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Three concepts of decidability for general subsets of uncountable spaces

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Abstract

There is no uniquely standard concept of an effectively decidable set of real numbers or real n-tuples. Here we consider three notions: decidability up to measure zero [M.W. Parker, Undecidability in $\mathbf{R}^n$: Riddled basins, the KAM tori, and the stability of the solar system, Phil. Sci. 70(2) (2003) 359–382], which we abbreviate d.m.z.; recursive approximability [or r.a.; K.-I. Ko, Complexity Theory of Real Functions, Birkhäuser, Boston, 1991]; and decidability ignoring boundaries [d.i.b.; W.C. Myrvold, The decision problem for entanglement, in: R.S. Cohen et al. (Eds.), Potentiality, Entanglement, and Passion-at-a-Distance: Quantum Mechanical Studies fo Abner Shimony, Vol. 2, Kluwer Academic Publishers, Great Britain, 1997, pp. 177–190]. Unlike some others in the literature, these notions apply not only to certain nice sets, but to general sets in $\mathbf{R}^n$ and other appropriate spaces. We consider some motivations for these concepts and the logical relations between them. It has been argued that d.m.z. is especially appropriate for physical applications, and on $\mathbf{R}^n$ with the standard measure, it is strictly stronger than r.a. [M.W. Parker, Undecidability in $\mathbf{R}^n$: Riddled basins, the KAM tori, and the stability of the solar system, Phil. Sci. 70(2) (2003) 359–382]. Here we show that this is the only implication that holds among our three decidabilities in that setting. Under arbitrary measures, even this implication fails. Yet for intervals of non-zero length, and more generally, convex sets of non-zero measure, the three concepts are equivalent.
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1. Introduction

There is no one standard, natural, and useful notion of an effectively decidable set of real numbers or real n-tuples. The obvious naïve definition would be,

Definition 1.1. A set $A \subseteq \mathbf{R}^n$ is (*naïvely*) *decidable* if and only if its characteristic function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

is a computable function on the reals (as defined by Grzegorczyk [6–8]).

Unfortunately, this notion is practically empty. Grzegorczyk-computable functions are continuous [6,7], and the only subsets of $\mathbf{R}^n$ that have continuous characteristic functions are $\mathbf{R}^n$ and $\emptyset$. However, some sets of reals are intuitively

more computable than others, and accordingly, various relaxed notions of a decidable or recursive set of reals have been introduced [4,9,10,12–15,21,24] (see [24] for other references). Unlike the several concepts of decidability over **N** introduced in the 1930s, most of these concepts are far from equivalent.

Here we adopt the main computability concepts of Weihrauch's type-2 theory of effectivity (TTE) [4,11,11,20,21,24], though our approach to computable sets will differ in some respects. In TTE, as here, computations are performed by a very Turing-like finite state machine that operates one symbol at a time on infinite strings representing points in a continuum. Much attention is given to the role of topology in computability.

In the TTE literature, though, studies of more or less computable sets are typically restricted to certain classes of nice sets, such as the convex [12], regular [24], or open or closed [4,21] sets. One defines recursive and non-recursive closed sets, for example, without classifying sets like the interval [0, 1). This approach has definite advantages. For one, it limits the cardinality of the class of sets under consideration to 2^ω, so that every set can be uniquely represented by a countably infinite string. The computability of sets then reduces to the computability of corresponding strings. However, in applications one may encounter sets that are not open, closed, convex, or regular, or one may simply not know whether a given set has such properties. Hence it also seems useful to discuss the computability of arbitrary sets in $\mathbf{R}^n$, as do Ko and others [2,3,9,10,13–15].

In TTE, notions of computability are usually defined in terms of representation systems and the computability of the strings representing a point or set. However, when we come to consider the decidability of arbitrary sets, such explicit reference to representations will not be as useful as it is for the restricted classes of sets usually considered in TTE. Since there are more than 2^ω arbitrary sets in a continuous space, there is no way to represent each such set with a distinct infinite string. Furthermore, much of the subtle dependence on representations, to which one must attend when considering, say, real-valued functions, falls away when we consider yes-or-no decision problems. Here we will adopt notation closer to Ko's than Weihrauch's and more convenient for discussing the decidability of arbitrary sets.

It is also worth noting that the TTE notions of recursive open and closed sets [4,9,21,23] are not notions of *decidability*. To give brief definitions, we first define r.e.: an *r.e. open* set is the union of a set of open balls whose centers and radii are given by a recursive sequence of rationals. An *r.e. closed* set is a closed set that contains a dense recursive sequence of rational points. A *recursive open/closed* set is then just an r.e. open/closed set with an r.e. closed/open complement. These sets have several properties analogous to those of recursive sets in **N**:

  (i) a recursive set is an r.e. set with an r.e. complement,
 (ii) the complement of a recursive set is recursive, and
(iii) a function defined on a recursive closed and bounded domain is recursive if and only if its graph is a recursive closed and bounded set [23].
     Furthermore,
 (iv) an open or closed interval is recursive if and only if its endpoints are computable numbers,

and there is yet more to recommend the notions of recursive open and closed sets as generalizations of the usual notion of recursive set [21].

However, *decidability* of a set *A*, as defined by Gödel in 1934 [5], consists in the existence of an effective procedure to decide in finite time whether *or not* a given object is in *A*. This is equivalent to the existence of a computable characteristic function, which as noted, is essentially impossible for sets of real numbers. Yet, we might still seek algorithms that *usually* tell us whether or not an object is in a set, perhaps permitting some limited errors or omissions. The TTE notions of recursive open and closed sets do not imply even such near-decidability. If a set *A* is recursive open (or just r.e. open) there does exist an algorithm that tells us when $x \in A$, and does not halt if $x \notin A$. However, this does not guarantee an algorithm to tell us with any reliability when a point is *not* in *A*. [1] In this respect, "recursive open" and "recursive closed" are rather asymmetrical and unlike the notion of a decidable set, despite their other virtues. [2]

Here we consider three notions that come closer to decidability: Myrvold's decidability ignoring boundaries (d.i.b.) [13], Ker-I Ko's recursive approximability (r.a.) [10], and decidability up to measure zero, or decidability "mod zero"

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[1] Admittedly, "recursive open" does guarantee an algorithm to tell us when a point is *near* another that is not in the set [21, p. 127], but that gives us no information, even of a probable kind, about whether the point itself lies outside the set. In this respect it is unlike a notion of decidability.

[2] I am grateful to Vasco Brattka and Qing Zhou for an enlightening discussion of these considerations at RNC'5.

(d.m.z.) [14]. [3] As defined here, these notions apply to subsets of any separable metric space with a dense computable sequence and equipped with an outer measure (a generalized notion of length, area, or n-dimensional volume). They can be further generalized to second-countable, T_0 topological spaces in the style of [4,21], or even to convergence spaces [16]; we assume a metric here for convenience only. As a way of getting to know these concepts, we demonstrate the logical relations between them.

Informally, a set A is d.i.b. if there is some algorithm that correctly decides membership in A except on the boundary of A, where the algorithm does not halt. This is closely related to the notion of a "weak characteristic function" [12], but the latter was applied in [12] only to convex sets, and in [24] to regular sets. D.i.b. is also close to but slightly weaker than the notion of a Δ-decidable set [9], i.e., a set A such that the distance functions for both A and its complement are computable.

A set A is r.a. with respect to a measure μ if there is an algorithm that correctly decides A except on some set with arbitrarily small μ-measure. That is, given a parameter $n > 0$, the algorithm correctly decides A except on some set with μ-measure less than 2^{-n}, where it halts but with incorrect output. This is intended to pick out those sets that can be decided with a probability of correctness arbitrarily close to 1 [10]. It does not, as some have thought, imply that a set can be decided with a probability of *exactly* 1.

Decidability up to measure zero does pick out those sets that can be decided with probability 1, given an appropriate measure [14]. Informally, A is μ-d.m.z. if there is an algorithm that decides A except on some set with μ-measure zero, where the algorithm may not halt or may give incorrect output. It has been shown [14] that certain interesting sets in physics that have been thought intuitively undecidable, such as "riddled" basins of attraction [1,18] and the unions of KAM tori for nearly-integrable systems, are not d.m.z. with respect to Lebesgue measure, [4] though there is reason to believe that such sets are r.a. and d.i.b. In fact, in view of the KAM results, d.m.z. represents a precise and meaningful sense in which the problem of the stability of the solar system may well be undecidable [14].

Here we will see that for intervals of non-zero length, d.m.z., r.a., and d.i.b. are all equivalent, and hold for precisely those intervals with computable endpoints. (Thus they satisfy condition (iv) above, and in fact they also satisfy (ii).) More generally, our three concepts are equivalent for full-dimensional convex sets in $\mathbf{R}^n$. Yet for general sets in a separable metric space with an arbitrary outer measure, they are independent: no two imply the third, nor the negation of the third. In the middle ground of *arbitrary* sets in $\mathbf{R}^n$ with Lebesgue measure, it has been shown that d.m.z. is strictly stronger than r.a. [14]. We will see here that this is the only implication that holds among our three relaxed decidabilities in that setting.

2. Preliminaries

We write int(A) for the interior of a set A and ext(A) for the exterior. For a set Σ, we let Σ^* denote the set of finite sequences of elements of Σ. However, a lowercase greek letter with asterisk μ^* denotes an outer measure, and μ the measure obtained by restricting μ^* to those sets on which it is countably additive. In particular, λ will always denote Lebesgue measure.

We are concerned here not with analog computation but with computation as performed by human beings and digital computers, and as conceived by Turing [19]: the systematic manipulation of finite symbol strings. In the context of an uncountable space, where there is no way to represent each element uniquely with a single finite string, we define computability in terms of sequences of finite approximations. For this it is helpful to have a notion of distance, and finitely coded elements arbitrarily near to any given element. Hence a separable metric space, i.e., one with a countable dense subset, makes a convenient context.

Fix a finite alphabet Σ. Then,

Definition 2.1 (*Weihrauch [21]*). Let (X, d) be a separable metric space and A a countable dense subset of X.

(1) A *notation* of A is a map $v : \Sigma^* \to A$ onto A.
(2) The quadruple $\mathbf{X} = (X, d, A, v)$ is called an *effective metric space* and the elements of X are called *points*.

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[3] In [14] this is called decidability in μ, or d-μ, but 'decidability up to measure zero' is more self-explanatory.

[4] Lebesgue measure is the standard notion of length, area, or n-dimensional volume. The precise definition can be found in any analysis text but is not critical to this paper.

("Effective" here means only that the space supports a theory of effectiveness or computability, since certain elements are given finite names on which computations can be performed. Neither the notation v nor any other component of an effective metric space need be effective.) Henceforth, X will denote the domain of some effective metric space $\mathbf{X} = (X, d, A, v)$, and most definitions will thus be implicitly relativized to a metric d and notation v.

We also fix effective one-to-one notations $v_\mathbf{N} : \Sigma^* \to \mathbf{N}$ for the natural numbers and $v_\mathbf{Q} : \Sigma^* \to \mathbf{Q}$ for the rationals. By "effective" here, we mean that over all strings $s_1, s_2 \in \Sigma^*$, the string relations $v_\mathbf{N}(s_1) < v_\mathbf{N}(s_2)$ and $v_\mathbf{Q}(s_1) < v_\mathbf{Q}(s_2)$ are recursive in the usual sense for string functions. We will sometimes ignore the distinction between numbers and their notations if no ambiguity arises. For example, given $n \in \mathbf{N}$ and a function $f : \Sigma^* \to \Sigma^*$, we might write $f(n) = q \in \mathbf{Q}$ where literally we mean that if $v_\mathbf{N}(s) = n$ then $v_\mathbf{Q}[f(s)] = q$. Wherever we discuss the real line $\mathbf{R}$ below, we have in mind the effective metric space $(\mathbf{R}, \delta, \mathbf{Q}, v_\mathbf{Q})$, where δ is the standard Euclidean metric.

Given finite names for the elements of a countable set that is dense in some space, arbitrary points in that space can be represented by rapidly converging sequences:

Definition 2.2. For any effective metric space $\mathbf{X} = (X, d, A, v)$ and $x \in X$, let

$$CF_x^< = \{\{\phi_n\} \in \Sigma^{*\mathbf{N}} \mid (\forall n \in \mathbf{N})d[v(\phi_n), x] < 2^{-n}\}.$$

If $\phi = \{\phi_n\} \in CF_x^<$, we call ϕ a *strictly regular Cauchy name* for x.

This differs from the usual notion of a Cauchy sequence in its requirement of rapid convergence—hence the word 'regular,' which we will sometimes omit. It also differs from Ko's definition of 'Cauchy function' in that we use a strict inequality (hence 'strictly regular') where Ko's is inclusive. Systems of regular Cauchy names are computationally equivalent whether this inequality is strict or inclusive [21, p. 88], so the choice does not affect the extension of our decidability concepts. However, choosing strict inequality gives a more elegant form to Proposition 2.7 below. Regular Cauchy representations are also equivalent to Weihrauch's "standard" representations, wherein a point x is represented by a list of open rational cubes containing x [21, p. 88].

Our computability and decidability concepts will be based on the notion of an algorithm or machine supplied with an infinite input string, and perhaps also a finite input string or parameter. (It will often be convenient to regard some finite parameter as distinct from the input.) We will use the following notation, similar to that of Ko [10]:

Definition 2.3. Let $\phi \in \Sigma^{*\mathbf{N}}, \Sigma^*$, or $\Sigma^{*\mathbf{N}} \times \Sigma^*$ and $s \in \Sigma^*$. If M is a machine that, given input ϕ, writes the finite output string s and halts, we write $M(\phi) \downarrow$ and $M(\phi) = s$, or to state both explicitly, $M(\phi) \downarrow = s$. If M does not halt given input ϕ, we write $M(\phi) \uparrow$.

We have given no account of what exactly an algorithm or machine can do with an infinite input. Ko [10] and Weirauch [21] do so by specifying the details of appropriately modified Turing machines. Here we assume that such details do not matter. Specifically, we adopt a modified Church–Turing thesis for functions on infinite arguments:

A generalized Church–Turing thesis: Given a finite alphabet Σ, the functions $M :\subseteq \Sigma^{*\mathbf{N}} \to \Sigma^*$ that can be computed by any finite-state machine or algorithm in finitely many steps are precisely those that can be computed by a two-tape Turing machine, where the first tape supplies the infinite input $\phi \in \Sigma^{*\mathbf{N}}$ while scratch work and output are printed on the second tape. [5]

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[5] This is only faintly stronger than the generalized Church–Turing thesis used in the study of r.e. degrees [17]. The latter states that the functions $M :\subseteq \Sigma^* \to \Sigma^*$ on finite strings that can be computed by any algorithm, using reference data supplied by a *particular* infinite sequence $\phi \in \Sigma^{*\mathbf{N}}$, are precisely those computed by some two-tape Turing machine supplied with that *same* particular sequence ϕ. (Such a machine is called an oracle Turing machine and ϕ an oracle.) Here we merely allow ϕ to vary and regard it as an input, thus computing functions on infinite arguments.

In fact, even a two-tape machine is not necessary; the same computations could be carried out by an ordinary one-tape Turing machine, given appropriate input and output conventions. It makes little difference what sort of finite-state, discrete computing machine we take as our model. [6]

The Use Principle of recursion theory [17], or part of it, has topological implications for recursive analysis and is useful for demonstrating certain kinds of undecidability. Essentially, the Use Principle says that if a machine halts, it uses only finite information:

Definition 2.4. (1) Let $s : \{0, 1, \ldots, k\} \to \Sigma^*$ be a finite sequence of strings. For any $\phi : \mathbf{N} \to \Sigma^*$, we write $s \subset \phi$ and $\phi \supset s$ if $s(i) = \phi(i)$ for all $i \in dom(s) = \{0, 1, \ldots, k\}$.
(2) We write $M(s, n) \downarrow = x$ if for every sequence $\phi \supset s$, $M(\phi, n) \downarrow = x$. Otherwise, we write $M(s, n) \uparrow$.

Proposition 2.5 (*Use Principle*). $M(\phi, n) \downarrow = x \Leftrightarrow (\exists s \subset \phi) M(s, n) \downarrow = x$.

Proof. See [17]. $\square$

This implies that, roughly speaking, if a machine M halts on input x, it halts on an open neighborhood of x, since M only makes use of approximate information about x. To show this (and other propositions), we will refer to the set $U(s)$ of points in X that have strictly regular Cauchy names beginning with the finite sequence s of finite strings. That is,

Definition 2.6. Given an effective metric space **X**,

(1) for any $x \in X$ and $\varepsilon \in \mathbf{R}$, let $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$;
(2) for any $s \in \Sigma^{**}$ (the set of finite strings of finite strings), let

$$U(s) = \bigcap_{i=1}^{length(s)} B(v(s_i), 2^{-i}).$$

Proposition 2.7 (*Topological Use Principle*). [7] *If $M(\phi, n) \downarrow = q$ for some $x \in X$ and $\phi \in CF_x^<$ then there is a neighborhood U of x (in the topology induced by the metric d) such that $(\forall y \in U)(\exists \psi \in CF_y^<) M(\psi, n) \downarrow = q$.*

Proof. Assume the antecedent. By the Use Principle, choose a finite sequence s such that $s \subset \phi$ and $M(s, n) \downarrow = q$. Then $U(s)$ will satisfy the consequent. To see this, let $y \in U(s)$. Choose $\theta \in CF_y^<$ and let

$$\psi(i) = \begin{cases} s(i) = \phi(i) & \text{if } i \leqslant length(s), \\ \theta(i) & \text{otherwise.} \end{cases}$$

Then $\psi \in CF_y^<$, and by the Use Principle, $M(\psi, n) \downarrow = M(s, n) = q$. $\square$

Finally, we will need the concepts of a computable point and a semi-recursive or r.e. open set:

Definition 2.8. (1) A point $x \in X$ is *computable* if there exists a Turing machine M such that for all $n \in \mathbf{N}$, $M(n) \downarrow \in X$ and $d[M(n), x] \leqslant 2^{-n}$.
(2) A set A is *r.e. open* if there is a machine M such that for all $x \in X$ and $\phi \in CF_x^<$, $M(\phi) \downarrow \Leftrightarrow x \in A$.

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[6] One caveat: Weihrauch permits his Turing-like "Type-2 machines" to output infinite strings, with the restriction that the head on an output tape can only move from left to right. This ensures that one need not wait for eternity in order to see whether a given output symbol will remain in the final output. Here, though, we adopt the approach of Ko and others, where the output of a computation is always finite; the computability of infinite sequences and of functions with infinite output is then defined in terms of sequences of finite outputs. The demands imposed by these definitions have the same effect as Weihrauch's one-way restriction on infinite outputs.

[7] This is closely related to well-known results, e.g., if $f :\subseteq \Sigma^\mathbf{N} \to \Sigma^*$ is computable then dom(f) is open (in the standard topology for $\Sigma^\mathbf{N}$) [21]. Proposition 2.7 applies not only to well-defined functions on metric spaces but to machines that might give different values for different Cauchy names for the same point, and it is stated with explicit quantifiers that have elsewhere been omitted.

3. Three decidabilities

Myrvold [13] uses r.e. open sets to define a relaxed notion of decidability for effective metric spaces:

Definition 3.1. A subset A of X is *decidable ignoring boundaries* (d.i.b.) if its interior and exterior are both r.e. open.

Myrvold motivates this definition by observing, "There are cases in analysis and physics in which the boundaries of a given region are of little concern…" [13]. This is especially true when the boundary in question has zero measure, as is often the case. In physical applications, the measure of a set of possible states may represent the probability that a state in that set will occur at a given time. In that case, a boundary with measure zero, or any other measure-zero set, is indeed of little concern because at any given time, it is virtually certain that the state of a system will not lie in that set. When the states of a class of systems are naturally associated with points in $\mathbf{R}^n$, sets of zero Lebesgue measure are often assigned zero probability, so in such cases we can say more specifically that sets of zero Lebesgue measure are of little concern. Of course, in some situations even a measure-zero boundary may have interest, but if the problem is to decide whether a point lies in a given set, and the boundary of that set has measure zero, one often need not worry about boundary cases.

On the other hand, if a boundary has positive measure, decidability ignoring boundaries may not be very reassuring. If the boundary of a set has non-trivial μ-measure, the set may be d.i.b. and yet it may be impossible to determine membership in the set for a μ-large portion of points. For such applications, a notion of decidability defined in terms of measure may be more useful.

Ko [10] defines such a notion. Essentially, he permits a machine to make errors, so long as those errors can be confined to a set of arbitrarily small positive measure. However, Ko also requires a machine to halt on every Cauchy name. We generalize his definition to an arbitrary effective metric space with any outer measure. (This should be useful in applications, since the space of states of a physical model is often not Euclidean n-space but a curved manifold or an infinite-dimensional Hilbert space.)

Definition 3.2. Let μ^* be an outer measure on X. A set $A \subseteq X$ is *μ-recursively approximable* (μ-r.a.) if there is a machine M such that for all $x \in X$, $\phi \in CF_x^<$, and $n \in \mathbf{N}$,

(1) $M(\phi, n) \downarrow$, and
(2) if $n > 0$ then $\mu^* E_{A,n}(M) \leqslant 2^{-n}$,

where

$$E_{A,n}(M) = \{x \in X \mid (\exists \phi \in CF_x^<) M(\phi, n) \neq \chi_A(x)\}.$$

Intuitively, $E_{A,n}(M)$ is the set of points where M miscalculates the membership of A given parameter n.

Ko [10] shows that on the interval [0, 1] with Lebesgue measure, r.a. is equivalent to Šanin's notion of recursive measurability [15]. This will be useful below, so we state the definition and result.

Definition 3.3. (1) A sequence $\{S_n\}$ of sets in $\mathbf{R}$ is a *recursive sequence of sets* if there is a recursive function $j(n)$ such that for each n,

(a) $j(n) = (k, \langle a_1, b_1, \ldots, a_k, b_k \rangle)$ for some $k(n) \in \mathbf{N}$ and dyadic rationals $a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_k < b_k$, and
(b) $S_n = \bigcup_{i=1}^k (a_i, b_i)$.

(2) A set $S \subseteq \mathbf{R}$ is *recursively measurable* (r.m.) if there exists a recursive sequence $\{S_n\}$ of sets in $\mathbf{R}$ such that for all $n > 0$, $\lambda(S \Delta S_n) \leqslant 2^{-n}$, where Δ is the symmetric difference operator.

Theorem 3.4 (*Ko [10]*). *A set $S \subseteq [0, 1]$ is λ-r.a. if and only if it is r.m.*

Proof. See [10]. $\square$

However, one can demand more of a set than to be recursively approximable or recursively measurable. In effect, to say that a set A is μ-r.a. means that one can make the probability of miscalculating a point's membership in A as small as one likes, but not necessarily 0. It would seem intuitively preferable, and closer in spirit to the classical notion of decidability, to have an algorithm for which the probability of error is *exactly* 0. In another respect, though, one might demand less: as long as we are permitting errors on a measure-zero set, the epistemological situation would seem no worse if our algorithm *failed to halt* on some measure-zero set. The algorithm would still have a probability 1 of halting with correct results, so in physical applications, we could safely bet that the cases we cannot correctly judge would never actually arise, barring special circumstances. Given that only trivial sets are completely decidable in the naïve sense, probability 1 is the closest thing to certainty available. This is the motivation for the following definition.

Definition 3.5. For a given measure μ on a class of subsets of X, a set $A \subseteq X$ is *decidable up to μ-measure zero* (or μ-d.m.z.) if there exists a machine M such that $\mu E_A(M) = 0$, where

$$E_A(M) = \{x \in X \mid (\exists \phi \in CF_x^<)[M(\phi) \downarrow \neq \chi_A(x) \text{ or } M(\phi) \uparrow]\}.$$

It is possible to define notions of decidability that are stronger than μ-d.m.z. For example, if "or $M(\phi) \uparrow$" were omitted from the above definition, only measure-theoretically trivial sets would be μ-d.m.z. [14]. However, no such strengthening of μ-d.m.z. would imply a greater *probability* of correctly determining set membership, strictly speaking. Decidability in μ guarantees an algorithm that succeeds with probability 1, provided only that the relevant probability measure is absolutely continuous in μ. In that respect, μ-d.m.z. is strong enough to express the kind of decidability desired in physical situations. It is shown in [14] that μ-d.m.z. is also strong enough to capture the apparent undecidability of certain sets of physical states such as riddled basins of attraction and the KAM tori—sets that some have suggested may be undecidable, but without giving a non-trivial definition of undecidability for subsets of $\mathbf{R}^n$ [18,22].

4. Logical relations among decidabilities

All three of our relaxed decidabilities succeed in avoiding the extreme triviality that afflicts the naïve notion of decidability in $\mathbf{R}^n$. For example,

Remark 4.1. If $\mu A = 0$ or $\mu A^C = 0$ then A is μ-d.m.z. and μ-r.a.

Proof. If $\mu A = 0$, apply the *Nancy Reagan algorithm*: just say no. That is, let $M(\phi) \equiv M(\phi, n) \equiv 0$ for all ϕ, n. If $\mu A = 0$, then $E_A(M) = E_{A,n}(M) = A$, so $\mu E_A(M) = \mu E_{A,n}(M) = 0$. If instead $\mu A^C = 0$, let $M(\phi) \equiv M(\phi, n) \equiv 1$. $\square$

For non-degenerate intervals on the real line, all three of our notions are equivalent, holding for precisely those non-degenerate intervals with computable endpoints.

Proposition 4.2. *If $a, b \in \mathbf{R}$ and $a < b$, then the following are equivalent*:

(1) *a, b are computable*,
(2) *$(-\infty, a)$, (a, b), and (b, ∞) are r.e. open*,
(3) *all intervals (open, half-open, or closed) with endpoints a and b are d.i.b.*,
(4) *all intervals with endpoints a and b are λ-d.m.z., and*
(5) *all intervals with endpoints a and b are λ-r.a.*

Proof. (1) $\Rightarrow$ (2): Assume a, b computable. To see that (a, b) is r.e. open, we sketch an algorithm for a machine M that halts on an input $\{\phi_i\} \in CF_x^<$ if and only if $x \in (a, b)$: By dovetailing, [8] simultaneously compute, for each $i \in \mathbf{N}$, approximations a_i and b_i to a and b such that $|a_i - a|, |b_i - b| < 2^{-i}$. At the same time, evaluate for each i whether

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[8] A machine can carry out a countable number of algorithms, in effect simultaneously, by 'dovetailing' them, i.e., dividing its time and tape space between the several algorithms. This is a standard technique in recursion theory; see [17].

$v_{\mathbf{Q}}(\phi_i) - a_i, b_i - v_{\mathbf{Q}}(\phi_i) > 2^{1-i}$. Let M halt when such an i is found. This will occur if and only if $x \in (a, b)$, so M witnesses that (a, b) is r.e. open. The cases for $(-\infty, a)$ and (b, ∞) are similar.

$(2) \Rightarrow (3)$: Assuming the intervals $(-\infty, a)$, (a, b), and (b, ∞) are r.e. open, we have machines M_0 and M_1 such that if $x \in \mathbf{R}$ and $\phi \in CF_x^<$ then $M_0(\phi) \downarrow$ if and only if $x \in (-\infty, a)$ and $M_1(\phi) \downarrow$ if and only if $x \in (b, \infty)$. Let M dovetail the algorithms of M_0 and M_1, and halt if either halts. Thus $(-\infty, a) \cup (b, \infty)$ is r.e. open (and more generally, the r.e. open sets are closed under finite union). Now suppose I is an interval with endpoints a, b. Then the interior of I is (a, b), which is r.e. open by hypothesis, and the exterior $(-\infty, a) \cup (b, \infty)$ is also r.e. open, in virtue of M. Therefore, I is d.i.b.

$(3) \Rightarrow (4)$: Suppose I is a d.i.b. interval with endpoints a, b. This means there are machines M_0 and M_1 such that if $\phi \in CF_x^<$ then $M_0(\phi) \downarrow \Leftrightarrow x \in (-\infty, a) \cup (b, \infty)$ and $M_1(\phi) \downarrow \Leftrightarrow x \in (a, b)$. Let M dovetail the algorithms of M_0 and M_1, output 0 if $M_0(\phi) \downarrow$, and output 1 if $M_1(\phi) \downarrow$. Then $E_I(M) = \{a, b\}$, so $\lambda E_I(M) = 0$ and I is λ-d.m.z.

$(4) \Rightarrow (5)$: This follows from the fact that λ-d.m.z. $\Rightarrow \lambda$-r.a. [14].

$(5) \Rightarrow (1)$: See [10, p. 163]. $\quad\square$

Thus, all three of our relaxed decidabilities hold for an interval like $(0, 1)$, so they are mutually consistent and non-trivial in extension.

Can the equivalence of our three decidabilities on intervals be extended to convex sets?[9] Strictly speaking, no. Just as the above result applies only to non-trivial intervals, it can be extended only to convex sets of non-zero measure. A convex set in $\mathbf{R}^n$ may have dimension lower than n; a singleton, line segment, or disc, for example, is convex. Such low-dimensional sets in $\mathbf{R}^n$ have Lebesgue measure zero, so for them, λ-d.m.z. and λ-r.a. are trivial, while d.i.b. is not. In particular, a singleton containing a non-computable point is d.m.z. and r.a. but not d.i.b. We prove this via the following lemma.

Lemma 4.3. *If $\{x\} \subseteq \mathbf{R}$ is d.i.b. then x is a computable point*.

Proof. Our strategy is this: given a machine M that halts on input y if and only if $y \notin \{x\}$, we effectively approximate x to within 2^{-n} for any n by tracking the open sets on which M halts.

Let $\{s_i\}$ be an effective enumeration of Σ^{**}. Let a, b be computable numbers such that $x \in (a, b)$, and let M be a machine that halts on input y if and only if $y \notin \{x\}$. To approximate x with error less than 2^{-n}, apply M in dovetailed fashion to all finite sequences s_i of finite strings. Keep a list $\{t_i\}$ of those finite sequences on which M halts. Eventually, the sets $U(t_i)$ (Definition 2.6) will cover some set $[a, a'] \cup [b', b]$, where a', b' are computable and $b' - a' < 2^{-n}$. (This is because such a set is compact and contained in the exterior of $\{x\}$.) When this happens, take a' (or any number in $[a', b']$) as an approximation of x. $\quad\square$

Proposition 4.4. *There exists a convex set in $\mathbf{R}$ that is λ-r.a. and λ-d.m.z. but not d.i.b.*

Proof. Let x be a non-computable point on the real line. By 4.1, $\{x\}$ is λ-r.a. and λ-d.m.z. By 4.3, $\{x\}$ is not d.i.b. $\quad\square$

Hence d.m.z., r.a., and d.i.b. are not equivalent on *all* convex sets. This also shows that in general λ-r.a. and λ-d.m.z. do not imply d.i.b. Still, if we restrict our attention to full-dimensional convex sets, then our three notions do become equivalent.

Proposition 4.5. *Let A be an n-dimensional convex subset of $\mathbf{R}^n$. Then the following are equivalent*:

(1) *A is d.i.b.*,
(2) *A is λ-d.m.z., and*
(3) *A is λ-r.a.*

Proof. Since convex sets have measure-zero boundaries, it is clear that for such sets, d.i.b. implies λ-d.m.z., and we know that the latter implies λ-r.a. [14]. We need only show that λ-r.a. implies d.i.b. for full-dimensional convex sets.

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[9] This question was raised at the RNC'5 conference by Martin Ziegler. A set A is *convex* if for any two points x, y in A, all points on the line segment $\overline{xy}$ are in A.

Suppose A is such a set and is λ-r.a., as witnessed by a machine M. We seek algorithms to tell us when $x \in \text{int}(A)$ and when $x \in \text{ext}(A)$.

To confirm that a given point x is in $\text{int}(A)$, our strategy is to find an open ball containing x where so much of the ball is contained in A that by convexity, x must be in $\text{int}(A)$. Let $q \in \mathbf{Q}^n$ and $r \in \mathbf{Q}$. Let $\mathbf{S}_r$ denote the n-sphere with radius r and center at the origin, and let

$$v(x, q, r) = \lambda\{(a_1, a_2, \ldots, a_n) \in \mathbf{S}_r \mid a_1 > \|x - q\|\}.$$

If $\lambda(B(q, r) \cap A) > v(x, q, r)$, then by convexity of A, $x \in \text{int}(A)$. Notice also that $v(x, q, r)$ is a straightforwardly computable integral. Our algorithm, then, is as follows: examine all quadruples $(q, r, m, k) \in \mathbf{Q}^n \times \mathbf{Q} \times \mathbf{N} \times \mathbf{N}$ in dovetailed fashion. For each such quadruple find finite sequences $s_1, s_2, \ldots, s_k \in \Sigma^{**}$ such that for each $i \leqslant k$, $M(s_i, m) = 1$, and compute $\lambda(\bigcup_{i=1}^k U(s_i) \cap B(q, r))$. (See Definition 2.6 for U.) If this is greater than $v(x, q, r) + 2^{-m}$ then output 1 and halt.

If this procedure halts, then by definition of r.a., $\lambda(B(q, r) \cap A) > v(x, q, r)$, so $x \in \text{int}(A)$. Conversely, if $x \in \text{int}(A)$, then since A is r.a., such a quadruple (q, r, m, k) exists and will eventually be found.

To confirm that a given x is in $\text{ext}(A)$, we begin by finding a large m and a rational ball $B(q, r)$ such that for each $y \in B(q, r)$, $M(y, m) = 1$. This implies that a slightly smaller ball $B(q, s)$ is contained in A, with s a computable function of r and m. We then look for a large integer j and for neighborhoods U_i lying strictly between x and $B(q, s)$ such that $M(y, j) = 0$ for all $y \in U_i$. If we find such neighborhoods with $\lambda \cup U_i > 2^{-j}$ then output 0 and halt.

If this procedure halts, then by r.a., some points between x and $B(q, s)$ must not be elements of A. Hence by convexity, $x \in \text{ext}(A)$. Conversely, if $x \in \text{ext}(A)$ then such U_i and j exist and will be found. $\quad\square$

We now proceed to more general settings. We next show that for an arbitrary measure μ (and even a regular measure), the conjunction of d.i.b. and μ-d.m.z. does not imply μ-r.a. For example,

Proposition 4.6. *There exists an outer measure μ^* on $\mathbf{R}$ and an interval $I \subseteq \mathbf{R}$ such that I is μ-d.m.z. and d.i.b., but not μ-r.a.*

Proof. For any Lebesgue-measurable set A, let $\mu A = \int_A |1/x| \, dx$. Let μ^* denote any outer measure extending μ. (This is inessential; we mention it only to ensure strict compliance with our generalized definitions of μ-r.a. and μ-d.m.z.) Note that the measure of $(0, 1)$ will be infinite. To see that $(0, 1)$ is not μ-r.a., suppose some machine M halts on every input $(\phi, n) \in CF_x^< \times \mathbf{N}$ for every $x \in \mathbf{R}$. Fix n. By the Topological Use Principle (2.7), fix $\psi \in CF_x^<$ and a neighborhood U of 0 such that $(\forall x \in U)(\exists \phi \in CF_x^<)M(\phi, n)\downarrow = M(\psi, n)$. Then there exists an interval $(-a, a) \subseteq U$. But either $(-a, 0)$ or $(0, a) \subseteq E_{(0,1),n}(M)$. By choice of μ, $\mu(-a, 0) = \mu(0, a) = \infty$, so $\mu^* E_{(0,1),n}(M) = \infty$. Therefore $(0, 1)$ is not μ-r.a. Yet by 4.2, $(0, 1)$ is d.i.b. Since $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ are r.e. open and $\mu\{0, 1\} = 0$, $(0, 1)$ is also μ-d.m.z. $\quad\square$

Here we have used a strange measure μ to show that μ-r.a. is not implied by μ-d.m.z., d.i.b., or both in the general case. In $\mathbf{R}^n$ with Lebesgue measure, this independence breaks down in that λ-d.m.z. implies λ-r.a. [14]. However, d.i.b. alone still does not imply λ-r.a., as we will see later (4.10).

Next we show by example that d.i.b. and λ-r.a. together do not imply λ-d.m.z. The example is the same positive-measure generalized Cantor set constructed in [14] to show that λ-r.a. does not imply λ-d.m.z. We merely show that this set is d.i.b.

Proposition 4.7. *There exists a set in $\mathbf{R}$ that is d.i.b. and λ-r.a. but not λ-d.m.z.*

Proof. Say a closed interval I is *maximal* in a set $S \subseteq \mathbf{R}$ if $I \subseteq S$ and for every closed interval $J \subseteq S$, $J \cap I \neq \emptyset \Rightarrow J \subseteq I$. For all $i \in \mathbf{Z}^+$ let $r_i = 2^{-i-1}$. We construct a generalized Cantor set C as follows:

(1) Let $C_0 = [0, 1]$.
(2) For each $i \in \mathbf{Z}^+$, let

$$C_i = C_{i-1} \setminus \bigcup\{B([a + b]/2, 2^{-i-1} r_i) \mid [a, b] \text{ is maximal in } C_{i-1}\}$$

where $B(x, \varepsilon)$ denotes the interval $(x - \varepsilon, x + \varepsilon)$ and '$\setminus$' denotes set difference.
(3) Let $C = \bigcap_i C_i$.

Part (2) dictates that we obtain C_i by removing the middle segment of length $r_i/2^i$ from each of the 2^i maximal intervals in C_{i-1}. Hence the limit of this process, the set C, has measure $1 - \sum r_i = 1/2$.

It is shown in [14] that such a set is λ-r.a. and not λ-d.m.z. To see that C is d.i.b., we show that $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ are r.e. open. Note C is nowhere dense by construction. Therefore, $\mathrm{int}(C)$ is empty and hence trivially r.e. open. Next we give an algorithm which, given input $\{\phi_i\} \in CF_x^<$, halts if and only if $x \in \mathrm{ext}(C)$. Recall that $v_\mathbf{Q}(\phi_j)$ is just the rational number coded by ϕ_j.

(1) Let $i = 1$.
(2) Let $k(i) = 2^{i-1}$ and let $0 < c_{i1} < d_{i1} < c_{i2} < d_{i2} < \cdots < c_{ik(i)} < d_{ik(i)} < 1$ be the boundary points of C_i. (These are rationals that can be found effectively using the definition of C_i.)
(3) Set j such that $2^{-j} < \min_{n \leqslant k(i)} (d_{in} - c_{in})/2$.
(4) If $c_{mn} + 2^{-j} < v_\mathbf{Q}(\phi_j) < d_{mn} - 2^{-j}$ for some $n \leqslant k(i)$, halt.
(5) If $v_\mathbf{Q}(\phi_j) < -2^{-j}$ or $v_\mathbf{Q}(\phi_j) > 1 + 2^{-j}$, halt.
(6) Let $i = i + 1$ and go to (2).

Note $x \in \mathrm{ext}(C)$ if and only if either $x < 0$, $x > 1$, or there is some i, n such that $c_{in} < x < d_{in}$. If $\{\phi_i\} \in CF_x^<$, then $x \in \mathrm{ext}(C)$ if and only if for sufficiently large j, either $v_\mathbf{Q}(\phi_j) < -2^{-j}$, $v_\mathbf{Q}(\phi_j) > 1 + 2^{-j}$, or there exist i, n such that $c_{in} + 2^{-j} < v_\mathbf{Q}(\phi_j) < d_{in} - 2^{-j}$. Hence this algorithm will halt if and only if $x \in \mathrm{ext}(C)$. Therefore $\mathrm{ext}(C)$ is r.e. open and C is d.i.b.　□

We conclude our study of the logical relations (or for the most part, lack thereof) among our three decidabilities by showing that d.i.b. does not imply λ-r.a. One way to do this is by defining some non-measurable dense subset of a simple set like the interval $[0, 1]$. A dense subset of $[0, 1]$ is d.i.b. by Proposition 4.2, but a non-measurable set is a fortiori not recursively measurable in the sense of Šanin [15], and therefore not λ-r.a. by 3.4.

However, it is well known that a non-measurable set cannot be defined without appeal to the Axiom of Choice. Therefore, we give a measurable and constructive example of a d.i.b. set that is not λ-r.a. by a slight modification of the generalized Cantor set in 4.7. In fact, the example shows that there are even open and closed sets that are d.i.b. but not λ-r.a. (and therefore not λ-d.m.z.).

We will need two lemmas:

Lemma 4.8. *If $S \subseteq \mathbf{R}$ is r.m. (Definition 3.3) then λS is a computable number (Definition 2.8).*

Proof. Assuming S is r.m., let $\{S_n\}$ be a recursive sequence of sets such that for all $n \in \mathbf{N}$, $\lambda(S \Delta S_n) \leqslant 2^{-n}$. By Definition 3.3 there is a machine M that, given n, computes the boundary points $a_1, b_1, \ldots, a_{k(n)}, b_{k(n)}$ of S_n. Let $M'(n) = \sum_{i=1}^{k(n)} (b_i - a_i) = \lambda S_n$. Then $|\lambda S - M'(n)| \leqslant \lambda(S \Delta S_n) \leqslant 2^{-n}$.　□

Lemma 4.9. *Say $y = \sum_{n \in K \subseteq \mathbf{N}} 2^{-n}$ is a computable number. Then $K \subseteq \mathbf{N}$ is recursive.*

Proof.

Case 1: $0 \leqslant y < 1$ is a dyadic rational, i.e., y can be written in binary notation as $0.a_1 a_2 \ldots a_m 1000 \ldots$, where each $a_i = 0$ or 1. Then there is exactly one other binary representation of y, namely $0.a_1 a_2 \ldots a_m 0111 \ldots$. Hence either $n \in K$ for all $n > +1m$ or $n \notin K$ for all $n > +1m$. In either case, K is recursive.

Case 2: y is not a dyadic rational. Then there is a unique binary name $0.a_1 a_2 \ldots$ for y. Now, a number $x \in \mathbf{R}$ is computable with respect to Cauchy names if and only if it is computable with respect to the binary representation (see [21, p. 93]). Since y is a computable number, the sequence $\{a_i\}$ of its binary digits is recursive. So to determine whether $n \in K$, merely compute the nth digit a_n. Then $n \in K \Leftrightarrow a_n = 1$.　□

Proposition 4.10. *There exist open and closed subsets of $\mathbf{R}$ that are d.i.b. but not λ-r.a. (and therefore not λ-d.m.z. either).*

Proof. Let $K \subseteq \mathbf{Z}^+$ be any non-recursive r.e. set, e.g., the set $K = \{e \in \mathbf{N} | \varphi_e(e) \downarrow\}$ of classical recursion theory, where $\{\varphi_e\}$ is an effective enumeration of the partial recursive functions. Let $\{k_i\}_{i \in \mathbf{Z}^+}$ be a recursive enumeration of K

without repetition. Then construct C just as in the proof of 4.7 but with $\{r_i\} = \{2^{-k_i}\}$. As such, $\lambda C = 1 - \sum_i 2^{-k_i} = 1 - \sum_{n \in K} 2^{-n}$. By Lemma 4.9, $\sum_{n \in K} 2^{-n}$ is non-computable, and it follows that $\lambda C = 1 - \sum_{n \in K} 2^{-n}$ is non-computable. Therefore C is not λ-r.a. by Lemma 4.8 and Theorem 3.4 (and since λ-d.m.z. $\Rightarrow$ λ-r.a., C is not λ-d.m.z.). The proof that C is d.i.b. is just as in 4.7. $\square$

5. Conclusions and further research

The logical relations established among the three decidability concepts considered here are summed up in the following theorem:

Theorem 5.1. (1) *The properties d.i.b., μ-r.a., and μ-d.m.z. are mutually consistent*;
(2) *The properties d.i.b., μ-r.a., and μ-d.m.z. are independent, i.e., no conjunction of two implies the third*;
(3) *In $\mathbf{R}^n$ with λ = Lebesgue measure,*

  (a) *λ-d.m.z. and λ-r.a. together do not imply d.i.b.,*
  (b) *d.i.b. and λ-r.a. together do not imply λ-d.m.z., and*
  (c) *d.i.b. does not imply λ-r.a., but*
  (d) *λ-d.m.z. implies λ-r.a.*

Proof. (1) Proposition 4.2.
(2) Propositions 4.4, 4.6, and 4.7.

(3) (a) Proposition 4.4.
   (b) Proposition 4.7.
   (c) Proposition 4.10.
   (d) See [14]. $\square$

The logical relations among decidabilities stated here pertain only to two settings, the very general setting of an arbitrary metric space with arbitrary measure and the specific setting of $\mathbf{R}^n$ with Lebesgue measure. It would also be worthwhile to see how our three concepts relate to one another in other spaces with other measures, especially those of interest to physics.

Also, there are many other notions of decidability that might be brought into the fold of this study. In particular, the following concept might be interesting for physical applications:

Definition 5.2. A set $A \subseteq X$ is *somewhere decidable up to μ-measure zero* (*somewhere μ-d.m.z.*) if there exists a machine M and neighborhood $U \subseteq X$ (in the metric topology) such that $\mu^*(E_A(M) \cap U) = 0$. Otherwise A is *nowhere μ-d.m.z.*

This is a weaker form of μ-d.m.z., and its negation is a strong undecidability. The basins of attraction for certain dissipative dynamical systems [18] seem to be nowhere λ-d.m.z., indicating that the asymptotic behavior of any one such system cannot be decided up to measure zero even over the smallest region of phase space. Hence, we cannot predict the fate of even a single such system with probability 1, no matter what state it is in.

It would be interesting to see whether many or all of the various proposed concepts of decidability and recursiveness could be related to one another in some coherent structure. There have been several efforts in that direction, especially using the TTE framework [4,11,20,21], but these have usually considered only notions of decidability defined for special subclasses of the Borel sets rather than for arbitrary sets. It might not be difficult to relate broader notions of decidability to the sets treated in the TTE framework. For example, it is easy to show that a set $A \subseteq \mathbf{R}^n$ is λ-d.m.z. if and only if A and its complement are both equal "mod 0" to r.e. open sets. Thus d.m.z., a notion for general sets, is strongly related to a TTE notion for open sets.

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