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Bootstrapping of integer concepts: the stronger deviant-interpretation challenge (and how to solve it)

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Abstract

Beck (Cognition 158:110–121, 2017) presents an outline of the procedure of bootstrapping of integer concepts, with the purpose of explicating the account of Carey (The Origin of Concepts, 2009). According to that theory, integer concepts are acquired through a process of inductive and analogous reasoning based on the object tracking system (OTS), which allows individuating objects in a parallel fashion. Discussing the bootstrapping theory, Beck dismisses what he calls the "deviant-interpretation challenge"—the possibility that the bootstrapped integer sequence does not follow a linear progression after some point—as being general to any account of inductive learning. While the account of Carey and Beck focuses on the OTS, in this paper I want to reconsider the importance of another empirically well-established cognitive core system for treating numerosities, namely the approximate number system (ANS). Since the ANS-based account offers a potential alternative for integer concept acquisition, I show that it provides a good reason to revisit the deviant-interpretation challenge. Finally, I will present a hybrid OTS-ANS model as the foundation of integer concept acquisition and the framework of enculturation as a solution to the challenge.

Keywords Bootstrapping · Integer concepts · Numerical cognition · Concept acquisition · Object tracking · Approximate number system · Cumulative cultural evolution · Enculturation

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1 Introduction

Currently, one of the most discussed accounts in the philosophical literature on number concept acquisition focuses on the bootstrapping of integer concepts.[1] The bootstrapping account is based on Quine's (1960) idea of how genuinely new concepts can emerge from previous conceptual content. The most famous theory of the bootstrapping of natural number concepts was presented in detail by Carey (2009) in her book *The Origin of Concepts*. In that highly influential book, Carey argues that the first four numerals in the counting list are determined by procedures that are "constraints on computations" due to the core cognitive *object tracking system* (OTS). Recently, this stage of Carey's bootstrapping theory has been successfully explicated by Beck (2017). I see Beck's interpretation of Carey's account, which I will call the "Carey–Beck" theory of bootstrapping, as the strongest formulation yet in the literature of the general argument that natural number concepts are based on our cognitive core systems. For that reason, I focus in this paper on that version of the bootstrapping theory. As will be seen, I do not think that the Carey–Beck argument is fully convincing as it is. However, it provides a fruitful framework for studying the question of number concept acquisition. In the latter part of this paper, I will suggest my own theory of the influence and role of core cognitive abilities in the process of number concept acquisition.

In Sect. 2, I present the Carey–Beck bootstrapping account and analyze what is meant by "constraints on computations" based on the OTS. Section 3 deals with two challenges presented against the bootstrapping account, namely the "deviant-interpretation" and the "circularity" challenges. Beck (2017) argues that the latter is a proper challenge while the former is a general challenge for any account based on inductive learning and therefore not specific to bootstrapping. In Sect. 4, I propose that the existence of another core cognitive system for treating numerosities, the *approximate number system* (ANS), requires us to reconsider the deviant-interpretation challenge in a stronger form. Namely, the ANS-based quantity estimations have logarithmic rather than linear character and as such provide a potential route to "deviant" number concepts. This argument is developed in detail in Sect. 5. In Sect. 6, I propose an OTS-ANS hybrid account as the most feasible model for number concept acquisition. I will then show in Sect. 7 that this model can meet the stronger deviant-interpretation challenge by being placed within the framework of *enculturation*. This framework introduces cultural factors to the model, which explains why the linear interpretation of number concepts is present in ontogeny.

[1] More specifically, the discussion is about positive integer concepts, i.e., the *natural number* concepts (1, 2, 3,…). From now on, I will not distinguish between "integers", "numbers" and "natural numbers". In mathematics zero is usually included in the set of natural numbers but here it is excluded.

2 The Carey–Beck bootstrapping theory

The Carey–Beck theory of bootstrapping is located in the *core cognitive* framework, which takes human cognition to begin with "highly structured innate mechanisms designed to build representations with specific content" (Carey 2009, p. 67). The mental structures that represent the content realize "core cognition". According to the core cognitive framework in numerical cognition, two of the core cognitive systems are used for detecting quantities in the world. One, the approximate number system (ANS), enables estimating quantities and it will play an important role in this paper. But let us focus first on the core cognitive system that the Carey–Beck theory of number concept acquisition is based on. This system is called the object tracking system (OTS) and it allows parallel individuation, the creation of models of a small set of individuals in the working memory (Carey 2009).[2] Both the ANS and the OTS are often described as non-symbolic to distinguish them from quantity representations in terms of numerical or linguistic symbols (see, e.g., Hyde 2011).

The OTS is already present in human infants and we appear to share the object tracking ability with many nonhuman animals (Starkey and Cooper 1980; Wynn 1992; Dehaene 1997/2011; Spelke 2000; Carey 2009). It is based on the ability to represent objects as persisting individuals and it is closely related to the ability to perform multiple-object tracking (Trick and Pylyshyn 1994; Spelke 2000). Importantly for numerical cognition, the OTS is thought to enable *subitizing*, the ability to determine the quantity of objects in the field of vision without counting, for up to three or four objects. In Carey's (2009) account, subitizing is explained by the quantity being recognized by forming distinct mental representations for each of the observed objects. These representations work by employing *object files*. Three dots in the field of vision, for example, are represented in three distinct object files.[3]

The object files are thus not numerosity-specific, but they are thought to allow detecting numerosity. Up to four items, detecting how many object files are employed allows determining the quantity of the items. However, the ontogenetic development related to recognizing quantities requires several stages in a process in which the implicit representation of numerosity is associated with number words. At the first stage of the process, the level of being *one-knower*, a child grasps that the numeral "one" is associated with observing one object. When the child becomes a *two-knower*, she also grasps that the numeral "two" is associated with observing two objects. Since object tracking stops working after four objects, for *four-knowers*, a

[2] It should be noted that in Carey (2009), she also discusses other cognitive systems as potential influences on the bootstrapping process. In this paper I follow Beck and focus mainly on the OTS-based account in presenting the work of Carey, although I will deviate from this policy when relevant. This topic will be discussed more in Sect. 4 where some of the empirical evidence for focusing on OTS is presented.

[3] The prevalent conception in the literature is that the OTS indeed enables the subitizing ability while the ANS enables an estimation ability. (see, e.g., Hyde 2011). The subitizing and estimating abilities have different behavioral signatures, and they have also been reported to have different neural correlates in an fNIRS (functional near-infrared spectroscopy) study (Cutini et al. 2014). It is now generally agreed that the OTS and the ANS are indeed two separate systems.

qualitative leap must be made so that the child can become a *cardinality-principle* (CP) knower, i.e., in order to acquire the ability to generally match the last numeral uttered in the counting sequence with the cardinality of a group of objects (Sarnecka and Carey 2008; Lee and Sarnecka 2011). In other words, instead of becoming a “five-knower”, the child at that stage of development also grasps the numerals six, seven, and so on (Lee and Sarnecka 2010).

Carey (2004, 2009; see also LeCorre and Carey 2007; Sarnecka and Carey 2008) has proposed a particular type of bootstrapping process as the solution to explaining this qualitative leap. Against the "building block" model of concept acquisition proposed by Fodor (1975, 1998), she argues that acquiring integer concepts is a process that cannot be reduced to composition from simpler concepts. Thus, bootstrapping must be a process that yields the acquisition and understanding of genuinely novel concepts. The solution to this process of creating new concepts seemingly out of thin air is provided by the “computational constraints” provided by the OTS. This kind of bootstrapping process consists of three stages. First is acquiring a list of placeholders void of semantic content. This is the stage at which children can recite part of the verbal count list without grasping the connection to quantities (see, e.g., Davidson et al. 2012). At this stage, a child can recite the numeral list, for example, up to “four”, but when asked for four things, she will choose an amount randomly. At the second stage, the OTS allows interpreting the first members of the placeholder list by being associated with the procedure of establishing one-to-one correspondence with mental models:

> The meaning of the word ‘‘one” could be subserved by a mental model of a set of a single individual {i}, along with a procedure that determines that the word ‘‘one” can be applied to any set that can be put in 1–1 correspondence with this model. Similarly, two is mapped onto a longterm memory model of a set of two individuals {j, k}, along with a procedure that determines that the word ‘‘two” can be applied to any set that can be put in 1–1 correspondence with this model. And so on for ‘‘three” and ‘‘four.” (Carey 2009, p. 477).

These procedures do not function based on explicit representations of the one-to-one correspondence. Rather, they exist as constraints on computation. These computational constraints, Beck argues, are analogous to inference rules in logic in that they refer to the way “any computational theory of mind that posit explicit representations also needs procedures that govern how those representations can be manipulated” (Beck 2017, p. 116). Thus establishing the one-to-one correspondence is based on constraints of how the mind manipulates the representations in the object files: “While the object files explicitly represent objects, they only implicitly represent that the objects are in one-to-one correspondence” (Beck 2017, p. 119).

The computational constraints are, according to Beck, internal to the mind. But he argues that also external constraints play an important role in the bootstrapping process. Children acquire number concepts through a “counting game” in which they learn to associate the last word uttered in an ordered numeral list with the cardinality of a collection. What begins as a sequence of meaningless words, in the presence of the counting game allows “endowing the words in the count list with new conceptual roles” (Beck 2017, p. 199). In particular, the numerals develop the conceptual role of being uttered when observing the corresponding number of objects: “three” is consistently uttered in the presence of three objects, etc. (ibid.).

Thus the internal constraints due to employing the OTS and the external constraints due to engaging in counting games combine to make it possible to successfully acquire the first four number concepts. Finally, in the third stage of bootstrapping a generalizing process is carried out in order to give semantic content to every member of the placeholder list. According to Beck, this process can be a combination of analogous and inductive reasoning: since the first four numerals associated with the OTS refer to quantities separated by “one” individual, the same numerals associated with the counting game are also separated by “one” individual (Beck 2017, p. 119). Extrapolating from this, children then inductively grasp that *every* term in the counting list designated a quantity “one” more than the quantity designated by the previous term (ibid.)

In sum, according to Beck (2017), after acquiring the numeral list (from an external source like a parent or a teacher), the computational constraints coming from the object files used for parallel individuation, together with external constraints coming from counting games, are used to grasp the first integer concepts, up to the concept FOUR. From this, by generalization using analogy and induction, the rest of the numerals are matched with the correct integer concepts and the child becomes a cardinality principle knower. This bootstrapping account has been discussed extensively both in the literature on numerical cognition and on concept acquisition in general (see, e.g., Gelman and Butterworth 2005; Rips et al. 2006; Sarnecka and Carey 2008; Kadosh and Walsh 2008; Spelke 2011a), as well as in the philosophy of mathematics (e.g., Piantadosi et al. 2012; Pantsar 2014, 2018).

3 The deviant-interpretation and circularity challenges

While the bootstrapping account has been widely discussed in the literature, as Beck (2017, p. 118) points out, too little attention has been given in the literature to the second stage of the process: how computational constraints can give semantic content to members of a verbal placeholder list. Most of the discussion has seemed to focus on the third stage of the bootstrapping account, the generalization process. As far as the third stage is concerned, much of that discussion has been focused on a particular form of what Beck (ibid., p. 112) calls the "deviant-interpretation challenge". This challenge is ancient and is best known among philosophers from Kripke’s (1982) account of Wittgenstein’s problem of rule-following, as well as Goodman’s (1955/1983) “grue” problem. The former "Kripkenstein" challenge asks how we can ever unassailably determine what rule a function follows if we only see a finite subset of its values. Thus, while the sequence 1, 2, 3, 4, 5, 6, 7, 8, 9, … would seem to have 10 as the next member, we cannot know that. The rule could actually entail something like "after 9, start again from 1". Goodman voiced a similar worry in his “new riddle of induction”. Imagine that the word “grue” is used as a synonym for “green” before some time *t*, after which it is used as a synonym for “blue” (the meanings of the words “green” and “blue” remaining constant over time). Now the question is, although "grue" is obviously an artificial construction, how do we know that our concepts do not change meaning like "grue" does?

In a contemporary version of the deviant-interpretation challenge, Rips and colleagues (2006, 2013) asked how the bootstrapping account can explain why a child that can count up to "twelve" continues to "thirteen" rather than, say, start again from "one" in a cyclical counting system. Some answers to Rips et al. are based on nativist views on numerical cognition that rely on presupposing stronger innate representational systems for quantities that reach beyond the OTS-range (e.g., Margolis and Laurence 2008). However, Rips and colleagues (2008) have responded that while they can accept determinate meanings for numbers in the OTS-range, counting systems beyond that rely on inductive inferences that are vulnerable to deviant interpretations.

I will return to that debate at the end of this paper, as I do believe that the rule-following challenge is a *bona fide* philosophical problem. However, I agree with Beck (2017, p. 113) that it is a problem for all inductive learning, not merely that of concept learning. More importantly, given the possibility that integer concept learning is particularly vulnerable to it, the rule-following challenge is definitely not exclusive to Carey's account of bootstrapping of integer concepts. Whatever rule we posit for the learning of integer concepts, there is always the chance that the rule deviates at some point. After grasping the integer concept 8,526,902 properly, the rule we use to grasp the next numeral might fail to acquire the concept of 8,526,903. There are only two possible scenarios in which the deviant-interpretation challenge does not apply. The first option is that all the integer concepts are innate. The second is that there is an innate recursive rule for integer concept acquisition. Since the number of integers is infinite and the brain is finite, the former position is impossible. The latter option cannot be ruled out as easily, but it is unsupported by evidence, which suggests that such recursive rules are a relatively late ontogenetic development and not universal (see, e.g., Sarnecka and Carey 2008; Pantsar 2019b).[4] As detailed in Sect. 7, there are cultures that do not have such recursive rules (Gordon 2004; Pica et al. 2004). Moreover, it would entail that the recursive rule is the product of evolution, which is a problematic position. The emergence of extensive written numeral systems is a recent phylogenetic development (see, e.g., Schmandt-Besserat 1996; Fabry 2020; Pantsar 2019b). While this does not preclude the possibility of older recursive systems of number words, the lack of evidence of recursive systems in non-literate cultures speaks against this possibility (see, e.g., Ifrah 1998; Everett 2017). Also this topic will be discussed in more detail in Sect. 7.

Based on the universal character of the deviant-interpretation challenge, Beck argues that Carey's account cannot be refuted by the Kripkenstein-Goodman argument, any more than accounts of inductive learning in general can be. This leads

[4] It is more plausible that instead of there being an innate recursive rule, there is some kind of innate *bias* for some recursive rule. In this case, the deviant-interpretation challenge still applies, but in a different form: now the question becomes why the bias is followed in some cases but not in others. The matter will be discussed further in Sect. 6. I thank an anonymous reviewer for their remark on this topic.

Beck to focus on what he calls the "circularity challenge" to bootstrapping. The circularity challenge was most famously presented by Fodor, according to whom:

> There literally isn't such a thing as the notion of learning a conceptual system richer than the one that one already has; we simply have no idea of what it would be like to get from a conceptually impoverished to a conceptually richer system by anything like a process of learning. (Fodor 1980, pp. 148–149).

Thus, when Carey argues that by bootstrapping we can acquire richer numerical concepts, Fodor (2010) and Rey (2014) hit back that, circularly, any such new concepts would need to be possessed by the learner already. In particular, as Rey (2014, p. 117) points out, Carey's notion that the successive numerals are connected by the relation "one greater than" (Carey 2009, p. 277) appears to assume that there is a pre-existing grasp of "one greater than", which is then used to define the successor relation. Beck sees this as a more worthwhile challenge to the bootstrapping theory than the deviant-interpretation challenge. He goes on to defend Carey's OTS-based bootstrapping theory by interpreting her account of internal computational constraints that can increase expressive power (Beck 2017, p. 119). As explained in the previous section, the computational constraints associated with the OTS allow a child to grasp that the numeral "one" is associated with one object file, thus acquiring the concept ONE. Then during the learning process she learns that combining a collection of "one" F with "one" F creates a collection of "two" F's, acquiring the concept TWO—and so on until four F's give the concept FOUR, which is the limit of the object tracking system. At this point, the child can then use analogy and induction in the extrapolation that also other words in the number sequence work in a similar manner: the next numeral is associated with the process of combining the previous collection of *n* F's with "one" F. Importantly, Beck argues that this theory avoids the circularity challenge. By the time the analogous and inductive process of bootstrapping happens, the placeholders in the numeral list already have been partially interpreted due to the computational constraints: the child has already grasped that the quantities designated by the words from "one" to "four" differ by "one" individual, which can be used to induce that the same difference of "one" characterizes also the rest of the numeral list (Beck 2017, p. 119). After this, Beck contends, the only problem left is the deviant-interpretation challenge, but since that is a problem for any theory of inductive learning and not specific to bootstrapping, Carey's account remains unharmed.

4 The Approximate Number System

In his explication of Carey's theory, Beck offers a clear and plausible philosophical account of how the bootstrapping process could work. In avoiding the circularity challenge, Beck manages to formulate Carey's account so that the move from computational constraints due to the parallel individuation ability to grasping the successor function requires no further conceptual input. But in this great strength of the Carey–Beck theory lurks also a potential problem. As is well established [and several times mentioned by Carey (2009)], there is also another core cognitive system for treating numerosities. This system, called *approximate number system* (or analogue magnitude system), is manifest in an estimation ability that—like the OTS—we possess already as infants and share with many nonhuman animals (Dehaene 1997/2011; Spelke 2000).

The estimations made with the approximate number system (ANS) have two important characteristics: the size effect and the distance effect. This means that the numerosity estimations are subject to the so-called *Weber's law*: they become less accurate as the numerosities become larger, and distinguishing between numerosities becomes easier as the numerical distance between them becomes greater (Dehaene 2003). Thus, the ANS-based estimations are characterized by their logarithmic rather than linear manner: establishing the difference between, say, sets of 5 and 6 objects is easier than that between 15 and 16 objects. Empirical data show that when asked to place numerosities on a number line, people in cultures with limited numeral systems (such as the Pirahã and the Mundurukú of the Amazon) recruit the ANS and place the numerosities in a way that is best modelled as logarithmic (see e.g. Dehaene et al. 2008; Pica et al. 2004).[5]

It is generally accepted that the ANS and the OTS are two separate cognitive systems for treating quantities (see, e.g., Dehaene 1997/2011; Feigenson et al. 2004; Agrillo 2015). Furthermore, although there are well-known accounts that ascribe limited importance for the ANS for the early stages of number concept acquisition (e.g., Carey 2009; Carey and Barner 2019), many researchers believe that the ANS plays a key role throughout the development of number concepts (see, e.g., Dehaene 1997/2011; Butterworth 1999; Spelke 2000; Halberda and Feigenson 2008a). Given this background, we cannot rule out the possibility that, just like the object tracking system, the ANS plays a role in the acquisition of number concepts. Indeed, there are accounts in the literature that take the ANS to be the primary core cognitive system in developing and acquiring number concepts, most influentially that of Dehaene (1997/2011).

Although it is agreed that the ANS-based estimations are logarithmic in character, there has been disagreement on how the ANS-based numerosities are represented in the mind. This is still an open question. Perhaps the most famous account in the literature, presented by Dehaene and Changeux (1993), takes the ANS-based representations to be logarithmic whereas the "accumulator" model of Meck and Church (1983) proposes that quantity information from sensory input is represented on a linear scale.[6] However, this question of how the quantities are represented should

[5] Although this does not appear to be always the case; see Núñez (2011). Perhaps most importantly, Núñez points out that 37% of the experimental runs on the Mundurukú reported in Dehaene et al. (2008) showed a bimodal response using only the endpoints of the number line. Thus the results reported by Dehaene and colleagues are perhaps better interpreted in a conditional manner: if subjects place numerosities on a number line in non-arithmetical cultures, it is in a logarithmic rather than linear manner.

[6] It is a fundamental question about these models what the logarithmic and linear representations mean on a neuronal level. The Dehaene-Changeux model, for example, is a connectionist network and it is not clear what "logarithmic representation" should be interpreted to mean on a neuronal level. Here a key distinction should be made. Dehaene (2003) argues for a logarithmic "mental number line", which I believe can be associated with both the Dehaene-Changeux and the Meck-Church model. On the other hand, in Dehaene (2001), instead of representations, he writes of logarithmic and linear "coding schemes" for quantities. These coding schemes, I understand, are the basic distinction between the two

not be confused with the fact that the ANS-based *estimations* follow a logarithmic rather than a linear structure (Nieder and Dehaene 2009). For the present purposes, the important matter is that the ANS is manifest in estimations that are characteristically logarithmic, whereas our natural number concepts follow a linear progression.[7] This is something a core cognitive account of number concept acquisition must be able to explain. Therefore, based on the accounts of ANS and OTS in the literature, there would seem to be three possible scenarios in the core cognitive framework. In the first one, the Meck–Church scenario, the quantitative information due to the ANS is represented in a linear fashion in the mind and number concepts are acquired based on these representations. In the second, the Dehaene–Changeux scenario, the logarithmic quantity representations due to the ANS are used in acquiring number concepts that follow a linear structure. In the third scenario, the linear structure of number concepts is based on the OTS, and in this process, it overrides the logarithmic character of the ANS-based numerosity estimations (and if we buy into the Dehaene-Changeux model, also the logarithmic character of the numerosity *representations*, or coding scheme).

Importantly for the present topic, the Carey–Beck OTS-based account as it is formulated by Beck (2017) would appear to be insufficient in all three cases. In the first two scenarios, it would be mistaken since the ANS rather than the OTS would be the primary cognitive system involved, and the computational constraints of the latter system would not be responsible for acquiring number concepts. This is a possibility that cannot be dismissed—even though Carey (2009, pp. 309–319) provides strong evidence against it—but given the growing amount of evidence that the OTS plays an important role in number concept acquisition, I will focus here on the third scenario. In Sect. 7, however, I will argue that all three scenarios are flawed. The linear structure of number concepts cannot be explained in a satisfactory manner if we focus exclusively on the core cognitive abilities.

5 The stronger deviant-interpretation challenge

Let us accept for now that the Carey–Beck account is fundamentally right and that our number concepts follow a linear progression due to the computational constraints of the OTS. Given the existence of the ANS, the Carey–Beck account needs to answer one important question: why do our number concepts follow the linear character of the OTS-based representations rather than the logarithmic character of the ANS-based estimations? Clearly some factors are responsible for the linear

Footnote 6 (continued)
models. However, it is possible that the same coding scheme can be instantiated by different types of neuronal-level models.

[7] It should be noted here that the ANS-based estimations are not fixed to a particular logarithmic model. Typically the estimation acuity increases gradually from infancy to adulthood (Halberda and Feigenson 2008b). It has also been established that by training, we can become better estimators, which also correlates with better performance with exact number concepts (Park and Brannon 2013). Curiously, there are reports that arithmetic training does not improve ANS acuity (Lindskog et al. 2016).

rather than logarithmic structure that our integers follow. The question is, does the Carey–Beck account provide a convincing argument that this process is explained by the computational constraints involved in applying the OTS? And if so, given the existence of two core cognitive systems for numerosities, why is it that in the bootstrapping process only the computational constraints given by the OTS are applied, whereas the ANS does not appear to play a role?

To be sure, there is a strong empirical case that this is indeed what happens. Carey (2009) provides strong evidence against the position of Dehaene (1997/2011) and others who argue that the ANS provides the primary cognitive foundation for natural number concepts.[8] For example, the data of several studies (e.g., Condry and Spelke 2008; LeCorre and Carey 2007) show that mapping numerals to ANS-based estimations is an additional ability that children need to acquire after learning to count. Before that, children choose numerals at random in estimation tasks, not showing any sign that larger numerals should be preferred over smaller ones in estimating the size of larger collections (Carey 2009, pp. 314–316). After becoming cardinality principle knowers, it takes children about six extra months to be able to map estimations of larger sets (>5) to numerals higher up the count list. This leads Carey (p. 316) to conclude that “These data absolutely rule out the possibility that mapping numerals in the range of 5 to 10 to analog magnitudes plays any role in the construction of the numeral list representation of natural number”.

However, even if we accepted this, it does not mean that that the ANS does not play any role in the general process of acquiring natural number concepts, beyond the range of one to ten.[9] As will be seen in the next section, several researchers have proposed models of number concept acquisition that involve both the OTS and the ANS. But even though I am a proponent of such a model, ultimately it is not crucial for the main argument of this paper. We can accept the main tenet of the Carey–Beck account that it is the OTS rather than the ANS that provides the computational constraints used at least in the early stages of acquiring natural number concepts. That is what the data strongly support. But the question I am interested in is *why* our natural number concepts follow the structure set by the computational constraints of the OTS also beyond the subitizing range of one to four. And in asking that question, we must ask why the natural number concepts follow a linear rather than a logarithmic structure.

By presenting this question, we return to the deviant-interpretation challenge. In this case, the proposed deviant interpretation is provided by the logarithmic character of the ANS-based estimations. But now the challenge is considerably stronger than the standard Kripkenstein-Goodman argument since instead of a mere theoretical possibility, it is based on a widely acknowledged core cognitive, quantity-specific ability. While Beck dismisses the deviant-interpretation challenge as having

[8] In the second edition of his book, Dehaene softens his position and allows for the possibility that the OTS is also important in number concept acquisition.

[9] This is in line with the analysis of Carey (2009, Chapter 9), who accepts that mapping the numeral list to the ANS-based estimations can be seen as part of the bootstrapping process. The important distinction is that this is a later stage in development than mapping the counting list to the OTS-based number concepts.

to deal generally with inductive learning, his criticism seems to be targeted to a particular type of the challenge, namely the general Kripkenstein-Goodman argument. Indeed, I agree with him that it is not fruitful to use the Kripkenstein-Goodman as an all-purpose tool to criticize any inductive account, including those on concept acquisition.

However, this does not mean that we should not consider possible deviant interpretations when there are genuine alternative theories backed up by empirical data. To use an analogy from the philosophy of science, no amount of observations of white swans can rule out the existence of black swans. But once we have evidence suggesting the existence of black swans, we certainly cannot dismiss the black swan challenge merely as a case of the Kripkenstein-Goodman argument. This way, I am not worried that the inductive process used to learn integers up to thousands (and beyond) will fail at some point. What I *am* worried about is whether the process of induction and analogy based on the computational constraints as described by Beck can be established as our mode of cognitive access to large integers.

Another analogy helps to see the difference. Let us compare the following sequences:

(a) 1, 2, 3, 4, 5, …
(b) 2, 3, 5, 7, 11, …

In both cases, the Kripkenstein-problem is present, but there appears to be an important difference. In the case of the sequence (a), bringing up the Kripkenstein-challenge would seem to be just the kind of all-purpose deviant-interpretation challenge Beck dismisses. But in the case of (b), there are several options for the sixth member of the sequence. Mathematically knowledgeable people probably associate the numbers in the sequence with the list of prime numbers, and thus the sixth member would be 13. But based on the small part of the sequence given here, the sixth member could equally well be, say, 15. This would follow, for example, from the rule that for the second member, add one. For the next two members, add two to each. For the next four members, add four to each, etc.[10] This kind of deviant interpretation is not simply rehearsing the Kripkensteinian remark about the inductive nature of the process. Instead, it provides a feasible alternative interpretation for the continuation of the number sequence.

Similarly, the logarithmic character of the ANS-based estimations provide a feasible alternative scenario that our number concepts could have taken. This way, I see the ANS as providing a stronger case of the deviant-interpretation challenge to the Carey–Beck OTS-based theory than the general Kripkenstein-Goodman argument does. If the ANS is a factor in our learning of natural numbers, the logarithmic character of ANS-based estimations could potentially show up in the progression of our number concepts. In this scenario, the distance between two successive numbers (in the sense of a number line) would become smaller as the numbers become larger.

[10] This is of course just one option for the rule. The On-line Encyclopedia of Integer Sequences recognizes 1005 integer sequences that include the progression 2, 3, 5, 7, 11. (oeis.org).

What this could mean in practice is perhaps hard to envision. But combined with an increasing lack of accuracy of the ANS as the numerosities become larger, one possibility is that number concepts would be used to refer to larger intervals of discrete numerosities. For example, the number concept FOUR could refer to the numerosity 4, but some single number concept X could refer to our numerosities 5 and 6 (or, to put it another way, the set {5,6}), and the next number concept Y to the numerosities 7, 8 and 9 (the set {7,8,9}). While this example is both naïve and hypothetical, it is certainly not a completely unrealistic scenario based on the research on the anumeric cultures such as the Pirahã and the Mundurukú (Gordon 2004; Pica et al. 2004). If a logarithmic number line models the progression of quantities, it is possible that number concepts would in this manner refer to larger intervals of numerosities.[11]

Since such number concepts did not develop in our culture, we are left with the three scenarios specified in the last section. First, according to the Meck–Church theory, although the ANS functions characteristically in a logarithmic manner, the numerosity representations based on it are linear. Second, according to the Dehaene–Changeux theory, the ANS-based numerosity representations are logarithmic but they are turned into linear number concepts. Or third, the ANS remains as a parallel system for treating numerosities but our number concepts get their linear structure from somewhere else (like the OTS). In the third scenario that we focus on here, the question is: why do children use analogical and inductive learning based on the OTS when they acquire integer concepts, while not employing the logarithmic character of the ANS-based estimations?

The question why the OTS overrides the ANS in number concept acquisition becomes particularly important when we remember that the ANS has one clear advantage over the OTS as the basic cognitive core system for quantities: while the OTS concerns only the numerosities from one to four, the ANS is in principle unbounded and can thus be used for large numerosities beyond the scope of the OTS. This wider scope of the ANS is particularly important when we note that the three steps (from ONE to TWO, from TWO to THREE, and from THREE to FOUR) that the Carey–Beck bootstrapping theory posits as the way to acquire natural number concepts is a very small amount of instances to base a general rule on. At the very least, the process on inductive and analogical reasoning at the foundation of the Carey–Beck account leaves the theoretical possibility of deviant interpretations. This theoretical possibility becomes a serious challenge when there's already a quantity-specific core cognitive system in place which follows such "deviant" rule. Why does that system, the ANS, not take over? We can see that this is not merely re-stating the deviant-interpretation argument in the mold of Kripke and Goodman. Instead of pointing out the logical possibility of a deviant development, I am asking why the characteristics of the estimations based on a well-established ability for treating numerosities are not present in the early development of number concepts. If the Carey–Beck theory is accepted, it implies that the computational constraints given by the OTS are strong enough to guarantee that input on numerosity-related

[11] A different, but related suggestion is made by Ball (2017) in which he developes an account of "approximate cardinal numbers" as representations of ANS magnitudes.

information coming from the ANS is ignored. The question is, why would that happen?

As I will aim to show in Sect. 7, I do not believe that this is simply due to some genetically determined psychological processes. Instead, I will argue that answer can be found in the way cultural factors shape the development of our cognitive processes. But before that can be discussed, it is necessary to specify what kind of role I see for the two core cognitive systems discussed above in number concept acquisition.

6 Hybrid models

Given the considerations in the previous section, and the strong empirical support for the OTS and the ANS, I contend that we should approach the question of number concept acquisition in a way that allows both core cognitive systems to play a significant role in the process. Currently, the empirical data do not support a strict commitment to a single core system as the foundation of number concepts. According to Carey (2009, p. 316), in early number concept acquisition, at least from ONE to TEN, it seems that the ANS does not play a role by mapping numerals to the ANS-based estimations. But as she also notes (p. 335), this does not imply that the ANS could not play any role in acquiring the general concept of natural number. Consequently, we should be ready to look for possible explanations of number concept acquisition that evoke both systems, possibly in addition to other factors. The idea of both ANS and OTS being important for the process is not new in the literature on numerical cognition. Several researchers have proposed that the ANS could play a role in the acquisition of number concepts in parallel with, or as a complement of, the OTS. Spelke (2011b), for example, has argued that Carey's bootstrapping undervalues the role of ANS in acquiring number concepts. The analysis of van Marle and colleagues (2018) implies that measures of both the OTS and the ANS predict knowledge-levels of cardinal numbers in children, but in the key stage of acquiring the cardinality principle, for example, only the ANS-measure continues to do so. Their "merge" theory thus suggests that the two systems both play a role in early acquisition of number concepts, after which the ANS plays a more important role. In their analysis of the subitizing ability, Clements and colleagues (2019) also suggest that both the OTS and the ANS should be included in the explanation. Hyde (2011) reviews several other empirical results which imply the need for both OTS and the ATS to be included in the model of number concept acquisition. In the philosophical literature on numerical cognition, I have suggested a similar hybrid model over single-system theories (Pantsar 2014, 2015, 2016, 2018, 2019b). I have proposed, among other things, that the initial notion of discrete quantity can come from the OTS whereas the notion that quantities form an indefinitely continuing progression can come from the ANS. Thus, when the task is explaining natural number concept acquisition in general, in that model both the OTS and the ANS play an important role.[12]

Recall that one problematic aspect with the Beck-Carey theory is that the necessary inductive and analogical reasoning is thought to be conducted based on three steps, by moving from ONE to TWO, from TWO to THREE, and from THREE to FOUR. Based on this, children are thought to make the induction step to become cardinality-principle knowers and consequently soon after that grasp the concept of natural number in general. But now the question is how we can account for the fact that children on this basis grasp that there are quantities greater than four. If knowledge of numerosities is based on being able to track distinct objects simultaneously, how do we manage to apply this ability to groups of objects that we are *not* able to track in this parallel fashion? To put it another way, assuming that Carey and Beck are right that the concept FOUR comes from computational constraints due to the OTS, how do the concepts FIVE, SIX, SEVEN, etc. even make sense? After all, they refer to sizes of the kind of collections that are not in the domain of OTS.

What must happen is that the use of number concepts is expanded to new domains of collections larger than four, and thus the concept of number is no longer tied only to the OTS and its computational constraints. But such an expansion of the domain is not a trivial matter. It requires a change in how we observe the world: instead of collections of maximum four objects, we start to treat also larger collections in terms of their exact numerosity. But more important than the quantitative change is the qualitative one. Instead of observing collections based on the OTS, a new way of treating observations of collections (at least those larger than four) needs to be introduced. Empirical data suggests that this change does not happen early in ontogeny: Corden and Brannon (2009) report data according to which infants can discriminate between a small and a large number only when the ratio is 1:4, making them unable to distinguish between, say, 2 and 6 objects.

However, we know that already long before being cardinality principle knowers, children can compare collections of larger cardinalities than four. Although this ability is approximate and gets increasingly inaccurate as the numerosities become larger, it is a numerosity-specific ability that has important correlations both with developed mathematical ability and early number concept acquisition. Studies on college students, for example, show that better mathematical skills are associated with higher acuity in estimation tasks (Cantlon et al. 2006; Brannon and Merritt 2011). Wagner and Johnson (2011) report that preschoolers that are not CP-knowers show signs of ANS-based representations in grasping the verbal numeral list. As mentioned above, van Marle and colleagues (2018) report a study according to which in acquiring the cardinality principle, it is indeed the measure of the ANS rather than the measure of the OTS that predicts knowledge-levels. However, data also suggest that the ANS alone cannot explain the early ability with numerosities. Estimates of small and large numbers differ in reaction times and accuracy, and individual variability in small number range does not correlate with individual

[12] I would like to thank Paula Quinon for helpful discussions in developing this view. In Quinon (forthcoming), she develops her own hybrid account of natural number concept acquisition.

variability in large number range (Revkin et al. 2008; Piazza et al. 2011; Hyde 2011). Therefore the OTS and the ANS both appear to be used in non-symbolic numerical cognition and a hybrid model connecting the two is needed.

This connection between the ANS and both early and developed numerical ability could explain why the OTS-based inductive extrapolation that Carey and Beck propose can be carried out. Based on the ANS, children already have a great deal of experience in assessing the sizes of collections larger than four. They realize that quantities can be assigned to small collections of four or less objects, but also to larger ones. Based on the ANS, they can also get the general idea that quantities form a continuing progression. From merely three inductive steps, that can be difficult to grasp. But if the child has seen much larger collections and been able to estimate their numerosities, there is nothing new in treating those collections in terms of quantities. The new knowledge and skill component required at this stage is the ability to associate numerals also with larger quantities, which is present at the application of the ANS.[13] Thus, both the OTS and the ANS can play important roles in the development of numerical cognition, including the crucial step of acquiring natural number concepts.

However, now the big question at hand is whether such a hybrid model could survive the stronger deviant-interpretation question presented in the previous section. It might seem that the hybrid model runs into the same trouble as the OTS-based Carey–Beck model, given that the ANS with the logarithmic character of its estimations is specifically included in the theory. Why does the hybrid theory not follow the ANS-based estimations and give us number concepts that are not linear? The answer I propose here is based on the fact that both the ANS and the OTS are present throughout the development of a child. If a child only had the ANS, it is possible that the resulting number system would follow a logarithmic structure, if such a system could develop at all. If the child only had the OTS, it is possible that the resulting number system would only be applicable to collections maximum of four items. But if both cognitive core systems are used to process quantitative information, there is no (theoretical) limit to the sizes of collections, nor do the number concepts need to be approximate.

In an ANS-based account, the latter is a problem. How does the logarithmic and increasingly inexact estimation ability with quantities transform into linear, exact number concepts? In the Carey–Beck account, the former poses an equally important problem: how is the ability to determine quantities up to four extended into general number concepts? In a hybrid model, there can be an explanation for both problems. The Carey–Beck account can be correct when it comes to the first number concepts. But for generalizing into larger number concepts, the influence of ANS can be the missing piece of the puzzle. The inductive and analogical reasoning

[13] It should be noted that this ANS-based association of numerals with larger quantities does not imply that children use ANS-based representations in the initial stage of figuring out how numerals represent natural numbers. There is data showing that when children figure out how counting represents natural numbers, they still pick numerals at random when estimating larger numerosities. It takes them approximately six months to learn to map numerals to larger estimated numerosities (Condry & Spelke, 2008; LeCorre & Carey, 2007).

involved in bootstrapping could function essentially as described in the Carey–Beck theory. But the ANS could explain why such reasoning into larger quantities makes sense.

7 Enculturation

Above I have argued that a hybrid theory can give us a more comprehensive explanation of the bootstrapping process. But the challenge remains: why do the resulting number concepts follow a linear structure derived from the OTS rather than a logarithmic structure derived from ANS-based estimations? As the way to meet this challenge, I contend that we should not look for answers only in the ontogeny but also in the phylogeny and the history of number concepts.[14] The development of linearly structured numerosity systems is not a universal characteristic, as seen, for example, in the case of the Pirahã (Gordon 2004). The emergence of number systems is a cultural phenomenon tied to the applications there have been for numerosity concepts. It is the applicability of exact numbers to trade and other cultural phenomena that has most likely made the difference in the development of numerical ability in different cultures (Everett 2017). Consequently, we should be prepared to look for the explanation for linear number concepts also in cultural factors. Seen this way, number concepts are the result of *cumulative cultural evolution*, the process of transmitting and developing knowledge and skills across generations (Boyd and Richerson 1985, 2005; Tomasello 1999; Henrich 2015; Heyes 2018).

Perhaps it is possible to have both linearly and logarithmically structured early number concepts, but something in cultural contexts favors the linear structure and allows for the development of general concept of natural number. It is of course possible that the preference for a linear structure is a product of biological evolution, due to an innate bias for a recursive, linear number system. However, given the knowledge we have about the importance of cognitive tools like symbol systems and writing tools in number concept acquisition, it is more likely that instead of an innate bias, the preference is due to cultural factors. One of the earliest known such cognitive tools, tallying systems of keeping track of quantities by simple stroke notation, for example, can be conducive to the emergence of linearly structured number concepts (Ifrah 1998). Each stroke represents one event or object and a progressing sequence of strokes results in a linear notation.[15] Over generations, such stroke notations can be developed into, or merged with, existing numeral systems. The resulting cumulative cultural evolution of numbers can be a long process in which each new generation uses the OTS and the ANS to acquire number concepts. But importantly,

[14] This inclusion of phylogenetic considerations is not a new idea in the literature. It was already included by Piaget (1980) and is discussed by Carey (2009). But as will be seen, I propose a specific theoretical framework for combining ontogenetic and phylogenetic considerations, building on the work of Menary (2015).

[15] Of course a system of tallying could also represent a logarithmic number structure. However, in the known systems of tallying the marking system has been linear (see, e.g., Ifrah 1998).

they are instructed of the culture-specific numeral system which determines how their number concepts develop.

Starting from the appearance and development of number words, the history of natural numbers is tightly connected to their applications and to the modes of cultural transmission (Pantsar 2019b). For hunter-gatherer cultures like the Pirahã, having a linear exact number system may not have provided important advantages. They engage in trade in a limited fashion, they do not practice agriculture, nor do they store food or other goods (Everett 2017). Consequently, they do not face many of the cultural conditions in which large exact number concepts have proven to be useful. However, the Pirahã have the same computational constraints based on the OTS as we do. The fact that their quantity concepts did not develop according to the Carey–Beck theory is not due to a difference in the cognitive core systems. The different ontogeny of their number concepts (if they can be called number concepts) is due to a different phylogeny of number concepts (if any), which in turn cannot be considered independently of the cultural settings. As detailed in Sect. 2, it is essential for the Carey–Beck theory of bootstrapping that in ontogeny there is access to a list of number words that act as placeholders, which can be filled with semantic content. But the existence of such number words is a cultural development and not the product of biological evolution. However, the importance of culturally-shaped factors does not end in number words. As was seen in Sect. 2, in Beck's (2017) account also "counting games" are considered to be central in the bootstrapping process.

The importance of linguistic factors like systems of numbers words and cultural practices like counting games fits well with the framework of Menary (2015) in which he presents mathematical cognition as a case of *enculturation*. Enculturation refers to the transformative process in which interactions with the surrounding culture influence the acquisition and development of cognitive practices (Menary, 2015; Fabry, 2018; Pantsar 2019b, 2020a). New cognitive capacities can be acquired due to the neural plasticity of the brain that makes both structural and functional variations possible (Dehaene 2009; Ansari 2008). Menary (2014) calls the mechanism for this "learning driven plasticity" and it makes culturally developed cognitive abilities like reading and writing possible by redeploying older, evolutionarily developed neural circuits for new culturally specific functions (Dehaene 2009; Menary 2014). As well as for reading and writing, the enculturation framework has been used for explaining the development of mathematical cognition (Menary 2015; Pantsar and Dutilh Novaes 2020). So far, the literature on enculturation with regard to mathematical cognition has focused primarily on arithmetical abilities (e.g., Jones 2020; Fabry 2020), but here I want to show that the enculturation framework has great potential also in explaining the earlier process of number concept acquisition.

In particular, the enculturation account can in the present context help explain the connection between the phylogeny of number concepts and their ontogeny. It is clear that the emergence of natural number concepts is not due to any innate physiological difference in our brains, compared to the brains of people in anumeric cultures. Although it is impossible to know when the first extensive number systems were introduced, the oldest known systems of symbolic numerals are approximately 5000 years old (Schmandt-Besserat 1996). This would appear to be on a totally different time-scale from the kind of periods it would take biological evolution to give rise to numerical abilities (Dehaene 1997/2011; Menary 2015; Fabry 2020). In addition, bi-lingual members of anumeric cultures—e.g., the Mundurukú—have shown that they can learn to count, supporting the position that the capacity to count is not due to some recent evolutionary development (Gelman and Gallistel 2004; Gelman and Butterworth 2005).[16]

Consequently, unlike the core cognitive abilities with quantities, although initially very likely based on the OTS and the ANS, the ability with developed number concepts is not due to brain regions having developed for that purpose. Instead, the plasticity of our brains allows old neural circuits to be redeployed to new functions. This principle is called *neuronal recycling* (Dehaene 2009; Menary 2014) and it is a specification of what Anderson calls *neural reuse* (Anderson 2010, 2015; Fabry 2018; Jones 2020). Neural reuse refers to the general process of "circuits [continuing] to acquire new uses after an initial or original function is established" (Anderson 2010, p. 245). The neuronally recycled functions, Menary (2014, 2015) argues, are dependent on cultural practices.

Because it provides a clear hypothesis of the mechanism how cultural practices influence the development of cognitive processes, I see the enculturation framework as having great potential in explaining number concept acquisition. The reason for this that we can see that the ontogeny of number concepts is made possible by neuronal recycling [or more extensive neural reuse, as argued by Jones (2020) and Fabry (2020)], which is determined by the interactions we have in our cultural setting. Starting from learning languages with extensive numeral systems and getting the appropriate instruction, the ontogeny of number concepts is made possible by children's neural circuits starting to be redeployed for new numerical purposes. There is empirical evidence that this is indeed the case. The intraparietal sulcus, for example, is used for both the core cognitive and arithmetical treatment of quantities (Dehaene and Cohen 2007), suggesting that in learning arithmetic (including natural number concepts), we redeploy evolutionary developed neural circuits for processing quantities. Without a conducive cultural setting, however, this redeployment does not happen. Aside from numeral systems, the development of numerical cognition is shaped by many other culturally determined factors, such as cognitive practices and tools (Fabry and Pantsar 2019).

While the enculturation framework as presented above may sound like an alternative to the Carey–Beck theory, it is—I contend—in fact consistent with both Carey's (2009) original work and Beck's interpretation of it. Carey (p. 414) accepts that numeral systems[17] are social constructions and learning them is a social process, but contends that emphasis on the social aspects is not enough to explain how new concepts can be mastered. I agree with this. I believe that our core cognitive systems play a crucial role in the ontogeny of acquiring number concepts. Recall also that

[16] Interestingly, in the studies by Gelman and colleagues, the bi-lingual Mundurukú continued to use the ANS in problem solving tasks even though they knew the Portuguese counting words which would have given them exact solutions. This supports the view that the cultural context is crucial for acquiring mathematical knowledge and skills even if the required linguistic capacity is in place (Pantsar 2019b).

[17] Curiously, she only mentions the rational number system here and not the natural numbers.

according to Beck, in addition to the internal computational constraints due to the OTS, *external* computational constraints determine the bootstrapping process. Beck used the game of repeating the counting list while pointing to objects as an example of this. This game, from having the counting list to the particular gestures used in it, is clearly a product of cumulative cultural evolution. Games and other cultural practices form the basis on which children learn to use words and acquire the appropriate concepts. Indeed, the importance of cultural input is mentioned already by Carey, according to whom "the capacity to represent the positive integers is a cultural construction that transcends core cognition" (Carey 2009, p. 287).

This way, the present account should not be seen as an effort to refute the Carey–Beck theory. However, whereas Carey only points out that representing numbers is a cultural construction, I have wanted to make a specific proposal to enhance the bootstrapping theory to include cultural factors. Similarly, Beck (2017) writes about counting games without providing a theoretical framework for how cultural factors can shape our cognitive processes. In order to place the bootstrapping account in a coherent theoretical framework that can include both genetically determined and culturally shaped influences, I have wanted to introduce the enculturation account based on neuronal recycling (or neural reuse) to the debate on number concept acquisition. In turn, I believe that applying the enculturation account to number concept acquisition can enhance our present understanding of mathematical cognition as the result of processes of enculturation.

Recognizing the importance of the cultural input for the ontogeny of number concepts by the way processes of enculturation shape the cognitive process of number concept acquisition completes the hybrid model proposed in this paper. The two core cognitive systems (with others possibly also involved) both play a role in number concept acquisition. But equally important is the role of the cultural setting which determines what kind of number concepts are learned and how. Crucially for the present context, this also provides us with a solution to the stronger deviant-interpretation challenge. Besides the core cognitive systems, the number concepts we acquire are determined also by the culturally determined manner in which we are taught to count and use numerals.[18] Simply put, our number concepts do not follow a logarithmic structure primarily because, as the result of cumulative cultural evolution, we are *enculturated* with linearly structured number concepts. It is this external computational constraint that can finally complete the answer to the stronger deviant-interpretation challenge posed by the ANS.

[18] This trajectory of culturally shaped practices continues throughout the acquisition of mathematical knowledge and skills (Pantsar 2019b). These influences range from the effect of numeral systems on arithmetical operations like addition (Buijsman and Pantsar 2020) to diagrams and spatial arrangement of symbols (Fabry and Pantsar 2019). Importantly, the culturally shaped practices also influence the *complexity* of mathematical cognition, i.e., the amount of cognitive resources required for completing mathematical tasks (Pantsar 2019a, 2020b).

8 Conclusions

To sum up, the main difference of my hybrid account to the Carey–Beck account is ascribing a larger role for the ANS in the development of number concepts, as well as proposing a theoretical framework in which to research the cultural influence. The present account is not meant to refute the Carey–Beck bootstrapping theory. Indeed, I find that theory in general highly plausible. However, I have shown that the bootstrapping theory should also include considerations on the approximate number system, as well as an explanation of the way that culturally determined factors set external constraints for number concept acquisition. Based on the three-part theoretical model (OTS-ANS-Enculturation), I have argued that the resulting hybrid system can solve the stronger deviant-interpretation challenge.

As a final remark, we should note that the present hybrid account can also solve other potentially problematic formulations of the deviant-interpretation challenge, such as that presented by Rips and colleagues (2006). As noted in Sect. 3, they present the possible deviant interpretation that a natural number system based on the bootstrapping process could have loops in it. For example, the bootstrapping process could work like the numbers on a clock: after TWELVE the next number concept could again be ONE. This might be seen as just the kind of Kripkenstein-Goodman deviant-interpretation challenge that Beck saw as targeting all inductive inferences. While that could be the case (Rips and colleagues explicitly deny this (Rips et al. 2006, p. B58) while Beck (2017, p. 113) disagrees), we can see that the ANS could prevent the possibility of such loops in the number concept structure. The ANS is not limited to small quantities and based on it we would expect numbers to form a progression rather than a looped system. Whether we agree about the relevance of the deviant interpretation proposed by Rips and colleagues, however, we have seen that the deviant-interpretation challenge is more relevant than Beck allows for. Indeed, if Beck's (2017) account of answering the circularity challenge is successful, the deviant-interpretation challenge may prove to be the most important remaining difficulty with Carey's bootstrapping theory. In particular, I have identified one form of this challenge as the most pertinent problem in developing the bootstrapping account: the ANS-based case that I have called the *stronger* deviant-interpretation challenge to distinguish it from the general Kripkenstein problem of inductive inferences. In this paper, I have argued that by extending the framework beyond the OTS, this challenge can be met.

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