status relevantly similar to that of a definition, it will be analytic, on the classical account, that n exists, for each finite cardinal n.

for objects. To this purpose, one could argue as follows. Insofar as the truth of the mere axioms of the system of second-order logic to which this principle is added to get FA does not require the existence of any object, the following options remain open: a) this import is distributed among this system of second-order logic, HP taken as such, the extension of comprehension to formulas including '#', and the explicit definitions used to single out natural numbers among cardinals; b) in the presence of this extension and of these explicit definitions, this system of second-order logic triggers the ontological import for objects of HP, without having itself any such import; c) vice versa, in the presence of this extension and of these explicit definitions, HP triggers the ontological import for objects of this system of second-order logic, without having itself any such import; d) taken together, both this system of second-order logic and HP trigger the ontological import for objects of this extension and of these explicit definitions, without having themselves any such import. The assessment of the ontological import of HP could provide a sound argument in favour of the weak neo-logicist analyticity thesis only insofar as it were suitable for positively supporting (c) or (d), or at least for discarding (a) and (b). But two problems arise. Firstly, granting (c) or (d) would result in ascribing an ontological import for objects to this system of second-order logic, and to this extension and these explicit definitions, respectively, and this would be at odds with the purpose of deriving the strong thesis from the weak one along the lines suggested above. Secondly, the existential argument is able neither to positively support (c) or (d), nor to discard (a) and (b), since all these options are left open by acknowledging that HP is not arrogant (insofar as its truth does not require the truth of '$\#P = \#Q$'), while contrasting the arrogance of Peano axioms with HP's avoiding arrogance is simply not relevant for choosing among the four options.

Moreover, one could also observe that deriving the existence of natural numbers in the neo-logicist setting hinges on the logical truth of appropriate instances of '$P \approx Q$' only in free logic. Since, if logic is not free, the mere introduction of the functional constant '#', *via* HP, and the extension of comprehension to formulas including this constant allow one to derive any instance of '$\#P = \#P$' from '$\forall x\,(x = x)$'([27], p. 447; that the logic underlying FA is to be free is also argued, among others, in [32], p. 108, and admitted by Hale and Wright in [22], p. 463-464). But assuming that the relevant logic is free seems to be at odds with alleging that the existence of some objects is analytic if it follows from logic plus some analytic principles, a principle being analytic insofar as it has no ontological import for objects. Indeed, the adoption of free logic seems to be naturally linked with the view that existence cannot be a matter of logic.

I'm far from considering these or similar remarks as knock-down objections against the suitability of the existential argument for supporting the neo-logicist claim that HP is analytic, this claim being intended either in agreement with the weak or with the strong neo-logicist analyticity thesis. For my present purpose, it is enough to show that this argument can be questioned. This should be enough to urge anyone regarding the neo-logicist definition of natural numbers with interest to look for other reasons to favour it over other definitions, and, possibly, to also view

HP, or better the whole FA, as analytic, namely reasons that are not based on the assessment of HP's ontological import for objects. My suggestion is that such a reason can be found in the epistemic economy of the former definition, which I regard as being independent of HP's having or not having this import.

Before arguing in favour of this suggestion a disclaimer is in order. The point I want to make is neither that the neo-logicist definition of natural numbers is definitely better than, or to be preferred to, any other current one, because it is epistemically economic, nor that its being so makes this definition better than, or preferable to, any other current one from an epistemic point of view. My point is rather that its being so gives this definition a particular epistemic virtue that other current definitions do not possess, which makes then former preferable to the latter in appropriate circumstances and according to some aims—though I admit that other virtues, either epistemic or not, could be differently distributed among the relevant definitions, and could bring on other choices under different circumstances and according to different goals.

4 Comparing FA and Z_2 with Respect to Epistemic Economy

Until now, I followed Hale and Wright and spoke of Peano axioms in general (in fact they often speak of Dedekind-Peano axioms, but this makes no relevant difference, of course). To settle some ideas, it is better to be more precise. In the present §, I shall consider Peano arithmetic under the form of Hilbert-Bernays second-order theory Z_2, in Simpson's version ([24], suppl. IV; [34]), and compare its epistemic cost with FA's.

Z_2's language—let us call it '$\mathfrak{L}^{\mathsf{Z}_2}$'—is two-sorted, and its two sorts of variables, 'i', 'j', 'k', 'm', 'n',... and 'X', 'Y', 'Z',... are intended to range over natural numbers and sets of natural numbers respectively. $\mathfrak{L}^{\mathsf{Z}_2}$ also includes the identity symbol for terms and six non-logical constants '0', '1', '+', '$\cdot$', '$<$ ', and '$\in$', the first five of which are governed by eight first-order "basic axioms", reminiscent of Peano's original system ([29]), and providing a minimal keystone for Peano arithmetic[15], while the sixth is governed by a second-order induction axiom and an unrestricted comprehension axiom-scheme, namely:

$$\forall X\left[\left[0\in X\wedge\forall n\left(n\in X\Rightarrow n+1\in X\right)\right]\Rightarrow\forall n\left(n\in X\right)\right], \qquad (\mathrm{Ind}[\mathsf{Z}_2])$$

and the universal closure of

$$\left[\exists X\forall n\left[n\in X\Leftrightarrow \overset{\mathfrak{L}^{\mathsf{Z}_2}}{\varphi}(n)\right]\right], \qquad (\mathrm{CompSc}[\mathsf{Z}_2])$$

[15] Adding to these axioms a first-order axiom-scheme of induction, one gets the system Z_1, which provides a convenient version of Peano first-order arithmetic ([34], pp. 7-8).

where '$\overset{\mathfrak{L}^{\mathsf{Z}_2}}{\varphi}(n)$' stands for any formula of $\mathfrak{L}^{\mathsf{Z}_2}$ in which 'X' does not occur freely.

This is a very strong theory, existentially speaking. But it is also very specific. It is true that intending its variables to range over natural numbers and sets of them does not *ipso facto* restrict their range, and CompSc[Z_2] is just the usual unrestricted comprehension scheme of full second-order logic extended to the whole $\mathfrak{L}^{\mathsf{Z}_2}$. Nonetheless, Z_2 is such that its models only include items behaving as natural numbers and sets of them are ordinarily required to do, not only for their forming a progression, but also for their being linked to each other by the order, additive and multiplicative relations that are ordinarily required to hold for natural numbers. In other words, Z_2 is specifically about natural numbers, as bearing these relations, and about sets of them. Understanding it involves understanding the conditions characterising both of these relations and two sorts of items, such that the items of the first sort bear these relations to one another, while those of the second sort are sets of items of the first sort, and calls for the appropriates notions. In particular, understanding the conditions characterising the second sort of items calls for more than the notion of a set of items of the first sort, since it also involves understanding how a particular such set is fixed through a condition expressed in $\mathfrak{L}^{\mathsf{Z}_2}$[16]. All this demands a large amount of intellectual resources. Moreover, as it is typical of structural definitions, these resources are all required simultaneously and from the start, in order to gain epistemic access to these items and begin to work consciously with them, since these definitions determine *ex novo* the relevant items as *sui generis* items characterised by the relations they are required to bear.

As is well known, the existential strength of Z_2 can be significantly reduced, while preserving much of its deductive strength, by restricting the comprehension axiom-scheme to formulas of an appropriate syntactical simplicity (cf. [34] for a comprehensive study). This results in systems like ACA_0, where comprehension is restricted to formulas containing no second-order quantifier, or RCA_0, where comprehension is restricted to Σ_1^0 and Π_1^0 formulas that are equivalent to one another and Ind[Z_2] is replaced by an axiom-scheme restricted to Σ_1^0 or Π_1^0 formulas. However, this does not lessen the intellectual resources required to understand the definition of natural numbers in Z_2. On the contrary, one could even argue that it increases these resources, since, on the one hand, restricting comprehension in a certain way does not result in limiting in the same way the complexity of the formulas involved in the relevant system, from the understanding of which depends the understanding of the system itself (to see it, remark that the universal closure of an instance of a restricted comprehension axiom-scheme can have a greater syntactical complexity than the formulas to which this scheme is restricted)[17], and, on the other hand, understanding restricted comprehension involves understanding the criterion that the restriction depends on, which is, of course, not necessary to understand unrestricted comprehension. Moreover, though

[16] Cf. footnote (18); below.

[17] The most evident case is that of a comprehension axiom-scheme restricted to formulas containing no second-order quantifier, as that involved in ACA_0: whatever such formula '$\varphi(n)$' might be, the syntactical complexity of '$[\exists X \forall n [n \in X \Leftrightarrow \varphi(n)]]$' is greater than that of this formula.

non-categorical (relatively both to their first and second order parts), ACA_0 and RCA_0 are, in a sense, even more specific than Z_2: the items they implicitly define are so specified as to be putatively suitable for supplying the building blocks to be used in the enterprise of recovering certain portions of "ordinary mathematics" ([34], p. 1) on the basis of set-theoretic existential assumptions that are as weak as possible.

The situation with FA is quite different. Its language—let us call it '$\mathfrak{L}^{\mathsf{FA}}$'—is much poorer than $\mathfrak{L}^{\mathsf{Z}_2}$, since it reduces to the language $\mathfrak{L}^{\mathsf{L}_2}$ of an appropriate system of full second-order logic L_2 including identity for terms and both monadic and dyadic predicate variables, supplemented by the single non-logical (functional) constant '#', introduced by HP[FA], which, written *in extenso* and replacing schematic predicates with predicate variables, takes the following form:

$$\forall F, G \left[\#F = \#G \Leftrightarrow \exists R \left[\begin{array}{l} [\forall x \left(F\left(x\right) \Rightarrow \exists! y \left(R\left(x,y\right) \wedge G\left(y\right)\right)\right)] \wedge \\ [\forall y \left(G\left(y\right) \Rightarrow \exists! x \left(R\left(x,y\right) \wedge F\left(x\right)\right)\right)] \end{array} \right] \right]. \qquad (\text{HP}[\mathsf{FA}])$$

Besides adding HP[FA] to the axioms of L_2, to move from L_2 to FA, one must also extend comprehension, both for monadic and dyadic predicates, to formulas including '#', which results in replacing the comprehension axiom-schemes of L_2 with the universal closures of the following schemes:

$$\exists F \forall x \left(F\left(x\right) \Leftrightarrow \overset{\mathfrak{L}^{\mathsf{FA}}}{\varphi}\left(x\right) \right) \qquad \text{and} \qquad \exists R \forall x, y \left(R\left(x,y\right) \Leftrightarrow \overset{\mathfrak{L}^{\mathsf{FA}}}{\varphi}\left(x,y\right) \right) \qquad (\text{CompSc}[\mathsf{FA}])$$

where '$\overset{\mathfrak{L}^{\mathsf{FA}}}{\varphi}(x)$' and '$\overset{\mathfrak{L}^{\mathsf{FA}}}{\varphi}(x,y)$' respectively stand for any formulas of $\mathfrak{L}^{\mathsf{FA}}$ in which 'F' and 'R' do not occur freely.

This already shows that FA does not involve sets[18], and, although dealing with cardinal numbers is not specifically about them. Indeed, L_2 merely deals with objects, their properties, or

[18]Notice that what matters here is not merely the way in which $\mathfrak{L}^{\mathsf{FA}}$'s predicates are informally conceived, in particular by neo-logicists, as opposed to the way in which $\mathfrak{L}^{\mathsf{Z}_2}$'s predicates are conceived. Indeed intending the second-order variables of Z_2 to range over sets of elements of the range of the first-order ones, and the constant '$\in$' as designating the set-theoretic relation of membership is not mandatory. One could rather intend the second-order variables of Z_2 to range over the monadic properties of the elements of the range of the first-order ones, and consider that '$n \in X$' is nothing but a typographic variant of '$X(n)$' or 'Xn' (that is, merely an alternative way to predicate the property X of the individual n). What matters is rather the way in which predicates work in FA and Z_2, respectively. Focusing on the mere definition of natural numbers, the difference is not really significant, since what Z_2's predicates do in relation to this definition can, *mutatis mutandis*, also be done by FA's monadic predicates. The difference becomes, instead, quite significant in relation to the definition of real numbers within these theories (which I shall consider in § 5, below). Indeed, if second-order variables of Z_2 are taken to range over monadic properties of the elements of the range of the first-order ones, rather than over sets of these same elements, one can hardly be happy with a definition of real numbers as some particular items within the range of the former of these variables, as that suggested by Simpson in relation to ACA_0 and RCA_0 (cf. § 5.1, below).

first-level concepts, and dyadic relations among them, and HP[FA], although suitable to implicitly define cardinal numbers, does not allow one to prove that $\forall x \exists F \, [x = \#F]$, with the result that the models of FA can be populated by objects other than cardinal numbers. *A fortiori*, FA does not involve sets of natural numbers and it is not specifically about these numbers.

Natural numbers have, rather, to be explicitly defined within FA. By relying on comprehension and HP[FA], one first defines 0 and the successor relation between cardinals:

$$\begin{array}{l} 0 =_{df} \# \, [n : n \neq n] \\ \forall x, y \, [\mathcal{S}(x, y) \Leftrightarrow \exists F \exists z \, (F\,(z) \wedge (y = \#F) \wedge (x = \# \, [n : F\,(n) \wedge n \neq z]))] \end{array} . \qquad \text{(NatNum[}\mathsf{FA}\text{](}i\text{-}ii\text{))}$$

Then, by relying on comprehension, again, one defines the strong ancestral $\mathcal{S}^*$ of $\mathcal{S}$, and defines natural numbers as those cardinals which are either 0 or bear the relation $\mathcal{S}^*$ with it:

$$\forall x \, [\mathcal{N}\,(x) \Leftrightarrow (0 = x \vee \mathcal{S}^*(0, x))] \, , \qquad \text{(NatNum[}\mathsf{FA}\text{](}iii\text{))}$$

where '$\mathcal{N}$' designates the property of being a natural number, of course. It follows that in FA, natural numbers are singled out among cardinal numbers, without exhausting the latter, since it is clear from NatNum[FA](i-iii) that there is at least one cardinal number, namely $\#\mathcal{N}$, which is not a natural number.

In many respects, this lack of specificity is not a virtue. But it is also the symptom of FA's epistemic weakness, since it makes clear that what FA's definition of natural numbers does is not fixing these very numbers and the sets of them *ex novo* as *sui generis* items, but rather fixing, firstly, cardinal numbers within the putative range of the individual variables of $\mathfrak{L}^{\mathsf{L}_2}$, that is, among objects in general (conceived as the inhabitants of this range), and, next, singling out natural numbers among cardinal ones. Understanding this definition requires a considerable amount of intellectual resources. But it seems to me that these are fewer, or at least more basic and more progressively appealed to, than those required to understand Z_2 as an implicit definition of natural numbers. Let us explains why.

First at all, it is clear that Z_2 includes, like FA, a system of second-order logic with full comprehension, with the result that understanding Z_2 involves understanding this system, just as it happens for FA. Still, the system L_2 included in FA is both distinct and differently conceived than the one included in Z_2. Understanding the former calls for: the notions of an object, of a first-level concept (or property of an object)[19], and of a first-level dyadic relation; the notions of falling of an object under a (first-level) concept, and of a pair of objects bearing a certain (first-level) relation to each other; the notion of identity for objects; the notions of a variable ranging over objects, concepts, and dyadic relations, respectively; the notion of full comprehension both for first-level concepts and first-level dyadic relations; and all the notions generically involved in

[19] Cf. footnote (5), above.

understanding a system of predicative logic. The difference with the system of second-order logic included in Z_2 depends on the fact that, in this last one, concepts are replaced by sets[20] and dyadic relations are avoided. Still, the notion of a dyadic relation is required to understand the basic axioms of Z_2, and, once the notion of a second-order variable is at hand, moving from this to the notion of a variable ranging over dyadic relations does not require too much. Hence, if understanding L_2 requires more than understanding the system of second-order logic included in Z_2, the difference does not seem to be significant.

Secondly, all HP[FA] does is appeal to a formula of $\mathfrak{L}^{\mathsf{L}_2}$ to fix some items—namely, cardinal numbers—in the putative range of its variables, by putting forward an identity condition for them. This definition is close to a structural one, in some sense. Since the items forming the putative range of individual variables of L_2 lack a sufficient characterisation to allow one to merely single out some of them among all of them, by merely specifying such a characterisation. Hence, one could say that HP[FA] introduces cardinal numbers *ex novo* as *sui generis* items, like the axioms of Z_2 do for natural numbers and sets of them. Despite this, and whatever HP[FA]'s existential import for objects might be, besides the notions mentioned above in connexion to the understanding of L_2, understanding HP[FA] merely calls for the notion of a many-one association between concepts and objects (better, between many concepts and a single object), which is needed to understands HP[FA]'s left-right side, plus the notion of the objects falling under a certain (first-level) concept being in bijection with those falling under another (first-level) concept, which is needed to understand the formula of $\mathfrak{L}^{\mathsf{L}_2}$ providing HP[FA]'s right-hand side. It follows that, apart for what pertains to the notion of a many-one association between concepts and objects, HP[HA]'s epistemic cost is structurally analogous to (*i. e.* is constituted in the same way as) the epistemic cost of any explicit definition of a sort of objects within any second-order theory[21]. Indeed, to provide such an explicit definition, one writes a formula of the appropriate language that involves a single unbounded first-order variable, then appeals to comprehension to guarantee that there exists a property that an item belonging to the putative range of this variable has, or a set to which such an item belongs, if and only if it satisfies this formula, and finally introduces an appropriate monadic predicate constant to designate this property or set. Analogously, in order to introduce a predicate constant designating the property of being a cardinal number on the base of HP[FA], one has to rely on comprehension (extended to '#'), to conclude that there exist a property that an object has if and only if it is uniquely associated to a certain concept F in such a way that it is the same object as any object equally associated to any concept G on condition that F and G satisfy the formula of $\mathfrak{L}^{\mathsf{L}_2}$ providing the right-hand side of HP[FA]), and then introduce such a predicate constant to designate this property.

Thirdly, though HP[FA]'s epistemic cost is minimal, understanding it allows one to have an

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[20]Cf. footnote (18), above.

[21]Remark also that the notion of a many-one association between appropriate sorts of items is involved in any definition of whatsoever functional constant.

epistemic access to cardinal numbers, and then begin to work consciously with them. Definitions NatNum[FA](i-iii) result from this work. The first and the second of them depend on HP[FA] and comprehension, but are mutually independent, and also independent of any other definition, whereas the third is, as such, independent of HP[FA], but depends on comprehension and the two previous ones. It is just this definition that singles out natural numbers among cardinal ones. Hence, if, once having been defined through HP[FA], cardinal numbers are taken to be particular items, natural ones are defined within FA by coding them with appropriate such items. It follows that the definition of natural numbers coming with FA is neither structural nor close to a structural definition in the sense in which that of cardinal numbers *via* HP[FA] is so: it does not result from defining a structure, but rather from providing a system exemplifying it and formed by items singled out among items that one has already an epistemic access to. Moreover, it succeeds in this without appealing to any relation, or operation on natural numbers themselves, except for the relations $\mathcal{S}^*$ and $\mathcal{S}$, already defined on cardinal numbers. It follows that, besides the notions mentioned above in connexion to the understanding of L_2 and HP[FA], understanding the definitions NatNum[FA](i-iii) merely calls for the notions of a first-level concept under which no object falls, for that of a (first-level) concept under which falls all the objects falling under another (first-level) concept plus a single one, and for that of the strong ancestral of a first-level dyadic relation.

Once one has obtained an epistemic access to natural numbers as defined though HP[FA] and NatNum[FA](i-iii), by appealing to the notions just mentioned, one can conscientiously work on them, in turn, namely define appropriate relations and operations on them, and prove that, under these relations and operations, they exemplify the same structure as that defined by the axioms of Z_2 (which is what is usually called 'Frege's theorem'). Hence, it seems to me that so defining natural numbers is not only epistemically more economic than doing it within Z_2, but also epistemically more economic than doing it within any version of Peano second-order arithmetic, even if this does not include axioms for addition, multiplication, and order. Indeed, any definition coming with any version of Peano second-order arithmetic is structural, and allows one to have access to natural numbers only insofar as at least one appropriate relation is defined on them. Moreover, in any such definition, zero is not singled out among items one has an independent epistemic access to, but is simply posited by fixing some conditions it has to meet. This should be enough to conclude that the definition of natural numbers coming with FA is epistemically economic, in my sense.

5 Epistemic Economy and the Definitions of Real Numbers

The epistemic advantage of appropriate abstraction principles over other forms of definitions become even clearer in the case of the definition of real numbers.

As is well known, there are several ways to define real numbers within second-order arithmetic. I shall firstly consider a definition within Z_2, which follows Cantor's classical approach, and I shall then compare it with three definitions within FA (or, strictly speaking, within appropriate extensions of it). Even though the former merely depends on supplying the axioms of Z_2 with some appropriate explicit definitions, whereas the latter all depend on adding new appropriate abstraction principles to FA's axioms, I shall argue that the epistemic cost of the former is comparable to that of two of the latter, while the third one is epistemically (more) economic.

5.1 Defining Real Numbers within Z_2

The first definition is proposed by Simpson ([34], § I.4) as a definition with ACA_0, which can also be repeated, with a minor change in its last step, within RCA_0. Its purpose is showing how to define real numbers within second-order arithmetic while considerably weakening the existential set-theoretic assumptions that Z_2 depends on. This is a crucial fact, underlying the whole program of reverse mathematics. But, for my present purpose, it can be ignored, and the definition can be merely taken as a definition within Z_2. There is certainly room for arguing that the mere possibility of achieving this definition within ACA_0, and one close to it within RCA_0, shows that this definition actually depends on much weaker existential assumption than those that, not only Z_2, but also FA, and consequently any definition of real numbers involving this latter theory, depend on. Still, this does not entail, in my view, that any definition of real numbers within FA has a greater epistemic cost than Simpson's within ACA_0 and RCA_0. This contrast between the advantage of Simpson's definitions in terms of existential parsimony and its lack of benefit in terms of epistemic economy is just part of the point I want to make. To this end, there is no essential difference between immersing Simpson's definition within ACA_0 or RCA_0 and immersing it within the whole Z_2[22].

This definition proceeds in four steps. In the first one, the set $\mathbb{N}$ of natural numbers is defined as the unique set X such that $\forall n\, (n \in X)$, which is licensed by comprehension and by the

[22]Once this definition is immersed within the whole Z_2, the items it defines—namely (the items re-casting) real numbers within Z_2—provably have many properties that the items it defines when it is immersed within ACA_0 or RCA_0—namely (the items re-casting) real numbers within ACA_0or RCA_0—do not provably have. The crux of reverse mathematics (to which [34] is entirely devoted) is just which properties of the former items are preserved once real numbers are defined within weaker sub-systems of Z_2, like ACA_0 or RCA_0. But, of course, this is not a matter I can consider here.

specificity of Z_2. In the second step, integer numbers are coded with elements of an appropriate subset $\mathbb{N}_{\mathbb{Z}}$ of $\mathbb{N}$ and order, addition, and multiplication are defined on them. Details left aside and supposing that ζ is whatever natural number, this results in coding the integers $+\zeta$ and $-\zeta$ with $\zeta^2 + \zeta$ and ζ^2, respectively. In the third step, rational numbers are coded with the elements of another appropriate subset $\mathbb{N}_{\mathbb{Q}}$ of $\mathbb{N}$ and order, addition, and multiplication are defined on them, too. Details left aside, anew, and supposing that ζ and ϑ are whatever pair of coprime natural numbers, the second of which is positive, the rational numbers $+\frac{\zeta}{\vartheta}$ and $-\frac{\zeta}{\vartheta}$ are respectively coded with $\left(\zeta^2 + \zeta + \vartheta^2 + \vartheta\right)^2 + \zeta^2 + \zeta$ and $\left(\zeta^2 + \vartheta^2 + \vartheta\right)^2 + \zeta^2$. In the fourth step, sequences of rational numbers are coded with appropriate subsets of $\mathbb{N}$, namely subsets meeting an appropriate condition, and real numbers by some of these subsets, namely those suitable for coding Cauchy sequences of rational numbers, in agreement with Cantor's 1872 definition ([7], pp. 123-124)[23].

This short description is enough to show that, according to this definition, integer, rational and real numbers are not introduced *ex novo* as *sui generis* objects, but rather singled out among items previously defined—namely natural numbers, for integer and rational ones, and sets of them, for real ones—through explicit definitions that are independent of any operation or relation on these very numbers. Firstly, integer numbers are singled out among natural ones, by appealing to addition, multiplication and order on the latter. Addition, multiplication and order are then defined on integer numbers, and appealed to in order to single out rational numbers among them[24]. Finally, addition, multiplication, order, and absolute value are defined on rational numbers, and appealed to in order to single out real ones among sets of them. A definition of addition, multiplication and order on real numbers only comes into play at this point, together with the proof that these numbers behave with respect to them so as to comply with the required structural conditions.

It is then clear that the intellectual resources needed to understand the whole definition are not involved simultaneously from the start. Understanding its different steps rather involves understanding a limited number of formulas at once, which are used to single out the relevant items among others that one already has an epistemic access to. Still, understanding the definition of

[23]It is easy to see the essential difference between this fourth step and the three previous ones: whereas in these three steps, the sets $\mathbb{N}$, $\mathbb{N}_{\mathbb{Z}}$, and $\mathbb{N}_{\mathbb{Q}}$ are explicitly defined, the subsets of $\mathbb{N}$ coding real numbers cannot be explicitly defined, in turn, and it is, *a fortiori*, impossible to define anything working as the set of real numbers. All that one can do is fix a condition that a subset of $\mathbb{N}$ has to met in order to code a single real number.

[24]In fact, integer numbers could be singled out among natural ones by appealing only to addition and multiplication on the latter, by merely stipulating that the former numbers are coded by those of the latter ones which are equal to $\zeta^2 + \zeta$ or ζ^2, for some natural number ζ. In this way, no justification could be offered for this choice, however. Analogously, rational numbers could be directly singled out among natural numbers, by only appealing, again, to addition and multiplication on the latter, by merely stipulating that the former numbers are coded by those of the latter ones which are equal to $\left(\zeta^2 + \zeta + \vartheta^2 + \vartheta\right)^2 + \zeta^2 + \zeta$ or $\left(\zeta^2 + \vartheta^2 + \vartheta\right)^2 + \zeta^2$, for some pair of coprime natural numbers ζ and ϑ, the second of which is positive. In this case also, no justification could be offered for this choice, however.

real numbers involves understanding the previous definitions of natural, integer and rational ones and of the relevant operations and relations on them, and calls then for all the resources that are needed to understand these previous definitions. Furthermore, it also involves the understanding of the notion of a Cauchy sequence of rational numbers, and the use of sets of natural ones (implicitly defined)[25] for coding other numbers, (that is, of second-order items to code first-order ones), and calls then for the corresponding resources. Even though this is a large amount of resources, it does not include those which are needed to understand the definitions of any operation and relation on real numbers themselves. These definitions come later and are not needed to have an epistemic access to these numbers. This is enough to show that the definitions of real number, as well as those of integer and rational ones, are not structural. If, once having been implicitly defined through the axioms of Z_2 as places in a structure, natural numbers and sets of them are taken to be particular items, the definitions of integer, rational and real numbers all consist in coding these numbers with some of these particular items, forming instances of other structures.

5.2 Defining Real Numbers within FA

I now return to FA. Nothing would prevent one from rephrasing Simpson's definition of integer and rational numbers within this theory, so as to code them with natural numbers appropriately singled out. Rendering Simpson's definition of real numbers within FA would be more problematic, since FA provides no definition of sets of natural numbers, and real ones could therefore not be singled out among these sets. One could, at most, state a certain condition that a property of natural numbers should meet in order either to code real numbers directly, or to belong to a range of (second-order) variables entering an abstraction principle implicitly defining these numbers as the values of a functional operator taking the elements of this range as its arguments. Insofar as, in FA's abstractionist setting, the former option would be at odds with the idea that real numbers are objects, just like natural, integer and rational ones, the latter would certainly be preferable. But going for it would be, in turn, more convoluted (and certainly epistemically less economic) than taking advantage from FA's lack of specificity and appealing to an appropriate abstraction principle to directly fix real numbers within the putative range of FA's individual variables—namely among objects, and especially among other ones than cardinal numbers—, thus replicating, *mutatis mutandis*, the same move already made to define cardinal numbers through HP[FA]. This is the option chosen by the three definitions of real numbers within FA, which I shall now consider.

To this purpose, I shall take, for short, '$\forall_\Phi$' and '$\exists_\Phi$'—where 'Φ' stands for a monadic predicate constant—to designate, respectively, the universal and the existential quantifiers restricted either

[25]Cf. footnote (23), above.

to items that have Φ or to their properties[26].

5.2.1 Shapiro's Rephrasing of Dedekind's Definition

The first definition is Shapiro's rephrasing of Dedekind's ([8]; [31], pp. 338-340) within FA. Like Simpson's, Shapiro's definition includes previous definitions of integer and rational numbers, but, unlike in Simpson's, these numbers are not singled out among natural ones. Like real ones, they are rather fixed within the putative range of the individual variables of $\mathfrak{L}^{\mathsf{FA}}$ through two abstraction principles, one for integer, the other for rational numbers. I suggest to call 'FDRA' (for 'Frege-Dedekind Real Arithmetic'), the extension of FA that is obtained by adding to its axioms these two principles together with what is required to define real numbers, and by extending comprehension to formulas including the functional constants so introduced.

The abstraction principle used to define integer numbers is the following:

$$\forall_{\mathcal{N}} x, x', y, y' \left[\mathrm{INT}\,(x,y) = \mathrm{INT}\,(x',y') \Leftrightarrow x + y' = x' + y \right], \qquad (\mathrm{INT}[\mathsf{FDRA}])$$

where 'INT' is the functional constant introduced by this principle, and '+' designates the addition on natural numbers. To be more precise, this principle merely provides an implicit definition of appropriate pairs of natural numbers; integer ones are then defined by coding them with these pairs: if p and q are whatever pair of natural numbers, $\mathrm{INT}\,(p,q)$ is taken, within FDRA, as an integer number. This results from explicitly defining a predicate constant—let us say '$\mathcal{Z}$'—designating the property of being such a number. Appealing to this constant, after having explicitly defined the multiplication on integer numbers, licenses a new abstraction principle:

$$\forall_{\mathcal{Z}} x, x', y, y' \left[\begin{array}{l} \mathrm{QUOT}\,(x,y) = \mathrm{QUOT}\,(x',y') \\ \qquad \Leftrightarrow \left[\begin{array}{c} [y = 0_{\mathcal{Z}} \wedge y' = 0_{\mathcal{Z}}] \vee \\ {[y \neq 0_{\mathcal{Z}} \wedge y' \neq 0_{\mathcal{Z}} \wedge x \cdot_{\mathcal{Z}} y' = x' \cdot_{\mathcal{Z}} y]} \end{array} \right] \end{array} \right], \qquad (\mathrm{QUOT}[\mathsf{FDRA}])$$

where 'QUOT' is the functional constant introduced by this principle, '$0_{\mathcal{Z}}$' abbreviates '$\mathrm{INT}\,(0,0)$'—the integer zero—, and '$\cdot_{\mathcal{Z}}$' designates the multiplication on integer numbers. This principle implicitly defines appropriate pairs of integer numbers. Rational numbers are then defined as the pairs whose second element is not $0_{\mathcal{Z}}$: if u and v are a pair of integer numbers and $v \neq 0_{\mathcal{Z}}$, $\mathrm{QUOT}\,(u,v)$ is taken, within FDRA, as an integer number. This also results from an explicit definition introducing a predicate constant—let us say '$\mathcal{Q}$'—designating the property of being an integer number. Appealing to this constant, after having explicitly defined the order relation on rational numbers, allows on to explicitly define the (second-order) relation that a property of rational numbers has with such a number if the latter is an upper bounded of the former:

[26] In other terms, I shall take '$\forall_\Phi x\,[\phi]$', '$\forall_\Phi X\,[\phi]$', $\exists_\Phi x\,[\phi]$', and '$\exists_\Phi X\,[\phi]$' to abbreviate '$\forall x\,[\Phi\,(x) \Rightarrow \phi]$', '$\forall X\,[\forall x\,[X\,(x) \Rightarrow \Phi\,(x)] \Rightarrow \phi]$', '$\exists x\,[\Phi\,(x) \wedge \phi]$', and '$\exists X\,[\forall x\,[X\,(x) \Rightarrow \Phi\,(x)] \wedge \phi]$' respectively.

$$\forall_{\mathcal{Q}} F \forall_{\mathcal{Q}} x \left[F \trianglelefteq x \Leftrightarrow \forall_{\mathcal{Q}} y \left[F(y) \Rightarrow y \leq_{\mathcal{Q}} x \right] \right], \qquad \text{(UpBound[}\mathsf{FDRA}\text{])}$$

where '$\leq_{\mathcal{Q}}$' designates the order relation on rational numbers. Finally comes a third abstraction principle:

$$\forall_{\mathcal{Q}} F, G \left[\text{CUT}(F) = \text{CUT}(G) \Leftrightarrow \forall_{\mathcal{Q}} x \left(F \trianglelefteq x \Leftrightarrow G \trianglelefteq x \right) \right], \qquad \text{(CUT[}\mathsf{FDRA}\text{])}$$

where 'CUT' is again a functional constant introduced by this principle. This new principle implicitly defines cuts on properties of rational numbers. Real ones are finally defined by coding them with appropriate such cuts: if P is a property of rational numbers, $\text{CUT}(P)$ is to be taken as (coding) a real number if and only if P is instantiated and has an upper bound, that is, it is such that $\exists_{\mathcal{Q}} x, y \left[P(x) \wedge P \trianglelefteq y \right]$).

This definition is quite natural from the neo-logicist perspective, but it structurally differs from Simpson's one within Z_2 only by appealing to cuts of rational numbers, rather than to (sets of natural numbers coding) Cauchy sequences of these same numbers, and by replacing the corresponding explicit definitions with abstraction principles working as new axioms. One could think that this latter circumstance makes Shapiro's definition epistemically more costly than Simpson's. But I do not think it is so.

Firstly, FA provides a basis for the former definition which is epistemically weaker than Z_2, or any sub-system of it, even though it remains that the mere definition of natural numbers within FA is not enough to license the subsequent definition of integer, rational, and real ones within this same theory, and has to be supplied by the definition of addition on natural numbers and of multiplication and order on rational ones.

Secondly, like HP[FA], these principles appeal to a formula of $\mathfrak{L}^{\mathsf{FA}}$—appropriately extended, in the case of QUOT[FDRA], and CUT[FDRA]—, to fix some items in the putative range of FA's individual variables by putting forward an identity condition for these items. Hence, understanding these principles does not call for more than understanding the explicit definition of a sort of objects depending on the introduction of a monadic predicate constant apt to designate the property that these items are required to have, or the set they form. I have already made a similar point with respect to HP[FA] in § 4. For INT[FDRA], QUOT[FDRA], and CUT[FDRA], the point is even clearer. To illustrate it with an example, compare Simpson's explicit definition of rational numbers within Z_2 with INT[FDRA]. To get the former one focuses on the open formula '$\exists m \left[h = m^2 \vee h = m^2 + m \right]$', relies on comprehension to conclude that $\exists X \forall h \left[h \in X \Leftrightarrow \exists m \left[h = m^2 \vee h = m^2 + m \right] \right]$, and takes rational numbers to be the elements of the set of natural numbers which have the property that satisfies this last condition. To define rational numbers through INT[FDRA] within FA, one states this principle, then relies on comprehension appropriately extended to 'INT' to conclude that $\exists F \forall z \left[F(z) \Leftrightarrow \exists_{\mathcal{N}} x, y \left[z = \text{INT}(x, y) \right] \right]$ and takes rational numbers to be objects that have the property that satisfies this last condition.

Finally, like Simpson's, Shapiro's definition defines real numbers by providing an example of the relevant structure, rather than defining this structure as such.

It seems, then, that the two definitions have analogous epistemic costs, or, even, that Shapiro's one is epistemically more economic.

5.2.2 Hale's Rephrasing of Frege's Definition

The second definition I shall consider results from adapting to FA Hale's rephrasing of Frege's definition of domains of magnitudes and of his plan to get a definition of real numbers as ratios on such a domain ([19], pp. 105-113; [15], part III, §§ II.55-245; [33]; [13], ch. 22; [30]). According to Frege, real numbers originate in measuring magnitudes, and they have to be defined as ratios of them, rather than as arithmetical items (like it happens in Cantor's and Dedekind's definitions, instead). Still, though independent of natural numbers, his definition of domains of magnitudes takes place in the same (inconsistent) system of logic in which he also defines these numbers, and could be rephrased within L_2 (appropriately strengthened). The successive definition of real numbers as ratios on such a domain can, then, be easily achieved by appealing to natural numbers as a useful auxiliary tool, but can also be freed from any recourse to these numbers (although then becoming a little bit more convoluted). In the former case, the definition would depend on an appropriate extension of FA; in the latter, it would depend on an extension of L_2, both different from, and independent of FA. In both cases, one should, however, pair the definition with an existence proof for domains of magnitudes, which Frege merely outlines informally. His idea is to obtain such a domain by starting from natural numbers, whose existence is taken for granted. Formally rendering his indications results, then, quite naturally, in a construction within FA. Faced with this situation, Hale provides an informal, algebraically shaped, definition of domains of magnitudes appealing to natural numbers (without specifying the way they are defined), then suggests an existence proof of these domains, deviating from Frege's indications, but still based on natural numbers and on the admission of their existence, and finally defines ratios of magnitudes by appealing to natural numbers. The simplest way to adapt his definition to FA, though openly departing from Frege's conception, is by directly defining positive real numbers as ratios on the specific domains of magnitudes drawn from natural numbers during the existence proof[27]. This is the plan I shall follow.

Hale first defines "normal quantitative domains" as pairs $\langle \mathbf{Q}, + \rangle$ where: $\mathbf{Q}$ is non-empty and closed under $+$; $+$ is associative, commutative and such that if $\mathbf{p}$ and $\mathbf{q}$ are distinct elements of $\mathbf{Q}$, there is another element $\mathbf{r}$ of $\mathbf{Q}$ for which either $\mathbf{p} = \mathbf{q} + \mathbf{r}$ or $\mathbf{q} = \mathbf{p} + \mathbf{r}$; for any $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{Q}$, there

[27] Hale's domains of magnitudes, unlike Frege's, only include, as we shall soon see, positive elements, with the result that only positive real numbers can be defined as ratios on them. Non-positive ones are, then, to be defined by extension.

is a $\mathbf{r}$ in $\mathbf{Q}$ and a positive natural number n such that $\mathbf{q}+\mathbf{r}=n\mathbf{p}=\underbrace{\mathbf{p}+\mathbf{p}+\ldots+\mathbf{p}}_{n \text{ times}}$. If a strict order $<$ is defined on $\mathbf{Q}$ by stipulating that $\mathbf{p}<\mathbf{q}$ if and only if there in $\mathbf{r}$ in $\mathbf{Q}$ such that $\mathbf{q}=\mathbf{p}+\mathbf{r}$, then $<$ is a total order on $\mathbf{Q}$, and it becomes easy to show that $\langle \mathbf{Q},+\rangle$ meets the Archimedean condition (and includes only positive elements). Ratios on normal quantitative domains are then defined by an abstraction principle rephrasing definition V.5 of Euclid's *Elements*, stating that, if $\langle \mathbf{Q},+\rangle$ and $\langle \mathbf{Q}^*,+\rangle$ are normal quantitative domains (non necessarily distinct), and $\mathbf{p}$, $\mathbf{q}$ are in $\mathbf{Q}$ and $\mathbf{p}^*$, $\mathbf{q}^*$ in $\mathbf{Q}^*$, then:

$$\mathrm{RAT}\left(\mathbf{p},\mathbf{q}\right)=\mathrm{RAT}\left(\mathbf{p}^*,\mathbf{q}^*\right)\Leftrightarrow\forall_{NN^+}h,k\left[h\mathbf{p}\lesseqgtr k\mathbf{q}\Leftrightarrow h\mathbf{p}^*\lesseqgtr k\mathbf{q}^*\right], \tag{EUCL}$$

where 'NN^+' designates the property of being a positive natural number (however defined). Next, domains of magnitudes (which Hale rather calls 'complete normal quantitative domains') are defined as normal quantitative domains that meet the fourth-proportional condition (for any $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in $\mathbf{Q}$, there is a $\mathbf{s}$ in $\mathbf{Q}$ itself, such that $\mathrm{RAT}\left(\mathbf{p},\mathbf{q}\right)=\mathrm{RAT}\left(\mathbf{r},\mathbf{s}\right)$) and are Dedekind-complete.

To prove that domains of magnitudes exist (*i. e.* that at least one such domain exists), Hale takes, as I have said above, the existence of natural numbers for granted, and relies on them to obtain such a domain by following a path analogous to that involved in Shapiro's foregoing definition. He begins by observing that positive natural numbers form, together with the addition on them, a normal quantitative domain, let us say $\langle \mathbf{N}^+,+\rangle$. One can then define ratios on it through EUCL, by taking $\mathbf{Q}$ and $\mathbf{Q}^*$ to coincide with each other and with $\mathbf{N}^+$, and easily verify that these ratios, together with an appropriate addition on them, form, in turn, a normal quantitative domain meeting the fourth-proportional condition, let us say $\left\langle \mathbf{R}^{\mathbf{N}^+},+\right\rangle$. This allows to define cuts on $\mathbf{R}^{\mathbb{N}^+}$ through a new abstraction principle, which is nothing but a restriction of Frege's Basic Law V:

$$\mathrm{CUT}\left(P\right)=\mathrm{CUT}\left(Q\right)\Leftrightarrow\forall x_{R^{\mathbf{N}^+}}\left[P\left(x\right)\Leftrightarrow Q\left(x\right)\right] \tag{CUT[$\mathbf{R}^{\mathbb{N}^+}$]}$$

where '$R^{\mathbf{N}^+}$' designates the property of being an element of $\mathbf{R}^{\mathbf{N}^+}$, and P and Q are whatever properties of the elements of $\mathbf{R}^{\mathbf{N}^+}$ that are non-empty, non-total, downward closed, and upward unbounded. These cuts form the required domain of magnitudes, let us say $\left\langle \mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}},+\right\rangle$.

The structure of domains of magnitude is categorical. Hence, any ratio on $\mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}}$ is identical with a ratio on any other domain of magnitudes (if any). This allows to define positive real numbers by coding them with ratios on $\mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}}$, since these ratios are the same as those on any other such domain (if any).

Like Shapiro's, Hale's definition of real numbers includes two stages: first some items are defined by appealing to an appropriate abstraction principle, then these items are appealed to in order to define real numbers. There are, however, important differences between the two definitions. The most evident is that the first stage of Hale's definition is not only more laborious than that of Shapiro's, but it is also informal and algebraic in spirit: it requires an existence proof and does not specify how the elements of a normal quantitative domain (and, then, also of a domain of magnitudes) are fixed, with the result that the only connection between this proof and the definition depends on verifying afterwards that the systems constructed meet the relevant condition. This does not make Hale's definition of real numbers structural. This definition does not define positive real numbers as places in a structure defined as such. But it is no more intended to code these numbers with the items defined in the first stage of the definition, namely with ratios on domains of magnitudes. In agreement with Frege's purpose, it is rather intended to identify these numbers as such, to disclose their ultimate nature, namely that of being these very ratios (which might be described in different ways, according to the particular domain of magnitudes that are chosen, provided that different such domain exist: taking these ratios, and then positive real numbers, to be ratios on $\mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}}$ is, indeed, nothing but a convenient way to name or describe them). This identification is, however, quite questionable, certainly more questionable than the neo-logicist idea (also inspired by Frege) that natural numbers are just the values of the function # singled out by NatNum[FA](iii). It is much more unquestionable to admit that ratios on domains of magnitudes merely code positive real numbers.

Together with the possibility of defining (or describing) these ratios as ratios on $\mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}}$, this suggests freeing Hale's definition from the definition of domains of magnitudes, and immersing it within FA. Indeed, all that matters for the definition to be appropriate, at least from a purely mathematical point of view, is that the ratios on $\mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}}$ provide suitable codes for real numbers, independently of their being regarded as ratios of magnitudes. It remains, however, that $\mathbf{C}^{\mathbf{R}^{\mathbf{N}^+}}$ cannot be defined, as such, within FA, since this cannot but be a set, and no set can be defined within FA. One has rather to replace it, as well as $\mathbf{N}^+$ and $\mathbf{R}^{\mathbf{N}^+}$, with appropriate properties. One firstly defines the property $\mathcal{N}^+$ of being a positive natural number, and rephrases EUCL, by replacing '$\mathbf{p}$', '$\mathbf{q}$', '$\mathbf{p}^*$', '$\mathbf{q}^*$' with first-order variables bounded by '$\forall_{\mathcal{N}^+}$', '$\forall_{NN^+}$' with '$\forall_{\mathcal{N}^+}$' again, and 'RAT' with '$\mathrm{RAT}_{\mathcal{N}^+}$'. This allows to explicitly define the property $\mathcal{R}^{\mathcal{N}^+}$ of being a value of the function $\mathrm{RAT}_{\mathcal{N}^+}$, and rephrase CUT[$\mathbf{R}^{\mathbf{N}^+}$], by replacing '$\forall_{R^{\mathbf{N}^+}}$' with '$\forall_{\mathcal{R}^{\mathcal{N}^+}}$', '$P$' and '$Q$' with second-order variables bounded by '$\forall_{\mathcal{R}^{\mathcal{N}^+}}$' again, and 'CUT' with '$\mathrm{CUT}_{\mathcal{R}^{\mathcal{N}^+}}$'. One explicitly defines, then, the property $\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}$ of being a value of the function $\mathrm{CUT}_{\mathcal{R}^{\mathcal{N}^+}}$ for the argument given by a non-empty, non-total, downward closed, and upward unbounded property of the items having the property $\mathcal{R}^{\mathcal{N}^+}$, and rephrases EUCL, again, by replacing '$\mathbf{p}$', '$\mathbf{q}$', '$\mathbf{p}^*$', '$\mathbf{q}^*$' with first-order variables bounded by '$\forall_{\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}}$', and 'RAT' with '$\mathrm{RAT}_{\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}}$' (by assuming that addition and strict order have been appropriately defined on the items having the property $\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}$). Finally, one codes positive real numbers with the values of the function $\mathrm{RAT}_{\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}}$. This results in supplementing FA with three abstraction principles, each of which is followed by a corresponding explicit definition.

In fact, the second rephrasing of EUCL could be avoided by directly coding positive real numbers with the items having the property $\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}$. In both cases, a further step, involving either a new abstraction principle, or a suitable explicit definition, is required, in order to move from positive real numbers to real numbers *tout court*.

Whatever opinion one might have of Frege's requirement that real numbers are to be defined as ratios of magnitudes, and, consequently, of the philosophical appropriateness of Hale's original definition of real numbers, it is doubtless that its epistemic cost is high, since its understanding involves the understanding of the definition of the structure of domains of magnitudes, and of the existence proof for domains of magnitudes coming with the definition of $\mathbf{C}^{\mathbf{R}^{\mathbb{N}^+}}$. But, if Hale's definition of real numbers is adapted to FA in the way suggested above, it becomes very close to Shapiro's, except for possibly involving an eliminable definition of ratios on cuts on ratios on positive natural numbers (*i. e.* on items having the property $\mathcal{C}^{\mathcal{R}^{\mathcal{N}^+}}$) and for obtaining non-positive real numbers in the end, rather than defining integer numbers in the beginning. On the one hand, the first rephrasing of EUCL replaces QUOT[FDRA], limitatively to positive natural numbers, and comes quite close to it, in fact, since, if 'x', 'x'', 'y', 'y'' range over these numbers, the two conditions that $\forall_{\mathcal{N}^+} h, k \left[hx \lesseqqgtr kx' \Leftrightarrow hy \lesseqqgtr ky' \right]$ and that $x \cdot_{\mathcal{Z}} y' = x' \cdot_{\mathcal{Z}} y$ are provably equivalent within FA. On the other hand, the rephrasing of CUT[$\mathbf{R}^{\mathbf{N}^+}$] replaces, limitatively to positive rational numbers, both CUT[FDRA] and the restriction to cuts of instantiated and upper bounded properties of these numbers. Hence, except for the subtle differences that one might discern among the respective epistemic costs of these two pairs of axioms and among that of the stipulation required to obtain non-positive real numbers and that of QUOT[FDRA], the epistemic cost of the two definitions is the same, and then, either analogous to that of Simpson's definition or smaller than it.

5.2.3 Real Numbers as Bicimal Pairs

Even though it openly departs from Frege's indication as well, the last definition I consider is suggested by one of his ideas, namely by his outline of the existence proof of domains of magnitudes (to which Hale does not conform, as I have said above).

Frege's heuristic suggestion goes as follows. Look at Cauchy's series of the form $\sum_{i=0}^{\infty} \lambda_i \frac{1}{2^i}$, where λ_0 is a natural number, and λ_i ($i = 1, 2, \ldots$) are either 0 or 1, but are not constantly 0 after a certain value of i. These series are in bijection both with positive real numbers (since any such series converges to such a number, any such number is the limit of a such a series, and distinct series converge to distinct numbers and vice versa), and with all the pairs $\langle \lambda_0, \mathfrak{S} \rangle$, where $\mathfrak{S}$ is the (infinite) set of positive natural numbers i such that $\lambda_i = 1$ (since, given any such series, one can get such a pair, and vice versa). There is thus a bijection between positive real numbers and these pairs. Now, looking at these pairs as such, one can define an internal addition on them, let us say $\uplus$, without any consideration of real numbers, then use it as a basis to define a family of permutations among these pairs: to any such pair α, one associates the permutation $\uplus_\alpha$ such that, for any two other pairs β and γ, $\uplus_\alpha(\beta, \gamma)$ if and only if $\beta = \alpha \uplus \gamma$. There is then also a bijection between these pairs and these permutations, and then between the latter and positive real numbers. The idea is, then, to show that the value-ranges of these permutations, together with those of their converses and of the identity permutation form a domain of magnitudes under the operation of composition (defined on permutations, then transferred to their values-ranges).

As I have said, Frege's purpose is to define real numbers as ratios of magnitudes. Hence, the heuristic interest of remarking that the relevant permutations are in bijection with positive real numbers is not that of showing that the latter can be coded with the former, or better with their value-ranges (and non-positive real numbers with the converses of these permutations together with the identity permutations, or with their value-ranges), but rather that of showing that there are enough relevant permutations for their value-ranges to form a domain of magnitudes. Still, Frege's outline naturally suggests to define real numbers, or, at least, positive ones, by coding them with pairs like $\langle \lambda_0, \mathfrak{S} \rangle$, or with some appropriate *alias* of them. Hilbert and Bernays do something close to this in their *Grundlagen der Mathematik*([24], vol. II, supplement IV, § C). My suggestion is to render this idea within FA, by generalising it *d'emblée* to non-positive real numbers. In particular, I suggest to extend FA, so as to define pairs like $\langle p, P \rangle$—where p is a natural number and P an infinite (that is, instantiated and upward unbounded) property of natural numbers—, and to code real numbers with these pairs.

To this purpose, it is enough to supplement FA with a single abstraction principle and a single explicit definition, and to extend comprehension to formulas including the constant introduced by this principle. I suggest to call 'FRA', for 'Frege Real Arithmetic', the extension of FA that is obtained in this way.

The abstraction principle is the following:

$$\forall_{\mathcal{N}} x, y \forall_{\mathcal{N}} X, Y \left[\langle x, X \rangle = \langle y, Y \rangle \Leftrightarrow \left[x = y \wedge \forall_{\mathcal{N}} z \left[X(z) \Leftrightarrow Y(z) \right] \right] \right], \qquad (\text{PAIR}[\mathsf{FRA}])$$

where '$\langle -, - \rangle$' is a dyadic functional constant introduced by this principle.

The explicit definition is required to impose that the properties entering the relevant pairs are infinite. It is the following:

$$\forall x \left[\mathcal{B}(x) \Leftrightarrow \exists_{\mathcal{N}} y \exists_{\mathcal{N}} Y \left[x = \langle y, Y \rangle \wedge \forall_{\mathcal{N}} z \exists_{\mathcal{N}} w \left[\mathcal{S}^*(z, w) \wedge Y(w) \right] \right] \right]. \qquad (\text{BicPAIR}[\mathsf{FRA}])$$

This definition introduces the monadic predicate constant '$\mathcal{B}$', designating the property of being a value of the function $\langle -, - \rangle$ when its second argument is an infinite property. I call 'bicimal pairs' the items having this property, and I suggest to code real numbers with them.

PAIR[FRA] and BicPAIR[FRA] are enough to complete the definition. But in order to show that this definition is appropriate, it is also necessary to define addition, multiplication and strict order on bicimal pairs, so that they behave with respect to these operation and this relation as real numbers are required to behave. Of course, this is the case of any definition of these numbers depending on coding them with appropriate items defined beforehand, and then also of the three definitions I have considered above. But in this last case, the definitions of these operations and this relation are less immediate than in the previous ones, though it should be clear that providing these definitions would be nothing but a question of logical routine. I have no space here to detail this routine. All I will say is how I suggest to distinguish positive from non-positive real numbers from the very beginning, that is, by relying neither on these last definitions, nor on the definition of the real zero.

The basic idea is as follows. Let q be any natural number, and Q and $\tilde{Q}$ any two properties of natural numbers such that $Q(0)$, $\neg\tilde{Q}(0)$, and $Q(n) \Leftrightarrow \tilde{Q}(n)$ for any positive natural number n. According to PAIR[FRA] and BicPAIR[FRA], the pairs $\langle q, Q\rangle$ and $\left\langle q, \tilde{Q}\right\rangle$ are distinct to each other, even though they correspond to the same pair $\langle q, \mathfrak{S}\rangle$, where $\mathfrak{S}$ is such that i belongs to it if and only if it is a positive natural number such that $Q(i)$. Hence, bicimal pairs are not in bijection with positive real numbers. But they are in bijection with real numbers *tout court*, since, if $\langle q, Q\rangle$ is associated with the positive real $\rho = q + \sum\limits_{i=1}^{\infty} \lambda_i \frac{1}{2^i}$, where $\lambda_i = 1$ if and only if $Q(i)$, $\left\langle q, \tilde{Q}\right\rangle$ can be associated to the non-positive real $\rho - 2q - 1 = \sum\limits_{i=0}^{\infty} \lambda_i \frac{1}{2^i} - q - 1$. This suggests taking a bicimal pair $\langle p, P\rangle$ to be positive if P is such that $P(0)$, and to be non-positive if P is such that $\neg P(0)$, and coding positive and non-positive real numbers with positive and non-positive bicimal pairs, respectively.

This is rendered by explicitly defining the two properties $\mathcal{B}^{+}$ and $\mathcal{B}^{0/-}$ as follows:

$$\forall x \left[\mathcal{B}^{+}(x) \Leftrightarrow \left[\mathcal{B}(x) \wedge \exists_{\mathcal{N}} y \exists_{\mathcal{N}} Y \left[x = \langle y, Y\rangle \wedge Y(0)\right]\right]\right], \qquad (\text{BicPAIR}^{+}[\mathsf{FRA}])$$

$$\forall x \left[\mathcal{B}^{0/-}(x) \Leftrightarrow \left[\mathcal{B}(x) \wedge \exists_{\mathcal{N}} y \exists_{\mathcal{N}} Y \left[x = \langle y, Y\rangle \wedge \neg Y(0)\right]\right]\right], \qquad (\text{BicPAIR}^{0/-}[\mathsf{FRA}])$$

and by coding positive and non-positive real numbers respectively with the items (belonging to the putative range of the individual variables of $\mathfrak{L}^{\mathsf{L}_2}$) that have these properties.

Once this is done, one can take the bicimal pair $\langle 0, \mathcal{N}^{+}\rangle$ (which is clearly such that $\mathcal{B}^{0/-}(\langle 0, \mathcal{N}^{+}\rangle)$) to code the real zero, and define addition and strict order on bicimal pairs so that a positive real ρ and a negative one $-\rho$ are respectively coded with the bicimal pairs $\left\langle \lfloor p\rfloor_{\rho}, P\right\rangle$ and $\left\langle |\lfloor p\rfloor|_{\rho}, \tilde{P}\right\rangle$, where: $\lfloor p\rfloor_{\rho}$ is the greatest natural number strictly smaller than ρ; P is such that $P(0)$ and that $P(i)$ if and only if $\sum\limits_{i=1}^{\infty} \lambda_i \frac{1}{2^i} = \rho - \lfloor p\rfloor_{\rho}$ is such that $\lambda_i = 1$; $|\lfloor p\rfloor|_{\rho}$ is the greatest natural number smaller or equal to ρ; and $\tilde{P}$ is such that $\neg\tilde{P}(0)$ and that $\tilde{P}(i)$ if and only if $\sum\limits_{i=0}^{\infty} \lambda_i \frac{1}{2^i} = \rho - |\lfloor p\rfloor|_{\rho}$ is such that $\lambda_i = 0$.

Defining real numbers this way allows to achieve the task without relying on any previous definition of integer and rational numbers. Moreover, just as positive and non-positive real numbers are discriminated, and the real zero is identified afterwards—once all real numbers have been defined at once and the same time by supplementing FA with PAIR[FRA] and BicPAIR[FRA]—, integer and rational numbers can be easily defined later, by discerning among real ones. To this purpose, it is enough to take integer numbers to be the real ones which are coded with bicimal pairs whose second element is either $\mathcal{N}$ or $\mathcal{N}^+$, and rational numbers to be the real numbers that are coded with bicimal pairs whose second element is a periodic property of natural numbers, that is, a property P of these numbers such that $\exists_{\mathcal{N}} x, y \forall_{\mathcal{N}} z \left[\mathcal{S}^*(x,z) \Rightarrow \left[P(z) \Leftrightarrow P(z+y)\right]\right]$. Once this is done, it is also quite simple to discern positive from non-positive integer and rational numbers: an integer number is positive if and only if it is coded with a bicimal pair whose second element is $\mathcal{N}$, and it is non-positive if and only if it is coded with a bicimal pair whose second element is $\mathcal{N}^+$; a rational number is positive if and only if it is coded with a bicimal pair whose second element is a periodic property of natural numbers that is enjoyed by 0, and it is non-positive if and only if it is coded with a bicimal pair whose second element is a periodic property of natural numbers that is not enjoyed by 0. Finally, the integer and the rational zero, simply coincide with the real one.

It does not only follow that understanding this definition of real numbers is independent of understanding any definition of integer and rational numbers, but also that understanding the definition of real numbers as such (which merely consist of PAIR[FRA] and BicPAIR[FRA]) provides most of what is required to understand several other subsequent definitions, namely that of the real zero, those of positive and negative real numbers, those of integer and rational numbers, and, among them, of positive and non-positive such numbers, and of the integer and the rational zero. Each of these definitions is not only quite simple, but it is also fully independent of the definition of any operation and relation on integer, rational, and real numbers themselves. Moreover, both the definitions of positive and non-positive real numbers and of the real zero, and those of integer numbers, and, among them, of positive and non-positive ones, and of the integer zero, are independent of the definition of any operation on natural numbers themselves, as well as of any relation on them other than the two relations $\mathcal{S}(x,z)$ and $\mathcal{S}^*(x,z)$, which already enter the definition of these last numbers. A previous definition of addition on natural numbers is only required to define rational numbers, and, together with multiplication, to define the usual operations and relations on real numbers (and, consequently on integer and rational ones), and to prove that these numbers behave with respect to these operations and relations as they are required to do[28].

This makes clear that understanding this definition of real numbers, as well as all those

[28]This proof is quite convoluted, but combinatorial in spirit, and epistemically quite economic, since, elementary arithmetic on natural numbers being taken for granted, it does not involve much more than propositional logic applied to predicate (second-order) formulas.

subsequent ones, requires only a small amount of intellectual resources. Indeed, apart from the notions that are needed to understand the relevant system of second-order logic, and the notion of a many-one association between pairs like $\langle p, P \rangle$ and objects, which is needed to understand the left-right side of PAIR[FRA], understanding the definitions of real numbers, and, among them, of positive and non-positive such numbers, and of the real zero, merely calls for: the notions of a natural number and of a property of natural numbers—which includes, in this setting, those of cardinal numbers, of the successor relation between these numbers, and of its strong ancestral—; the notions of a variable ranging over these numbers and over their properties, respectively; the notion of the natural zero, as distinguished from any other natural number; the notions of the identity and the strict order relation among natural numbers; the notion of a natural number having a property; and, finally, the notions of a property of natural numbers being enjoyed or not by a certain natural number, and being enjoyed by exactly the same natural numbers as another such property. Besides these notions, to understand the definitions of integer numbers, and, among them, of positive and non-positive such numbers, and of the integer zero, only the notion of a positive natural numbers—*i. e.* of a natural number other than zero—is called for. Finally, to understand, the definitions of rational numbers, and, among them, of positive and non-positive such numbers, and of the rational zero, it is enough to add the notion of addition on natural numbers. These resources are not only much fewer than those required to understand the three previous definition of real numbers, but they are also very basic. Noting this seems to me enough to conclude that the definition of real numbers coming with FRA is epistemically economic, in my sense.

6 Conclusions

The previous comparative assessments of two definitions of natural numbers (§ 4) and four definitions of real ones (§ 5) were intended to show that: *i*) alternative formal definitions of the same mathematical items can be discriminated according to their respective epistemic cost, and a choice among them can be made so as to prefer the one whose epistemic cost is the smallest, which (provided that the comparison involves a comprehensive and representative enough sample of formal definitions of the relevant items) I suggest to qualify as epistemically economic *tout court* (though, of course, other choices can be legitimately made on the basis of some other criteria); *ii*) in the cases under consideration, *i. e.* in relation to (second-order) definitions of natural and real numbers, the appeal to appropriate abstraction principles, within appropriate settings, namely to HP[FA] and to PAIR[FRA], allows one to obtain epistemically economic definitions; *iii*) this does not depend on the existential strength of these principles, and thus, *a fortiori*, on their existential parsimony or ontological neutrality (that, by the way, I do not think they benefit from), which shows that existential parsimony or ontological neutrality and epistemic economy are independent virtues. From all this, it follows that the epistemic economy of the definitions of natural and real numbers respectively coming with FA and FRA provides a possible reason to favour FA over other versions of (second-order) arithmetic, both as such, and as a base for real analysis: a reason which is independent of FA's existential strength.

My considerations leave, however, many related issues open. I wish to address two of them in conclusion.

6.1 Analyticity

The first issue concerns the relation between epistemic economy and analyticity.

Famously, Dedekind opened the preface to the first edition of *Was sind und was sollen die Zahlen* ([9]) by declaring that arithmetic is "the simplest science", namely a "part of logic", and that it is so insofar as "the number-concept [is][...] an immediate result [*Ausfluss*] from the pure laws of thought", since "numbers are free creations of the human mind [...][which] serve as a means of apprehending more easily and more sharply the difference of things", and "counting an aggregate or number of things" depends on "the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible" ([10], p. 14, with a slight modification)[29]. H. Benis-Sinaceur takes this to mean that arithmetic is a part of logic because "numbers [...] are rooted in the constitution of the mind or, as Dedekind writes to Keferstein (February 27, 1890), they are 'subsumed under more general notions and under activities [...] of the understanding [...] without which no thinking is possible' " ([2], §1.3.1; [35], p. 272; [36], p. 100). The same point is also clear in this other passage drawn from the fragment *Zum Zahlbegriff* (which Benis-Sinaceur quotes only partially: [2], §1.6): "Of all the resources that the human mind [is endowed with] for relieving its life, that is, [for fulfilling its] task, none is so effective and so inseparably connected with its inner nature as the concept of number [...], since every thinking man, even if he is not clearly aware of that, is a number-man, an arithmetician" ([12], app. LVIII, p. 315)[30].

Dedekind's logicism then seems to consist in the thesis that the resources we use to count, and, more generally, to deal with natural numbers, are just the same as, or part of those we use to think *tout court*, and in the identification of logic with the intellectual activity exercising these resources. Even though Dedekind and Frege have often been associated as two partisans of logicism—generally presented as the thesis that arithmetic can be reduced to logic by adopting

[29]Though he generically speaks of numbers (*Zahlen*), Dedekind's claims seem to be directly referred to natural numbers. Still, he also seems to consider that his views on these numbers extend to any other sorts of numbers, insofar as theories of the latter come from an extension of the theory of the former (or arithmetic, as usually intended). This is made clear by the parenthesis in the following claim: "In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept [*Zahlbegriff*] entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought" ([10], p. 14).

[30]I thanks Emmylou Haffner for drawing my attention to this passage.

an appropriate definition of natural numbers—this point of view is quite different from Frege's (emphasising this difference is the main purpose of Benis-Sinaceur's paper just mentioned; on this matter, cf. also the Foreword of [3]). Still, one would misunderstand Frege's point if one regarded it as the mere affirmation that arithmetic can be recovered within his logical system. Since it was part of this point that this system is epistemically basic, insofar as it pertains to the basic components of any thought[31] On the other side, Dedekind's purpose was not merely to account for the intrinsic features of arithmetic; it was also to prescribe the right definition of natural numbers: "upon this unique and therefore absolutely indispensable foundation [...] must, in my judgement, the whole science of numbers be established", he also writes in the mentioned preface, just after the passage quoted above ([10], p. 14). Doubtless, Frege's logic and thought are not activities, and, for him, natural numbers are certainly not "free creations of the human mind", as for Dedekind. Still, this crucial difference should not obscure a more fundamental agreement: that the logicality of arithmetic depends on its generality, which results, in turn, from its dealing with the building blocks of any other possible science (be it an exercise of human reason, as for Dedekind, or a system of truths, as for Frege); and that the definition of natural numbers has to conform to this.

It is then tempting to associate Dedekind's regarding the logicality of arithmetic as its involving the (or same as the) basic resources of thinking with Frege's view that arithmetical truths are analytic insofar as their proof only depends on "logical laws and definitions", and to suggest that a definition of some mathematical items is analytic insofar as its understanding only calls for logical resources. If this were admitted, there would also be room for assenting to Dedekind's view that the creative import of a definition does not preclude its being logical in nature. Exegetically speaking, one could doubt that Dedekind's regarding numbers as human mind's creations amounts to ascribing an ontological import for objects to whatsoever definition of them. Still, this would be independent from being ready to consent to the idea that admitting that a definition of natural numbers has such an import should not prevent one from considering that its understanding only calls for logical resources, and that it is then analytic in the tentative sense just mentioned. Of

[31]Look at the two following quotes, from the *Grundlagen* ([14], §14 and [16], p. 21), and from a coeval paper ("Über Formale Theorien der Arithmetik", [17], pp. 103-111, esp. p. 103; English translation in [18] pp. 112-121, esp. p. 112), respectively:

> The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought.
>
> As a matter of fact, we can count about everything that can be an object of thought: the ideal as well as the real, concepts as well as objects, temporal as well as spatial entities, events as well as bodies, methods as well as theorems; even numbers can in their turn be counted. What is required is really no more than a certain sharpness of delimitation, a certain logical completeness.

course, it would still remain to explain what it means for the understanding of a definition to only call for logical resources. This could be difficult to do in a precise way, but the previous considerations about the definitions of natural and real numbers coming with FA and FRA suggest that this construal of the notion of analyticity leaves room to argue that natural numbers, and possibly also real ones, admit an analytic definition. This would result in a vindication of Frege's views, although in a philosophical setting essentially different both from Frege's himself, and from the neo-logicist one.

To defend the strong neo-logicist analyticity thesis, one could maintain, then, that understanding HP[FA] and NatNum[FA](i-iii) only calls for logical resources by arguing as follows. If it were admitted that second-order logic is logic, or, more precisely, that L_2 is a genuine system of logic, it would be natural to maintain that understanding the right-hand side of HP[FA] only calls for logical resources. Hence, if it were also admitted that understanding the universal closure of a double implication '$\boldsymbol{S}(f) \Leftrightarrow \boldsymbol{S}$', introducing the new constant 'f' (non occurring in '$\boldsymbol{S}$'), only calls for logical resources if this is the case for understanding '$\boldsymbol{S}$', it would follow that understanding the whole HP[FA] only calls for logical resources. Furthermore, if it were equally admitted that understanding an explicit definition in $\mathfrak{L}^{\mathsf{L}_2} + \{f\}$ only calls for logical resources if this is the case for understanding the universal closure of '$\boldsymbol{S}(f) \Leftrightarrow \boldsymbol{S}$', it would also follow that understanding NatNum[FA](i-iii) only calls for logical resources, as well. It would then be enough to be ready to make the three mentioned admissions to conclude that the definition of natural number coming with FA is not only epistemically economic, as I have argued in § 4, but it is also analytic, in the foregoing sense.

If all this were conceded, it should also be possible to go ahead in a similar vein and argue that understanding PAIR[FRA], BicPAIR[FRA], BicPAIR^{+}[FRA] and BicPAIR$^{0/-}$[FRA] only calls for logical resources, in turn, with the result that the definition of real numbers coming with FRA would also be analytic. But then, why should it not be possible to fashion a similar argument supporting the claim that this is also the case of Shapiro's definitions of real numbers, as well as of that expounded in § 5.2.2, deriving from adapting Hale's one to FA? In the face of this option, one could adopt three different attitudes.

First, one could look for reasons to block the argument in relation to these two latter definitions, while admitting it in relation to the former one, in order to conclude that, whereas the former definition is analytic, the latter two aren't. For this purpose, one could advance, for example, that there is a relevant difference in the epistemic cost of the universal closure of a double implication '$\boldsymbol{S}(f) \Leftrightarrow \boldsymbol{S}$' introducing the new constant 'f', according to whether '$\boldsymbol{S}$' is a formula of $\mathfrak{L}^{\mathsf{L}_2}$, or '$\boldsymbol{S}$' includes some constants that do not belong to $\mathfrak{L}^{\mathsf{L}_2}$ (and this independently of whether this universal closure is unrestricted or admits a restriction involving some predicate constants), with the result that from admitting that understanding such a universal closure in the former case only calls for logical resources does not entail that this is also the case for understanding it in the latter case[32].

Second, one could question that an argument similar to the previous one, supporting the conclusion that the definition of natural numbers coming with FA is analytic, applies to FRA, and then maintain that neither the definition of real numbers coming with FRA, nor Shapiro's, nor the adaptation of Hale's to FA are analytic. For this purpose, one could argue, for example, that the mere fact that an abstraction principle incorporates a universal quantification restricted by using a predicate constant entails that the understanding of this principle calls for more than only logical resources. More radically, one could also question that understanding an explicit definition in $\mathfrak{L}^{\mathsf{L}_2} + \{f\}$ only calls for logical resources if this is the case for understanding the definition of 'f', so as to block, in this way, also the previous argument bringing to the conclusion that the definition of natural numbers coming with FA is analytic.

Finally, one could accept that Shapiro's definitions of real numbers and the adaptation of Hale's one to FA are both analytic, after all, just like the one coming with FRA, while conceding that distinct analytic definitions of the same items could have significantly different epistemic costs, with the result that only one of them is epistemically economic.

6.2 Exemplarist Definitions

I now turn to the second question.

At the end of the paper where he presents his definition, Shapiro touches on Heck's distinction between interpreting arithmetic "in some analytically true theory" and showing that "the truths of arithmetic, as we ordinarily understand them, are analytic", and he wonders whether the "cuts on bounded, instantiated properties of rational numbers [as defined by CUT[FDRA]] are the real numbers that we all know and love?" ([31], pp. 360-361; [23], p. 596). This last question is, as such, independent of the admission that FA, and possibly also FDRA, are analytic, and can be repeated for any definition of some mathematical items depending on coding these items with other ones previously defined, that is, as one could say, for short, for any exemplarist definition. By only considering the definitions of natural and real numbers respectively coming with FA and FRA, the question becomes: should we regard these theories as genuine theories of natural and real numbers, as we ordinarily understand them, or, merely, as theories within which arithmetic

[32] To see the point, remark, firstly, that $\mathfrak{L}^{\mathsf{L}_2}$ includes no predicate constant, so that a restriction involving some predicate constant cannot be stated in $\mathfrak{L}^{\mathsf{L}_2}$. Remark, then, that in PAIR[FRA], the restricted quantifier '$\forall_{\mathcal{N}} z$' can be equivalently replaced by an unrestricted one (its entering this principle is only motivated by easiness of understanding). Finally, remark the difference between the open formula '$x = y \wedge \forall z \left[X(z) \Leftrightarrow Y(z) \right]$', providing the right-hand side of PAIR[FRA], and the other open formulas '$x + y' = x' + y$', '$[y = 0_{\mathcal{Z}} \wedge y' = 0_{\mathcal{Z}}] \vee [y \neq 0_{\mathcal{Z}} \wedge y' \neq 0_{\mathcal{Z}} \wedge x \cdot_{\mathcal{Z}} y' = x' \cdot_{\mathcal{Z}} y]$', '$\forall_{\mathcal{Q}} x \left(F \trianglelefteq x \Leftrightarrow G \trianglelefteq x \right)$', '$\forall_{\mathcal{N}^+} h, k \left[hx \lesseqgtr ky \Leftrightarrow hx^* \lesseqgtr ky^* \right]$' and '$\forall_{\mathcal{R}^{\mathcal{N}^+}} \left[P(x) \Leftrightarrow Q(x) \right]$', providing the right-hand sides of the abstraction principles involved in Shapiro's definitions of real numbers, and in the adaptation of Hale's to FA: whereas the first of these formulas is a formula of $\mathfrak{L}^{\mathsf{L}_2}$, this is so for none of the others.

and real analysis are interpretable?

As I conceive it, the question is not whether FA and FRA truly define, respectively, natural and real numbers, that is, if the items that, within these theories, are taken to be these numbers are actually these very numbers. So conceived, the question makes little or no sense, it seems to me, unless one looks at the matter from a hyper-realist ontological perspective, which I neither share, nor believe could be adopted by default. In my view, the question is rather whether the definitions of these numbers respectively coming with FA and FDRA have what is essential to the nature we attribute to these numbers built into themselves. Of course, if one maintains that there is nothing essential to this nature beside these number's meeting the relevant structural conditions, then this question makes little sense as well. But maintaining this is far from mandatory: there is room to consider that what is essential to this nature also depends—or even depends only—on the place we attribute to these numbers in our mathematical knowledge as a whole (for example, on the mutual relations between natural and real numbers that follow from our way to conceive them), or, more generally, in the whole system of our knowledge.

In this perspective, the relevant question with respect to FA is whether we should consider that taking natural numbers to be trademarks of the cardinality of finite concepts reflects what is essential to their nature. In this form, the question has been often discussed, for example considering whether defining natural numbers through FA meets the application constraint. There is then no need to come back to it, here. I confine myself to observe that nothing ensures, in general, that a definition whose understanding calls for less and/or more basic resources than others, or even an analytic definition in the foregoing sense, has what is essential in the nature we attribute to the relevant items built into it. Hence, there is no general reason to think that FA's being epistemically economic goes together with its having what is essential in the nature we attribute to natural numbers built into it. Its possibly having both virtues would then be a supplementary epistemic advantage that this definition would have over alternative ones, since this would make it able to show that what is essential in the nature we attribute to natural numbers can be grasped by appealing to few and quite basic resources. Moreover, if it happened that these resources could be considered as merely logical, this would mean that logical resources are enough to grasp what is essential in the nature we attribute to natural numbers, which could be taken as a proper way to state a logicist thesis for someone who, like me, merely considers mathematics as a result of our intellectual activity.

It is then relevant to wonder whether something similar could also be said of FRA, supposing that it be regarded as providing an epistemically economic definition of real numbers, or even an analytic definition of them in the foregoing sense.

Against the idea that the definition of real numbers depending on FRA has all what is essential in the nature we attribute to these numbers built into it, one could observe that the explicit definitions making bicimal pairs behave as real numbers are guided by a previous understanding of the relevant structure, together with the admission that real numbers exemplify this structure, and that this suggests that taking real numbers to be (coded with) bicimal pairs might be considered appropriate only if these numbers are independently conceived to exemplify this structure. Now, it is certainly not built into the definition of bicimal pairs that they exemplify this structure: this can only be verified *a posteriori* with respect to the definition itself. Hence, if what is essential to the nature we ascribe to real numbers includes their exemplifying this structure, defining these numbers as (coded with) bicimal pairs cannot have all what is essential to this nature built into it.

This is a plausible argument, but there are reasons to resist it. One could argue, indeed, that what is essential to the nature we ascribe to real numbers does not include their exemplifying the relevant structure, but only their possibly doing so, should appropriate operations and relations be defined on them. To make an analogous point with respect to natural numbers, one could argue that what is essential to the nature we ascribe to them is not their behaving with respect to order, addition and multiplication as they do, but their being so that this relation and these operations can be defined on them so that they behave in this way. Arguing that natural numbers are essentially cardinal numbers (or numbers of concepts), rather than elements of a progression, is, after all, a way to make this point. Could one not make an analogous point for real numbers? If this were conceded, it would be relevant to remark that having an epistemic access to bicimal pairs is independent of having an epistemic access to the relevant structure. Indeed, this would leave room to maintain that the mere resources needed to understand PAIR[FRA] and BicPAIR[FRA] are enough to grasp what is essential to the nature we attribute to real numbers. Surely. But leaving room to maintain that this is so is still not the same as providing reasons for it. Hence the question remains: are the resources needed to understand PAIR[FRA] and BicPAIR[FRA] enough to grasp what is essential to the nature we attribute to real numbers? In order to argue that this is not so, and so contest that the definition of real numbers depending on FDRA have what is essential to this nature built into it, one could observe that it is a fact that our ordinary understanding of real numbers does not involve bicimal pairs. The following considerations should, however, be enough to overcome this objection and support a positive answer to the question.

Insofar as the notions of limit, convergence, continuity and cut are in no way appealed to in the definition of real numbers depending on FRA (though proving that bicimal pair form a complete group cannot but require appealing to some of them), this definition suggests that there is a way to understand the key notions of real analysis—which are certainly involved in our ordinary understanding of real numbers—that results from our acquaintance with an instance of the structure of real numbers, rather than the other way around. Hence, one could maintain that taking real numbers to be (coded with) bicimal pairs allows us to have an epistemic access to these numbers in such a way that we can, then, and only then, get our ordinary understanding of them through working on them by exploiting the possibilities embodied in the very nature of these pairs. If it were conceded that what is essential to the nature we ascribe to real numbers merely includes their possibly exemplifying the relevant structure if some appropriate operations and relations are defined on them, it would follow that this would already be implicitly built into these numbers being coded with bicimal pairs, and it would only be a question of making it explicit.

In the paper mentioned above, Benis-Sinaceur argues that Dedekind's original definition of real numbers, unlike Cantor's, "is not based on the concepts of limit and convergence", and that this definition rather "shows how to derive the concept of limit, and thus the usual theorems of real analysis, from the purely arithmetical definition of the concept of real number" ([2], §1.1). This suggests that Dedekind's definition already makes clear that understanding some key notions of real analysis can result from our acquaintance with an instance of the real numbers structure, rather that with this structure itself. Nonetheless, understanding Dedekind's definition involves understanding a previous definition of rational numbers and of cuts on (and then sets of) them, which is not the case of the definition depending on FRA. *Mutatis mutandis*, the same also happens for Shapiro's definition and for the adaptation of Hale's one to FA. Hence, if it were conceded that the definition depending on FRA has what is essential to the nature we ascribe to real numbers built into it, this would show that there is a way to grasp what is essential to this nature that is epistemically more economic than the way displayed by these alternative definitions.

But there is even more. Insofar as FRA is an extension of FA, and defining real numbers through FRA depends on defining natural numbers through FA, so defining real numbers brings forward an idea of these last numbers as reifications of properties of cardinal numbers combinatorially steered. This is not the idea of real numbers that Frege attached to them, and is certainly not the idea that arises from looking at their applications in geometry and science. But it is, it seems to me, a rather natural view of them both arithmetically speaking, and from a logicist perspective. Were it admitted that this very idea embodies what is essential to the nature we ascribe to real numbers—which is not only a possibility that the previous considerations leaves open, but also a very natural admission, if it were conceded that we essentially conceive real numbers as being numbers in the same sense as that in which natural numbers are so—, there would be no doubt that the definition of real numbers depending on FRA has what is essential to the nature we ascribe to these numbers built into it. And were it also admitted that this definition is analytic, in the foregoing sense, this would result in a version of the logicist thesis also for real numbers.

Still, even if all this were admitted, and this version of logicism were endorsed, this would not entail, in my view, that FRA provides the right or the best definition of real numbers. In my view, philosophy of mathematics should not aim at deciding which is the best way of defining certain mathematical items, or of structuring or founding mathematics or certain branches of it. It should rather aim (among other things) at identifying different philosophical virtues of different ways of defining mathematical items, and structuring or founding some branches of mathematics. My purpose was to isolate one of these virtues, namely epistemic economy, and to show that appealing to appropriate abstraction principles is suitable to obtain definitions that enjoy this virtue.

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