THEORIA

THEORIA, 2021
doi:10.1111/theo.12366

On What Ground Do Thin Objects Exist? In Search of the Cognitive Foundation of Number Concepts

by

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Abstract: Linnebo in 2018 argues that abstract objects like numbers are “thin” because they are only required to be referents of singular terms in abstraction principles, such as Hume’s principle. As the specification of existence claims made by analytic truths (the abstraction principles), their existence does not make any substantial demands of the world; however, as Linnebo notes, there is a potential counter-argument concerning infinite regress against introducing objects this way. Against this, he argues that vicious regress is avoided in the account of arithmetic based on Hume’s principle because we are specifying numbers in terms of the concept of equinumerosity, or its ordinal equivalent. But far from being only a matter for philosophy, this implies a distinct empirical prediction: in cognitive development, the principle of equinumerosity is primary to number concepts. However, by analysing and expanding on the bootstrapping theory of Carey in 2009, I argue in this paper that there are good reasons to think that the development could be the other way around: possessing numerosity concepts may precede grasping the principle of equinumerosity. I propose that this analysis of early numerical cognition can also help us understand what numbers as thin objects are like, moving away from Platonist interpretations.

Keywords: thin objects, bootstrapping, number concepts, Platonism, Linnebo, numerical cognition, epistemology of arithmetic

1. Thin Objects and Specification by Reference

Linnebo’s notion of “thin objects” has a two-fold Fregean foundation. First, he subscribes to the Frege (1953) idea that an analytic truth can make existence claims (Linnebo, 2018, p. 3). Second, he follows Frege (1892) in that objects are primitive notions that cannot be analysed logically. However, Linnebo does believe that the concept of an object is something that can be “glossed or characterized, for instance by relating it to other concepts and by explaining its role in our thought and reasoning” (Linnebo, 2018, p. 21). As far as abstract objects are considered, in Linnebo’s account they play a very particular role in our thought and reasoning; namely, they play referential roles in *abstraction principles*. Following Frege’s (1953) conception, an abstraction principle is a principle of the form:

$$\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta,$$

where α, β are variables that range over some items; $\sim$ is an equivalence relation on those items; and § is an operator that maps the items to objects (Linnebo, 2018, p. 8). An example of this from Frege (1953, p. 74) is the following abstraction principle:

> The direction of line A is the same as the direction of line B if and only if A and B are parallel.

In short, through the equivalence relation of lines being parallel, we are able to introduce new abstract objects, namely directions, by an analytic truth. Similarly, natural numbers as abstract objects can be introduced by the so-called Hume's principle, which states that the number of F is equal to the number of G if and only if F and G are *equinumerous*, that is, there is a one-to-one correspondence between F and G (Boolos, 1998, p. 181):

$$\#F = \#G \leftrightarrow F \approx G.$$

This kind of reference to abstract objects has been used by the neo-Fregeans Hale and Wright (2001, 2009) to argue for epistemologically unproblematic access to abstract objects. Simply by "carving up" previous propositional content (e.g., about equinumerosity) we get a criterion of identity for new concepts (e.g., natural numbers).

This is what Linnebo (2018) means by "thin objects": abstract objects like numbers and directions are referents of singular terms in the abstraction principles, and that is all they are required to be. As the specification of existence claims made by analytic truths, their existence does not make any substantial demands of the world. However, Linnebo argues that rather than carving up the propositional content, existence claims based on abstraction principles are due to *reconceptualization*:

> This reconceptualization must not be conflated with the notion of recarving of content. Where recarving is tied to the symmetric idea that one and the same worldly fact can be "carved up" as two different contents, reconceptualization (as I use the term) is tied to an asymmetric conception of abstraction; in particular, ontological commitments can increase as a result of reconceptualization. (Linnebo, 2018, p. 23, footnote 3)

This is an important point because Linnebo's account of thin objects is explicitly based on the asymmetry of abstraction principles:

> Why should the self-identity of a certain direction suffice for the parallelism of two particular lines? While abstraction on any suitably oriented line yields the relevant directions, there is no way to "retrieve" any particular line from this direction. Abstraction is a one-way road. What distinguishes one line from any of its parallels is irretrievably lost in the abstraction that takes us from a line to its direction. (Linnebo, 2018, p. 19)

As Linnebo (2018, pp. 39–40) points out, there is a potential counter-argument concerning infinite regress against introducing objects this way. Directions as abstract objects are specified by identity conditions referring to parallel lines. But how are lines specified? Linnebo claims that his account manages to avoid infinite regress by denying that the subject's relation to a specification is necessarily by reference (ibid., p. 40). In the case of reference to physical bodies, for example, the regress of reference ends because ultimately the last specification in the line of regress is by a purely causal relation between a piece of matter and a subject. Similarly, Linnebo argues, vicious regress is avoided in the account of arithmetic based on Hume's principle because we are specifying them in terms of the (Fregean) concept of equinumerosity, and "the appropriate relation to a concept is one of grasping or possessing, not one of referring" (ibid., p. 40).

Importantly, it is through this kind of implicit grasp of concepts in criteria of identity that mathematical *knowledge* is possible in Linnebo's account (ibid., p. 199). In his epistemological account, Linnebo argues that, first, abstraction principles establish mathematical truths; and second, competent mathematicians have beliefs because those beliefs are true (ibid., p. 201). Ultimately, explaining mathematical knowledge as knowledge of abstraction principles in this manner implies that there are abstraction principles that are fundamental to an area of mathematical knowledge, as is the case with Hume's principle and arithmetic. Thus, there is no infinite regress in Linnebo's epistemology of mathematics, and the foundation of mathematical knowledge is found in abstraction principles that – due to the asymmetry of abstraction – are grounded on particular concepts.

In accounts of arithmetic based on Hume's principle, such as the neo-Fregean account of Hale and Wright (2001), the concept in question is that of equinumerosity, that is, there being a one-to-one correspondence between collections of items. Infinite regress is avoided because natural numbers are specified by reconceptualizing the content of the principle of equinumerosity. Hume's principle treats natural numbers as cardinals, but Linnebo prefers an ordinal conception of natural numbers. The key idea behind Linnebo's (2018, p. 177) account of the ordinal conception is formulating a criterion similar to Hume's principle:

$$\mathrm{N}(\overline{\mathrm{m}}) = \mathrm{N}(\overline{\mathrm{n}}) \leftrightarrow \overline{\mathrm{m}} \sim \overline{\mathrm{n}}.$$

Here the operator *N* maps *numerals* to their associated numbers, thus grounding numbers not on the concept of equinumerosity but on the concept of numerals occupying the same positions in their respective orderings (pp. 177–178). Thus, the Roman numeral "VII" and the Hindu-Arabic numeral "7" specify the same number because they occupy the same position in the Roman and Hindu-Arabic counting lists, respectively. There are differences between the cardinal and ordinal accounts, some of which will be treated in section 3 of this paper, but for now it suffices to note that Linnebo believes that both the cardinal and ordinal conceptions of natural numbers are legitimate, at least in the context of abstraction principles (Linnebo, 2018, p. 178).

The important point in the present context is that, far from being a matter only for philosophy, the accounts of arithmetic based on abstraction principles imply specific empirical predictions: namely, that possessing or grasping the principle of equinumerosity (or the ordinal equivalent of it) is cognitively primary to possessing or grasping natural number concepts. I accept that this prediction carries intuitive appeal, but I will show that it is potentially problematic in terms of the state of the art in the empirical and philosophical literature on number concept acquisition. As detailed in the next section, there are good reasons to believe that the connection, at least in some stages of early cognitive development, can be the other way around: we possess some kind of early numerosity "proto-concepts" before we grasp the principle of equinumerosity or its ordinal equivalent.

The term "proto-concept" is used here cautiously. In the next section, we will see that the kind of early abilities with numerosities that I am concerned with are already possessed by infants and shared with many non-human animals. Hence, I do not want to suggest that they comprise conceptual cognition, let alone conceptual knowledge. Instead, by "proto-concepts" I am referring to cognitive capacities that are thought to be precursors of proper number concepts. This corresponds to the distinction between proto-arithmetical and arithmetical that I have introduced elsewhere (Pantsar, 2014, 2018, 2019). To avoid terminological confusion, I have suggested that abilities with numerosities that precede the acquisition of a rich enough system of exact number concepts should be called *proto-arithmetical* to distinguish them from proper arithmetical abilities with natural numbers. Therefore, by the term *proto-concepts* I refer to the functioning characteristics of the proto-arithmetical ability with numerosities, which may or may not be conceptual.[1] But regardless of the exact characteristics of proto-concepts, I will argue, in order to explicate the epistemological foundation of numbers, we should study the trajectory of cognitive development of arithmetical cognition. In some parts, our best understanding of this trajectory is potentially in conflict with Linnebo's account of the epistemology of arithmetic.

2. Bootstrapping of Number Concepts

As we saw in the previous section, in Linnebo's account the question of reference and knowledge concerning abstract objects comes down to there being grounding concepts that stand in a relation other than reference to the cognizing subject.

1 See, for example, Zahidi (2021) for an enactivist view against such early cognitive capacities being conceptual.

When we ask in what sense abstract objects exist, we thus need to trace the semantic connections and identify the basic concepts that are grasped or possessed in this non-referential manner. In order to achieve this, I argue, we cannot be satisfied with an a priori semantic or logical analysis. We need to move the focus to the basic cognitive abilities involved in number concept acquisition, which is a pursuit that cannot be conducted independently of empirical studies. Linnebo appears to be sympathetic toward this move into the study of cognition. He writes, for example, concerning arithmetical knowledge:

> But in my view the more pressing question is to what extent actual arithmetical practice provides such knowledge. In particular, how and to what extent do ordinary, educated laypeople achieve such knowledge?. (Linnebo, 2018, p. 179)

To answer that question, we need to trace the development to the very beginning of the ontogenetic acquisition of number concepts and basic arithmetical skills. Grasping abstraction principles like Hume's principle or its ordinal equivalent is only possible at a relatively advanced stage of cognitive development, by which time children already possess extensive numerical abilities (see, e.g., Pantsar, 2018, 2021a). Equally importantly, arithmetical practices are the product of millennia of development in culturally dependent contexts, and they vary between cultures (Ifrah, 1998; Everett, 2017; Pantsar, 2019). If our focus is on modern arithmetical practices, such as formal systems like the Dedekind-Peano axiomatization, our target phenomenon is only the tip of an iceberg of the development of arithmetical cognition. Therefore, both in terms of individual ontogeny and historical phylogeny, we must try to trace the cognitive development of arithmetic into primary capacities and abilities.

Again, Linnebo would appear to be sympathetic. Although the matter is only briefly discussed in a footnote, he acknowledges the relevance of the work of empirical researchers for his project:

> Indeed, most cognitive psychologists appear to think that our capacity for exact representations of numbers (other than very small ones) is based on our understanding of some system of numerals. See for example the influential account developed by (Carey, 2009). (Linnebo, 2018, p. 182)

Linnebo clearly sees the work of Carey and other cognitive psychologists as supporting his claim that exact representations of numbers are not the kind of primary concepts that are possessed in a non-referential manner. Rather, there are more basic forms of concepts that are cognitively primary and which make abstraction principles like Hume's principle possible, therefore giving us reference to abstract numbers. This has important consequences in terms of the empirical literature because it goes directly against *nativist* accounts of numerical cognition, according to which we possess innate number concepts (e.g., Gelman and Gallistel, 1978; Gallistel, 2017).

Nativist accounts, however, are increasingly unpopular in the empirical study of numerical cognition, and the state of the art in that field is leaning into the direction that more basic innate cognitive capacities involving numerosities are applied in acquiring number concepts. The work of Carey (2009) that Linnebo refers to provides one of the most prominent such accounts. Carey's account is called "bootstrapping," but it is important to distinguish it from Frege's "bootstrapping argument" that Linnebo discusses in his book. Indeed, Linnebo already implicitly points out this difference:

> Frege's famous "bootstrapping argument" [provides an argument for] the principle that every number has an immediate successor. Mathematically speaking, this argument is extremely elegant and interesting. But it is implausible as an account of people's actual arithmetical reasoning or competence. The argument was developed only in the 1880s and is complicated enough to require even trained mathematicians to engage in some serious thought. So this is unlikely to be the source of ordinary people's conviction that every number has an immediate successor. (Linnebo, 2018, p. 182)

In Carey's bootstrapping account, the purpose is different from Frege's. Instead of providing a mathematically compelling argument, Carey aims to explain precisely the thing Linnebo calls for: how ordinary people come to understand the principle that every natural number has an immediate successor.[2]

Here, Linnebo's remark in the footnote (p. 182) concerning the different status of "very small" numbers in terms of exact representation is central. What Linnebo is referring to is that, in Carey's account, the first three or four number concepts have a different cognitive status. They are acquired through the *object tracking system* (OTS), which is an innate "core cognitive" system that allows tracking multiple objects in the field of vision simultaneously (Knops, 2020).[3] The OTS is thought to be the cognitive basis for the ability of *subitizing*, that is, determining the amount of objects in our field of vision without counting, which is present already in neonates and shared with many non-human animals (Starkey and Cooper, 1980; Dehaene, 1997/2011). However, the subitizing ability stops working when the number of objects exceeds three or four, which is thought to be the limit of the object tracking system (Carey, 2009).

In Carey's bootstrapping account, number concepts in the subitizing range (from one to four) thus have a special place. Empirical data from cognitive

2 See Heck (2000) for an account of Fregean bootstrapping in a developmental context, which aims to avoid the cognitive implausibility that Linnebo points out in the Fregean account.

3 The object tracking therefore allows for parallel individuation of objects; hence, it is also sometimes called the *parallel individuation system*. The system is called *core cognitive* because it is thought to be innate and independent of other cognitive systems (Carey, 2009).

psychology backs this up. Children appear to acquire their first number concepts in an ascending order. At roughly 2 years of age, they become *one-knowers*, that is, they can consistently associate the numeral word “one” with one object. In steps typically taking about four or five months each, children then become *two-knowers, three-knowers*, *and four-knowers* (Knops, 2020). However, because the object tracking ability stops working after four objects, the child needs to make a cognitive leap in order to acquire number concepts larger than four. Instead of becoming a *five-knower*, children at this stage of development also typically grasp the meaning of the numerals for six, seven, and so on (Lee and Sarnecka, 2010). When they are able to generally match the last numeral uttered in the counting sequence with the cardinality of a group of objects, they are said to be become *cardinality-principle* knowers (Sarnecka and Carey, 2008; Lee and Sarnecka, 2011). In this manner, children are thought to “bootstrap” the successor principle (at least in an early form) and gain a general understanding of natural numbers (Carey, 2009; for more details and discussion, see Beck, 2017 and Pantsar, 2021a).

I presume that the bootstrapping account is what Linnebo is primarily referring to when he writes that: “cognitive psychologists appear to think that our capacity for exact representations of numbers (other than very small ones) is based on our understanding of some system of numerals” (Linnebo, 2018, p. 182). I also presume that by making the distinction between very small and other numbers, he is referring to numbers in the subitizing range. However, if this is correct, Linnebo’s interpretation is potentially at odds with Carey’s bootstrapping account in two ways: First, the system of numerals plays a role also for acquiring number concepts in the subitizing range. From research on anumeric cultures like the Amazonian people of Pirahã and Munduruku, it is known that having small exact number concepts is not a universal trait (Gordon, 2004; Pica et al., 2004). Not having stable numeral words for “one,” “two,” “three,” and “four” appears to imply that a child does not acquire those number concepts. This is predicted by Carey’s (2009) account, in which acquiring a placeholder list of numeral words plays a key role. The first stage of number concept acquisition is learning to recite a numeral word list. At this stage, a child can recite the list up to, say, “four,” but when asked for four things, she will give an amount randomly (Davidson et al. 2012). This corresponds to what Benacerraf (1965) calls *intransitive* counting, which refers to simply reciting the counting list “one,” “two,” “three,” “four,” and so on. In intransitive counting, there is no need to understand what the items on the counting list mean: they could just as well be “eenie,” “meenie,” “miney,” “mo,” and so forth. It is at the second stage of the bootstrapping process, the argument goes, that the meanings of the first four items of the counting list are connected to OTS-based mental models, resulting in the first exact number concepts (Carey, 2009; Beck, 2017).[4] But unlike suggested by Linnebo, at this stage access to the numeral word list already has played a role.

Ironically, this anti-exceptionalism concerning small numbers is actually in line with Linnebo's wider account because it emphasizes the importance of a system of numerals in number concept acquisition. This leads us to the more important way in which the bootstrapping account is potentially at odds with Linnebo's remark, namely the characterization of the cognitive process involved in number concept acquisition. Whereas access to a numeral word list is seen as a precondition for acquiring natural number concepts, it is not necessarily the case that the capacity for exact representations of numbers is based on our *understanding* of some system of numerals. In the second stage of the bootstrapping process, as detailed above, the child only needs to be able to *recite* the numeral word list. By counting games and similar practices, children learn to associate the first numeral words with OTS-based mental models (Beck, 2017; Pantsar, 2021a). Thus, children understand the meaning of numerals up to four, but they do not need to understand anything further about the system of numerals. Importantly, no further understanding is needed also in the third stage of the bootstrapping process, that is, the realization that when the quantity of objects is increased by one, the numeral word next in the counting list is associated with that quantity (Carey, 2009; Beck, 2017).[5] What is required of the learning process is access to a numeral word list and appropriate cultural practices, such as counting games (Beck, 2017).

It is therefore important to ask what kind of understanding of a numeral system is required. Recall that in Linnebo's account based on the ordinal equivalent of Hume's principle, reference to numbers is based on the concept of numerals occupying the same position in their respective orderings (Linnebo, 2018, p. 178). However, it is not clear that even this level of understanding is required for the initial acquisition of number concepts in the bootstrapping account. In acquiring number concepts early in ontogeny, a child may not grasp anything more than associating a numeral with a certain stage of a (transitive) counting procedure. Granted, when children acquire extended facility with numbers, understanding the numeral system becomes important because most languages show a recursive structure in their numeral words (Ifrah, 1998; Everett, 2017). But if the bootstrapping account is along the right lines, by this stage children already possess

4 Unfortunately, it is not possible here to go into the details of the "mental models" ascribed in the bootstrapping account, but the general idea is that each observed object occupies a separate mental "object file." In general, mental models can be thought to serve the kind of role I described for "proto-concepts" in the first section. See Carey, 2009; Beck, 2017; and Pantsar, 2021a for more.

5 In Benacerraf's (1965) terminology, this kind of ability to count items is *transitive* counting.

many number concepts.[6] Therefore, due to different interpretations of "understanding" in this context, it is potentially misleading to say that, under the bootstrapping account, acquiring exact number concepts requires understanding a numeral system. If it does, this is only in a very minimal sense of "understanding."

Aside from the above ontogenetic considerations about number concept acquisition, the issue of numerals and number concepts also concerns the phylogenetic emergence of them within cultures. As pointed out by Pelland (2018) in a partly different context, any satisfactory phylogenetic account of number concepts should account for the emergence of *first* number concepts. Unless we buy into nativist explanations, there existed a moment in time when people started possessing number concepts when there previously were none. But if acquiring number concepts requires understanding a numeral system, we must ask how numeral systems could have emerged before there were number concepts.

Unfortunately, it is not possible to give this topic a full treatment here; however, I see great promise in the account proposed by Wiese (2007), according to which number concepts and numeral words have co-evolved. Words originally not used for signifying quantity acquired new numerical purposes, as evidenced by, for example, in the Hup language the word for "two" meaning *eyes* and the word for "three" being also the word for a rubber plant seed that has three chambers (Epps, 2006; see also Dos Santos, 2021). If this co-evolution account is correct, we cannot answer the chicken-and-egg question which came first, number concepts or numeral words. However, regardless of the success of that account, it seems implausible that numeral words would have emerged independently of number concepts. But the question is, can we trace the development of number concepts (and numeral words) to some more basic cognitive principles?

3. What Are the Primitive Concepts in Arithmetic?

The considerations above prove to be surprisingly relevant for Linnebo's project of explaining the thin existence of abstract objects such as natural numbers. Recall that Linnebo avoided infinite regress in his accounts of reference and knowledge because at some point subjects simply grasp or possess concepts. He accepted both the principle of equinumerosity and, his preferred version, its ordinal equivalent as legitimate accounts for this purpose. Therefore, Linnebo's account appears to imply that both equinumerosity and its ordinal equivalent are epistemologically primary concepts over natural number concepts. However, if

6 It depends most likely on the language how many number concepts can be acquired without understanding the recursive structure of the numeral system. In English, numeral words start showing recursivity after 13, whereas in Mandarin, for example, this starts after 10.

the bootstrapping account presented in the previous section is along the right lines, it provides reason to doubt the cognitive primariness of both the concept of equinumerosity and its ordinal equivalent. With the concept of equinumerosity, this problem is clearly present. In ontogeny, the first number concepts are acquired by connecting OTS-based mental models to words in a numeral word list. Certainly, this process can be *described* in terms of applying a principle of equinumerosity. In successful (transitive) counting procedures, children count to the proper word in the numeral word list and associate the final word with the number of objects. But there is no reason to believe that they possess an earlier conceptual understanding of the principle of equinumerosity. Indeed, as Buijsman (2019) notes, experiments show that, although children are able to recognize that two sets can be put in a one-to-one correspondence, before they learn the cardinal interpretation of numbers words they are unable to make even simple inferences about the cardinality of sets.[7] Buijsman argues that this makes it unlikely that the concept of equinumerosity is behind children's conception of numbers.

I agree with this and consider it more likely that, when engaging in counting games, children parallelly acquire the first number concepts and an early understanding of the concept of equinumerosity. Indeed, based on the bootstrapping process as described by Carey (2009) and Beck (2017), it would appear that number concepts in the subitizing range could actually be cognitively primary to any understanding of equinumerosity. This would be the case if the numbers words would be attached to the mental models based on the object tracking system in a more direct manner, which is a possibility in the present understanding of the bootstrapping account.

An important consequence of all this is that we can no longer be certain of the direction in which Hume's principle works in terms of reference. Because equinumerosity is not necessarily primary to number, it is conceivable that it is actually the concept of equinumerosity that is specified by the cognitively primary number concepts, reversing the order. This would be at odds with Linnebo's account of thin objects because numbers as abstract objects would not be specified through an abstraction principle. Instead, numbers as objects would be the referents of number concepts that are possessed earlier in the ontogenetic development. If that were the case, we would need to ask what kind of ontology is required for numbers. Hume's principle would still work as specification by

7 More specifically, children fail to make the inference from "these sets have the same number of elements" and "this set has *n* elements" to "the other set has *n* elements" before they learn to have a cardinal interpretation of *n* (Buijsman, 2019; for reports of the experiments, see Sarnecka and Gelman, 2004, Sarnecka and Wright, 2013, Izard et al., 2014).

reference, just as Linnebo intended. But it would be reference based on an order that is the reverse of the actual cognitive basis of concepts.

However, whereas Linnebo accepts the account based on Hume's principle as legitimate, his own preferred account of number concepts is based on an ordinal conception. Could the ordinal conception be better in line with the empirical data on children's number concept acquisition? It certainly is the case that the above problem noted by Buijsman (2019), for example, does not apply to the ordinal conception. To the best of my knowledge, there are no comparable empirical studies that have specifically tested children's understanding of numbers in terms of the ordinal conception.[8] Nevertheless, if Carey's (2009) bootstrapping account is along the right lines, it would suggest that the first numerosity concepts are in fact cardinal. If the early numerosity concepts are based on the object tracking system, it would seem more likely that they are cardinal. The subitizing ability, for example, does not include counting or other kind of ordered process. This primariness of cardinality over ordinality also receives support from the empirical data (see, for example, Colomé and Noël, 2012 for a study of ordinal and cardinal numeral word acquisition in 3–5 year olds).

However, when counting starts to play an increasingly important role, the numerosity concepts are more likely to be ordinal. Hence, it could be that the early numerosity concepts are initially cardinal, but in the bootstrapping process they become ordinal – or that early on in ontogeny children possess different types (ordinal or cardinal) of number concepts for numbers in the subitizing range (one to four) and larger numbers. Obviously, more empirical data is needed before any such claims could be made. But I believe that the question of ordinality and cardinality, just like the question of epistemology of numbers in general should be approached philosophically with an awareness of the empirical research on number concept acquisition. Indeed, as I have argued in this paper, this kind of approach can complement philosophical accounts like the one developed in Linnebo (2018).[9]

It should be noted that I am not claiming that Linnebo's account of number concepts based on abstraction principles is mistaken. Indeed, there could be a way to

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8 This is not to say that children's understanding of ordinal numbers has not been studied empirically, although the focus has mostly been on cardinal numbers. The findings, however, are mixed. Colomé and Noël (2012) report data supporting that the acquisition of cardinal and ordinal meanings are related, whereas Brannon and Van de Walle (2001) present data that verbal cardinal numerical competence is largely unrelated to ordinal competence. Generally, there is extensive evidence that there is a double dissociation between the neural activation in cardinal and ordinal judgement tasks (Knops, 2020, p. 13); however, it should also be noted that Lyons and Beilock (2013) argue that the differences between ordinal and cardinal judgements are important for the symbolic representation of numerosities. Certainly, there are important possibilities for future research regarding, for example, the differences between ordinal and cardinal numeral words in different languages.

9 See Buijsman (2021) for an account that develops the semantic individuation of natural numbers in the context of Linnebo's work.

interpret Linnebo's account in a manner that is consistent with the empirical data. Whereas Linnebo's approach is focused on conceptual understanding, the proto-arithmetical abilities may not be conceptual in the same sense. It could be, for example, that the concepts discussed in the bootstrapping account are "proto-concepts" in relation to actual number concepts, and fully formed exact number concepts arise after there is a precise and general understanding of the principle of equinumerosity or its ordinal equivalent. In general, as mentioned earlier, it is important to distinguish between proto-arithmetical (concerning *numerosities*) and arithmetical abilities (concerning *numbers*). Even if the bootstrapping account were along the right lines, we would need to make sure that the numerosity concepts in that theory are sufficiently similar to number concepts as discussed by Linnebo.

However, perhaps the best way to understand Linnebo's approach in the present context is to focus on *numerals* instead of number concepts. Given that in the bootstrapping account the acquisition of a numeral word list is important already for small numerosities, it could be that an aspect of ordinality enters number concept acquisition originally through learning a stable order of numeral words. Whereas the first number concepts may be cardinal, as suggested by the bootstrapping account, this initial aspect of ordinality could already play a role in acquiring the first number concepts. Thus, grasping the ordinal characteristics of the numeral word list would, after all, precede the acquisition of cardinal number concepts. Given that Linnebo's ordinal equivalent of Hume's principle is based on numerals occupying the same positions in their respective orderings (Linnebo, 2018, pp. 177–178), this approach could provide fruitful in combining Linnebo's account with cognitive data on number concept acquisition.[10]

Ultimately, the relation between ordinality and cardinality in number concept acquisition is a topic in which both empirical and philosophical research can help. What I have been arguing for in this paper is that the cognitive basis of number concepts according to the bootstrapping account is potentially in conflict with Linnebo's account. However, as detailed above, this is not necessarily the case, and there could be ways to dissolve the tension. Rather being damaging for either the bootstrapping account or the thin objects account, I see this potential conflict as a reason for further research on the topic, both empirically and philosophically.

4. Conclusion: The Ontology of Numbers

Above I briefly discussed cardinality and ordinality of numbers as an important question about the epistemology of thin objects in terms of the cognitive foundations of number concepts. Perhaps the most fundamental question about thin

10 I want to thank an anonymous reviewer for pointing this out.

objects, however, concerns Platonism and the ontological status of mathematical objects. In Linnebo's account:

> The most straightforward understanding [of Platonism] is as a form of counterfactual independence. Had there been no intelligent agents, there would still have been mathematical objects, related to one another just as they in fact are. Is this counterfactual true? I believe the answer is a resounding "yes". Pure mathematical objects exist necessarily, if at all, and their relations to one another obtain by necessity. It follows that, however things had been, pure mathematical objects would have existed and been related to one another just as they in fact are. Indeed, the universe of pure mathematical objects would have remained the same even in circumstances far more remote than the nearest ones in which there is no intelligent life. (Linnebo, 2018, p. 189)

Based on my considerations above, can the answer still be a resounding "yes"? Had there been no intelligent agents, the cognitive trajectory outlined in the previous section would never have taken place. If number concepts are in some integral way grounded on evolutionarily developed cognitive capacities, such as the object tracking system, it would appear to follow that without agents with those capacities, number concepts would not exist.[11] In that scenario, would there still have been mathematical objects, that is, could numbers exist without number concepts? Of course, that is possible, but unless there is an underlying platonist commitment, what reason is there to believe in the existence of numbers independently of the existence of number concepts? I have proposed that arithmetical knowledge, including that of the abstraction principles, is (at least partly) based on evolutionarily developed proto-arithmetical abilities. If that is the case, we can explain the existence of numbers as thin objects without evoking a Platonist ontology. If numbers are only required to be referents of singular terms in abstraction principles, and we can explain the emergence of the relevant abstraction principles through contingent cognitive capacities, with Occam-like reasoning there is no reason to believe in the necessary existence of numbers.

To be sure, I do not want to claim that Linnebo's account of thin objects is necessarily in conflict with my reasoning here. Linnebo writes, for example:

> *It is possible to have knowledge of abstract mathematical objects, provided that these objects are regarded as thin*. (Linnebo, 2018, p. 203; italics original)

I have no qualm with that, but the way I understand thin objects includes the possibility that they would not have existed because their existence may be tied to cognitive processes for which evolutionary appearance was contingent.

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11 Another evolutionarily developed capacity discussed as the basis of our arithmetical abilities is the *approximate number system*, which allows for rough estimations of sizes of collections without counting (see, e.g., Dehaene, 1997/2011). See Pantsar (2021a) for a way of connecting the approximate number system to the bootstrapping account.

Admittedly, this can make my account incompatible with Linnebo's, who stresses that the existence of thin objects makes no substantial demands of the world. My account makes one such demand, namely, that there are agents that possess number concepts. But how important is this difference? Metaphysically, there is of course an important difference between numbers existing in all possible worlds and them existing in all possible worlds with intelligent agents who possess number concepts. But approaching the epistemology of arithmetic from a cognitive point of view, we are always limiting the examination – in a variation of the anthropic principle – to possible worlds in which there are agents that can possess number concepts. In these possible worlds, I contend, numbers have thin existence in the sense that, given there are agents that possess number concepts, the existence of numbers makes no *further* ontological demands of the world.

This also gives us the answer to the question presented in the title of this paper: on what grounds do thin objects exist? In the present account, this ground is the existence of cognitive agents that possess number concepts. In any such possible worlds, however, the existence or non-existence of numbers does not make demands of the world. Thus, in the present context, the existence of numbers is best described as thin, while acknowledging the potential difference to Linnebo's account regarding possible worlds without cognitive agents that possess number concepts.

One final matter should be addressed. Because in the account I am proposing mathematical objects are tied to the contingent fact that certain evolutionary processes (and cultural developments) have taken place, is there a danger of losing all that is considered to be central to the character of mathematical knowledge, namely, apriority, necessity, and objectivity? I have proposed elsewhere that these notions should be understood in a contextual sense (Pantsar, 2014, 2015). Because the object-tracking system, for example, is a universal characteristic of humans, its role in the ontogeny of number concepts determines that number concepts – if indeed developed – develop along similar trajectories across cultures. In earlier work (Pantsar, 2014), I have called such knowledge *maximally intersubjective*. Seen in this way, arithmetical knowledge does not survive the counterfactual challenge presented by Linnebo. But as I have argued elsewhere, it can still retain apriority (Pantsar, 2014), necessity (Pantsar, 2016), and objectivity (Pantsar, 2021b) in a strong sense of each term, applying to all human cultures who have developed or will develop arithmetic. And although the understanding of mathematical objects in my theory is not Platonist, I have proposed that it can still be compatible with Linnebo's general account of thin objects.

Acknowledgements

I want to thank Stefan Buijsman and Øystein Linnebo for extremely helpful comments on the first version of this paper. I also wish to thank Regina E. Fabry and Bahram Assadian for discussions that have helped me formulate the main arguments. This research has been supported by the Finnish Cultural Foundation (Suomen Kulttuurirahasto). Open Access funding enabled and organized by Projekt DEAL.

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