Finite information logic

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Abstract: we introduce a generalization of *Independence Friendly (IF)* logic in which Eloise is restricted to a finite amount of information about Abelard's moves. This Logic is shown to be equivalent to a sublogic $\exists\forall$ of first order logic, has the finite model property, and is decidable. Moreover, it gives an exponential compression relative to $\exists\forall$ logic.

Partial information logic is a generalization of both first order logic and Hintikka-Sandu [3] IF-logic. We motivate this logic by means of an example. Suppose we have a model $\mathcal{M}$ on some domain D and some formula $A = (\forall x)(\forall y)(\exists z)R(x,y,z)$ where R is atomic. Then to this formula corresponds a game between two players Abelard and Eloise. Abelard chooses two elements a, b from D. Then Eloise chooses a third element c from D. If the formula $R(a,b,c)$ holds in $\mathcal{M}$ then Eloise has won, else Abelard has. Now it can be shown that the formula A is *true* in $\mathcal{M}$ iff Eloise has a winning strategy.

The game as we have just desribed tells us how classical first order logic works. To look at IF-logic we consider a slight variant. Let B be the variant of A obtained by writing $B = (\forall x)(\forall y)(\exists z/x)R(x,y,z)$. Now the game proceeds as before with Abelard choosing a, b and Eloise choosing c, but now, the choice of c has to be *independent* of a because the quantifier $\exists z$ has now been marked by a $/x$, indicating independence of x, or as we might say, *ignorance* of x.

But we could just as easily say that Eloise's knowledge is restricted to the value of y, i.e. to b. Instead of concentrating on what Eloise does *not* know we concentrate on what she *does*. Similar restrictions might of course apply to Abelard in case he too has a move which follows the move of Eloise.

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Now we introduce an innovation which will turn out to be interesting. IF-logic allows Eloise to know the value of x, or of y or of both or neither. Could we consider other possibilities? E.g. suppose x, y are integers. We might restrict Eloise to know the value of their *sum*. Or for another example, suppose you meet on the airplane an attractive woman who tells you only her first name (until she knows you better). Now if x is the name variable whose value is Eloise Dzhugashvili and she only tells you 'Eloise', then you do not know x but neither are you ignorant of it. You know it *in part*.

This opens up the possibility of more general kinds of knowledge of the values of variables than allowed by IF-logic and we will see that it leads to interesting possibilities.

As usual we have variables, predicate symbols, certain special function symbols. Atomic formulas are defined as usual. Literals are atomic formulas or their negations. For simplicity we will apply negation only to atoms.

Definition 1 1 *Literals are formulas of PI.*

2a *If $\varphi(\vec{x}, y)$ is a formula of PI and f is one of the special function symbols, then $(\exists y /\!\!/_{f(\vec{x})})\varphi(\vec{x}, y)$ is a formula of PI.*

2b *If $\varphi(\vec{x}, y)$ is a formula of PI and f is one of the special function symbols, then $(\forall y /\!\!/_{f(\vec{x})})\varphi(\vec{x}, y)$ is a formula of PI.*

3a *If $\varphi(\vec{x}), \theta(\vec{x})$ are formulas of PI then $\varphi(\vec{x}) \vee /\!\!/_{f(\vec{x})} \theta(\vec{x})$ is a formula of PI.*

3b *If $\varphi(\vec{x}), \theta(\vec{x})$ are formulas of PI then $\varphi(\vec{x}) \wedge /\!\!/_{f(\vec{x})} \theta(\vec{x})$ is a formula of PI.*

Intuitively, the $\exists y$ in $(\exists y /\!\!/_{f(\vec{x})})\varphi(\vec{x}, y)$ is Eloise's move but because of the restriction $/\!\!/_{f(\vec{x})}$ she only knows $f(\vec{x})$ when she makes her move. We may, more generally, allow her also to know the values of two or more functions f, g of $\vec{x}$ so that in the extreme case she could know all the projection functions and hence know $\vec{x}$ precisely. That case corresponds to our usual first order logic. In an intermediate case, she could know *some* of the projection functions on $\vec{x}$, i.e. some but not all of the variables in $\vec{x}$. That case corresponds to IF-logic.

In $(\forall y /\!\!/_{f(\vec{x})})\varphi(\vec{x}, y)$ the move is Abelard's and he too is restricted in a similar way.

Let us consider $\varphi(\vec{x}) \vee /\!\!/_{f(\vec{x})} \theta(\vec{x})$. Since we have a disjunction here, it is for Eloise to choose which of the two formulas φ, θ to play. But when she chooses, she only knows the value $f(\vec{x})$ or perhaps more than one such value, but her knowledge of $\vec{x}$ might not be complete.

On the other hand, in $\varphi(\vec{x}) \wedge /\!\!/_{f(\vec{x})} \theta(\vec{x})$ the move is Abelard's but the restrictions are similar to those in 3a above.

Compositional sematics can be defined for PI in just the same way as they have been defined for IF-logic by Hodges [4, 5], Väänänen [10], etc. Moreover PI-logic can be interpreted into second order logic in the same way.

Now we come to a special kind of PI-logic where the special functions f allow only a *finite* amount of information about the arguments. Thus if a, b are integers and Eloise has to make a choice based on them, she might be allowed only to know whether $a < b$ or whether $a + b$ is odd, or whatever. Knowing the precise value of a, b or even of $a + b$ is out of the question.

Why consider such a restricted case? We have two reasons. One is that this special case of PI-logic which we shall call FI-logic, or *finite information logic* has very elegant logical properties. The other is that since quantifiers correspond to moves in games, the games which FI-logic represents arise all the time in social algorithms and are deeply related to how social human interations work.

For example a passport official at an airport only wants to know whether you have a valid visa or not. If you do, she lets you in, if not, she sends you back on the next flight. Or perhaps she classifies you among four classes, those who are citizens, those who come from friendly countries whose citizens do not require a visa, those who have a visa, and the remaining who are the ones sent back. In any case she only wants a finite amount of information about the variable, namely you.

Or a young man looking for a date might want to know if the prospective date is blonde or brunette. If she is, he is not interested, he wants to date brown hair only. If she does have brown hair, he wants to know if she is tall. If not, he is again not interested. So he seeks a finite amount of information about the prospective date. Naturally she may have similar questions about him. But each will seek only a finite amount of information.

We repeat the definitions which we had above for formulas of PI-logic, indicating where the difference arises between PI-logic in general and its special case, FI. Since only a finite amount of information is available at each step, it could easily be represented by one or more booleans, i.e. by formulas. Thus our special functions f drop out. Our main result is Theorem 8 which says that every consistent FI-sentence has a finite model. We use a strong form of this result to show that FI is exactly the existential-universal fragment of first order logic, if considered as a classical logic. However, FI is actually a non-classical logic with a rich many-valued semantics (this aspect will not be pursued in this paper). The reduction to first order logic is non-trivial in the sense that there is a trade-off: the first order expression seems to be exponentially longer than its FI representation.

In this section we define the *finite information logic* FI and discuss its semantics. It turns out that it makes sense to pay attention to what kind of θ we allow in $/\!\!/_\theta$, as the following informal result demonstrates:

Lemma 2 *The following conditions are intuitively equivalent in any model* $\mathfrak{A}$ *with at least two elements, whatever sentence* θ *is:*

1. $\mathfrak{A} \models (\forall x)(\exists y /\!\!/_{(x=c\vee\theta)})(y \neq x)$.
2. $\mathfrak{A} \models \neg\theta$

Proof. Suppose θ is true and $\exists$ knows it. Then the information that $(x = c \vee \theta)$ is true tells $\exists$ nothing about x. Also the information that $(x = c \vee \theta)$ is false tells nothing because this information is impossible, i.e. never given in this case. Thus in this case $\exists$ cannot possibly have a winning strategy for choosing $y \neq x$. On the other hand, suppose θ is false and $\exists$ knows it. Then she can make the following inference: If I am told that $(x = c \vee \theta)$ is true, I know that it is true because $x = c$, and then I know what x is. If I am told that $(x = c \vee \theta)$ is false, I know it is because $x \neq c$, and I can choose $y = c$. □

In the proof we used the assumption that although the information that $\exists$ has is limited as to the values of the variables, $\exists$ knows "generally known" things. For example, it follows that if $\exists$ has a winning strategy, she knows what it is. Also, if it is known that $\neg\theta$ (in a given model), then $\exists$ knows it too.

Lemma 2 shows that if we allow θ in $/\!\!/_\theta$, we are committed to have also the negation of θ. On the other hand, games of imperfect information may very well be non-determined. Therefore we should be cautious with negation.

In social software it seems that the information we use in decisions is often atomic ("man", "woman") or existential ("has a ticket", "has a visa, which is valid") or boolean combinations of such ("is retired or has exactly three children"). Accordingly we start by allowing θ in $/\!\!/_\theta$ to be any boolean combination of existential formulas.

Definition 3 *The set of formulas of FI is defined as follows:*

(1) *Atomic and negated formulas are FI-formulas.*

(2) *If $\varphi(\vec{x})$ and $\psi(\vec{x})$ are FI- formulas and $\theta(\vec{x})$ is a boolean combination of existential formulas, then*

$$\varphi(\vec{x}) \wedge /\!\!/_{\theta(\vec{x})} \psi(\vec{x})$$

and

$$\varphi(\vec{x}) \vee /\!\!/_{\theta(\vec{x})} \psi(\vec{x})$$

are FI-formulas.

(3) *If $\varphi(\vec{x}, y)$ is an FI-formula and $\theta(\vec{x})$ is a boolean combination of existential formulas, then*

$$(\forall y /\!\!/_{\theta(\vec{x})}) \varphi(\vec{x}, y)$$

and

$$(\exists y /\!\!/_{\theta(\vec{x})}) \varphi(\vec{x}, y)$$

are FI-formulas.

We now define semantics for FI. Suppose $\mathfrak{A}$ is a model and X is a set of functions s such that

(1) dom(s) is a finite set of variables

(2) $s, s' \in X \Longrightarrow \mathrm{dom}(s) = \mathrm{dom}(s')$

(3) $\mathrm{ran}(s) \subseteq A$.

Intuitively X is a set of plays i.e. assignments of values to variables. To incorporate partial information we have to consider sets of plays rather than mere individual plays. A partition $X = X_0 \cup X_1$ is $\theta(\vec{x})$-**homogeneous**, where $\theta(\vec{x})$ is first-order, if for all $s, s' \in X$

$$(\mathfrak{A} \models_s \theta(\vec{x}) \Longleftrightarrow \mathfrak{A} \models_{s'} \theta(\vec{x})) \Longrightarrow (s \in X_0 \Longleftrightarrow s' \in X_0).$$

Let

$$X[a:y] = \{(s \backslash \{\langle y,b\rangle : b \in A\}) \cup \{\langle y,a\rangle\} : s \in X\}$$
$$X[A:y] = \{s \cup \{\langle y,a\rangle\} : s \in X, a \in A\}.$$

We define the concept

$$\mathfrak{A} \models_X \varphi$$

for $\varphi \in FI$ as follows:

(S1) $\mathfrak{A} \models_X \varphi$ iff $\mathfrak{A} \models_s \varphi$ for all $s \in X$, if φ is atomic or negated atomic.

(S2) $\mathfrak{A} \models_X \varphi(\vec{x}) \wedge /\!\!/_{\theta(\vec{x})} \psi(\vec{x})$ iff $\mathfrak{A} \models_X \varphi(\vec{x})$ and $\mathfrak{A} \models_X \psi(\vec{x})$. ($\theta(\vec{x})$ plays no role)

(S3) $\mathfrak{A} \models_X \varphi(\vec{x}) \vee /\!\!/_{\theta(\vec{x})} \psi(\vec{x})$ iff there is a $\theta(\vec{x})$-homogeneous partition $X = X_0 \cup X$, such that $\mathfrak{A} \models_{X_0} \varphi(\vec{x})$ and $\mathfrak{A} \models_{X_1} \psi(\vec{x})$.

(S4) $\mathfrak{A} \models_X (\exists y /\!\!/_{\theta(\vec{x})}) \varphi(\vec{x}, y)$ iff there is a $\theta(\vec{x})$-homogeneous partition $X = X_0 \cup X$, and y_1, y_2 such that $\mathfrak{A} \models_{X_0[y_1:y]} \varphi(\vec{x}, y)$ and $\mathfrak{A} \models_{X_1[y_2:y]} \varphi(\vec{x}, y)$.

(S5) $\mathfrak{A} \models_X (\forall y /\!\!/_{\theta(\vec{x})})) \varphi(\vec{x}, y)$ iff

$$\mathfrak{A} \models_{X[A:y]} \varphi(\vec{x}, y)$$

($\theta(\vec{x})$ plays no role).

There is an asymmetry between $\wedge /\!\!/_{\theta(\vec{x})}$ and $\vee /\!\!/_{\theta(\vec{x})}$ on one hand and between $(\forall y /\!\!/_{\theta(\vec{x})})$ and $(\exists y /\!\!/_{\theta(\vec{x})})$ on the other hand. This is because in this paper we consider truth from the point of view of $\exists$ only, i.e. "classically". Thus we are concerned about the knowledge that $\exists$ has. As $\exists$ has to be prepared to play against all strategies of $\forall$, $\exists$ has to consider also the case that $\forall$ plays "accidentally" with perfect information. If we considered FI "non-classically" the symmetry would be preserved.

Suppose $\mathfrak{A} \models_{\{\emptyset\}} \varphi$. Now $\exists$ has a winning strategy in the obvious semantic game, namely, while $\exists$ plays she keeps $\mathfrak{A} \models_X \varphi$ and the play $\in X$ true. More exactly:

(G1) Suppose we are at an atomic or negated atomic formula φ. Since $\mathfrak{A} \models_X \varphi$ and the play is in X, $\exists$ wins by (S1).

(G2) We are at $\varphi(\vec{x}) \wedge /\!\!/_{\theta(\vec{x})}\psi(\vec{x})$. Now $\forall$ plays choosing, say, $\varphi(\vec{x})$. We use (S2) to conclude $\mathfrak{A} \models_X \varphi(\vec{x})$.

(G3) We are at $\varphi(\vec{x}) \vee /\!\!/_{\theta(\vec{x})}\psi(\vec{x})$. We can by (S3) divide $X = X_0 \cup X_1$ in a $\theta(\vec{x})$-homogeneous way and $\mathfrak{A} \models_{X_0} \varphi(\vec{x})$ and $\mathfrak{A} \models_{X_1} \psi(\vec{x})$. The play is in X so it is in one of X_0 and X_1, but $\exists$ does not know in which. We let $\exists$ make the choice on the basis of the following inference. If $\theta(\vec{x})$ is true and some $\vec{x}\,'$ in X_0 satisfies $\theta(\vec{x}\,')$, then she chooses X_0. In this case homogeneity gives $\vec{x}\,' \in X_0$ and we also have $\mathfrak{A} \models_{X_0} \varphi(\vec{x})$. If $\theta(\vec{x})$ is true and some $\vec{x}\,'$ in X_1 satisfies $\theta(\vec{x}\,')$, then she chooses X_1. Again homogeneity gives $\vec{x}\,' \in X_1$ and we also have $\mathfrak{A} \models_{X_1} \varphi(\vec{x})$. Similarly, if $\theta(\vec{x})$ is false and some $\vec{x}\,'$ in X_0 satisfies $\theta(\vec{x}\,')$, then she chooses X_1, otherwise X_0.

(G4) We are at $(\forall y /\!\!/_{\theta(\vec{x})}))\varphi(\vec{x}, y)$. $\exists$ knows $\mathfrak{A} \models_{X[A:y]} \varphi(\vec{x}, y)$ and the play so far is in X. Whatever $\forall$ plays, the play is in $X[A : y]$.

(G5) We are at $(\exists y /\!\!/_{\theta(\vec{x})})\varphi(\vec{x}, y)$. There is a $\theta(\vec{x})$-homogeneous partition $X = X_0 \cup X$, and y_1, y_2 such that $\mathfrak{A} \models_{X_0[y_1:y]} \varphi(\vec{x}, y)$ and $\mathfrak{A} \models_{X_1[y_2:y]} \varphi(\vec{x}, y)$. As in the case of disjunction, player $\exists$ chooses y_1 or y_2 according to whether some $\vec{x}\,'$ in X_0 satisfies $\theta(\vec{x}\,')$ or not. □

Examples 4 1° $(\forall x /\!\!/)(\exists y /\!\!/_{P(x)})(x = y)$ *says that both P and its complement have at most one element*

2° $(\forall x /\!\!/)(\exists y /\!\!/_{P(x)})(x \neq y)$ *says that both P and its complement are non-empty.*

Lemma 5 *If $\mathfrak{A} \models_X \varphi$ and $X_0 \subseteq X$, then $\mathfrak{A} \models_{X_0} \varphi$.*

Proof. Trivial. □

Lemma 6 *Every FI-sentence is first order definable.*

Proof. Suppose $\phi \in FI$. Let n be the length of ϕ. It suffices to show that truth of ϕ is preserved by n-equivalence. Suppose therefore that M and M' are models and $\emptyset \neq I_0 \subseteq I_1 \subseteq ... \subseteq I_n$ is a sequence with the back-and-forth property. Suppose X is a set as above. For $s \in X$ let s' be the result of applying the back-and-forth sequence to s. Let X' be the set of all s' where $s \in X$. It suffices to prove the equivalence of

$$M \models_X \phi \quad (1)$$

$$M' \models_{X'} \phi \quad (2)$$

for all FI-formulas ϕ. This is an easy induction on ϕ. □

A first order formula is existential-universal $\exists\forall$ if it is of the form

$$(\exists x_1) \ldots (\exists x_n)(\forall y_1) \ldots (\forall y_m)\varphi$$

where φ is quantifier-free. A formula is Δ_2 if it is equivalent to an $\exists\forall$-formula and its negation is too. An example of a Δ_2 formula is

$$(\exists x_1)(\exists x_2)(x_1 \neq x_2) \wedge (\forall x_1)(\forall x_2)(\forall x_3)(x_1 = x_2 \vee x_1 = x_3 \vee x_2 = x_3).$$

which says that there exactly three elements. Boolean combinations of existential formulas are, of course, Δ_2.

Lemma 7 *The following conditions are equivalent for any first order sentence φ:*

(1) *φ is equivalent to an $\exists\forall$-formula.*

(2) *If $\mathfrak{A} \models \varphi$ and $\mathfrak{A}$ is the union of a chain $\mathfrak{A}_\alpha$ ($\alpha < \beta$) of models, then there is an $\alpha < \beta$ such that $\mathfrak{A}_\alpha \models \varphi$.*

(3) *If $\mathfrak{A} \models \varphi$ and $B \subseteq A$ is finite, then there is $C \subseteq A$ finite such that $B \subseteq C$ and for all finite $\mathfrak{D}$ with $C \subseteq D \subseteq A$ we have $\mathfrak{D} \models \varphi$.*

Proof. Clearly (1) → (3) → (2). We prove (2) → (1). By (2) the sentence $\neg\varphi$ is closed under unions of chains of models. By the Łoś-Suszko lemma, $\neg\varphi$ is universal-existential, whence φ is equivalent to an $\exists\forall$ formula. □

Theorem 8 *Every FI-sentence has the finite model property.*

Proof. We prove condition (3) of Lemma 7. We use induction on φ to prove:

($\star$) If $\mathfrak{A} \models_X \varphi$, $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is finite and $\forall s \in X$ $(ran(s) \subseteq A_0)$, then there is a finite $\mathfrak{A}_1$, s.t. $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}$ and for all $\mathfrak{A}_2 \subseteq \mathfrak{A}$ with $A_1 \subseteq A_2 \subseteq A$ we have $\mathfrak{A}_2 \models_X \varphi$.

(S1) φ is atomic or negated atomic. We can choose $\mathfrak{A}_1 = \mathfrak{A}_0$.

(S2) Conjunction: Again we choose $\mathfrak{A}_1 = \mathfrak{A}_0$.

(S3) Disjunction: Suppose $\mathfrak{A} \models_X \varphi(\vec{x}) \vee\!\!/\!\!/_{\theta(\vec{x})} \psi(\vec{x})$ and $\forall s \in X$ $(ran(s) \subseteq A_0)$. Let $X = X_0 \cup X_1$ such that $\mathfrak{A} \models_{X_0} \varphi(\vec{x})$, $\mathfrak{A} \models_{X_1} \psi(\vec{x})$ and the partition is $\theta(\vec{x})$-homogeneous. Remember that $\theta(\vec{x})$ is Δ_2. Let $\mathfrak{A}_1^*$ be finite such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1^* \subseteq \mathfrak{A}$ and $\mathfrak{A}_1^* \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}$ implies for all $s \in X$

$$\mathfrak{A}_2 \models_s \theta(\vec{x}) \iff \mathfrak{A} \models_s \theta(\vec{x}).$$

By induction hypothesis we have $\mathfrak{A}_1^0$ for X_0 and $\varphi(\vec{x})$, and $\mathfrak{A}_1^1$ for X_1 and $\psi(\vec{x})$. Let $A_1 = A_1^0 \cup A_1^1 \cup A_1^*$. If $\mathfrak{A}_2 \subseteq \mathfrak{A}$ with $A_1 \subseteq A_2 \subseteq A$, then $\mathfrak{A}_2 \models_{X_0} \varphi$ and $\mathfrak{A}_2 \models_{X_1} \psi$, whence $\mathfrak{A}_2 \models_X \varphi(\vec{x}) \vee\!\!/\!\!/_{\theta(\vec{x})} \psi(\vec{x})$. Why? Because $X = X_0 \cup X_1$ is $\theta(\vec{x})$-homogeneous in $\mathfrak{A}_2$ [If $\mathfrak{A}_2 \models_s \theta(\vec{x})$, $\mathfrak{A}_2 \models_{s'} \theta(\vec{x})$, where $s, s' \in X$, then $\mathfrak{A} \models_s \theta(\vec{x})$ and $\mathfrak{A} \models_{s'} \theta(\vec{x})$ whence $s \in X_0 \iff s' \in X_0$].

(S4) Universal quantification: $\mathfrak{A} \models_X (\forall y)\varphi(\vec{x}, y)$ and $\forall s \in X$ $(ran(s) \subseteq A_0)$. Thus $\mathfrak{A} \models_{X[A:y]} \varphi(\vec{x}, y)$. Choose $\mathfrak{A}_1 = \mathfrak{A}_0$. Suppose $\mathfrak{A}_2 \subseteq \mathfrak{A}$ with $A_1 \subseteq A_2 \subseteq A$. Then $X[A_2 : y] \subseteq X[A : y]$, whence $\mathfrak{A}_2 \models_{X[A_2:y]} \varphi(\vec{x}, y)$. Now $\mathfrak{A}_2 \models_X (\forall y)\varphi(\vec{x}, y)$ follows.

(S5) Existential quantification: $\mathfrak{A} \models_X (\exists y /\!\!/_{\theta(\vec{x})})\varphi(\vec{x}, y)$ and $\forall s \in X$ $(ran(s) \subseteq A_0)$. Let $X = X_0 \cup X_1$ be $\theta(\vec{x})$-homogeneous and y_1, y_2 such that $\mathfrak{A} \models_{X_0[y_1:y]} \varphi(\vec{x}, y)$ and $\mathfrak{A} \models_{X_1[y_2:y]} \varphi(\vec{x}, y)$. Remember that $\theta(\vec{x})$ is Δ_2. Let $\mathfrak{A}_1^*$ be finite such that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1^* \subseteq \mathfrak{A}$ and $\mathfrak{A}_1^* \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}$ implies for all $s \in X$

$$\mathfrak{A}_2 \models_s \theta(\vec{x}) \iff \mathfrak{A} \models_s \theta(\vec{x}).$$

Let $\mathfrak{A}_1$ be such that $A_1^* \cup \{y_1, y_2\} \subseteq A_1 \subseteq A$ and $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}$ implies $\mathfrak{A}_2 \models_{X_0[y_1:y]} \varphi(\vec{x}, y)$ and $\mathfrak{A}_2 \models_{X_1[y_2:y]} \varphi(\vec{x}, y)$. Now $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}$ implies $\mathfrak{A}_2 \models (\exists y /\!\!/_{\theta(\vec{x})})\varphi(\vec{x}, y)$. Why? Because $X = X_0 \cup X_1$ is $\theta(\vec{x})$-homogeneous in $\mathfrak{A}_2$ [If $\mathfrak{A}_2 \models_s \theta(\vec{x})$, $\mathfrak{A}_2 \models_{s'} \theta(\vec{x})$, where $s, s' \in X$, then $\mathfrak{A} \models_s \theta(\vec{x})$ and $\mathfrak{A} \models_{s'} \theta(\vec{x})$ whence $s \in X_0 \iff s' \in X_0$.] and $\mathfrak{A}_2 \models_{X_0[y_1:y]} \varphi(\vec{x}, y)$, $\mathfrak{A}_2 \models_{X_1[y_2:y]} \varphi(\vec{x}, y)$.

$\square$

The above theorem has an alternative proof using the concept of a D-structure (see [6, 7, 8, 9], which build on [1]).

Example 9 *The sentence*

$$(\forall x /\!\!/)(\exists y /\!\!/_{x=x})(x \leq y)$$

says that the linear order $\leq$ *has a last element. It has no negation in FI as the negation does not have the finite model property.*

The finite model property would be true even if we allowed any Δ_2 formula θ to occur in $/\!\!/_\theta$. However, allowing $\exists\forall$-formulas θ leads us to new avenues: Let $FI(\exists\forall)$ be this generalization.

Theorem 10 $FI(\exists\forall)$ *does not have the finite model property.*

Proof. Let φ be the sentence

$$(\forall x /\!\!/)(\exists y /\!\!/_{\psi(x)})(y \neq x)$$

where $\psi(x)$ is the $\exists\forall$-formula

$$x = 0 \vee (\exists u)(\forall v)(v \leq u).$$

The vocabulary consists of $\leq$ and the constant 0. Let φ' be the conjunction of φ and the universal (hence FI) axioms of linear order.
Claim 1 $\langle \omega, \leq, 0 \rangle \models \varphi'$. The task of $\exists$ is choose $y \neq x$ knowing only whether $\psi(x)$ is true or not. She argues as follows: If I am told $\psi(x)$ is true, I know it is because $x = 0$, so I choose $y = 1$. If, on the other hand, I am told that $\psi(x)$ is not true, I know $x \neq 0$, so I choose $y = 0$.
Claim 2 φ' has no finite models. Suppose $\mathfrak{A} = \langle A, \leq, 0 \rangle$ were one. Now $\psi(x)$ is true independently of x. So $\exists$ has no way of choosing $y \neq x$ on the basis of whether $\psi(x)$ is true or not. More formally, suppose $\mathfrak{A} \models_X \varphi'$, where $X = \{\emptyset\}$. Then $\mathfrak{A} \models_{X[A:x]} (\exists y /\!\!/_{\psi(x)})(y \neq x)$. Let $X[A : x] = X_0 \cup X_1$ be a $\psi(x)$- homogeneous partition and $y_0, y_1 \in A$ such that $\mathfrak{A} \models_{X_0[y_0:y]} y \neq x$ and $\mathfrak{A} \models_{X_1[y_1:y]} y \neq x$. Since $\psi(x)$ is always true, $X_0 = \emptyset$ or $X_1 = \emptyset$. Say $X_1 = \emptyset$. Thus $\langle x, y_0 \rangle \in X_0$, whence $\mathfrak{A} \models_{X_0[y_0:y]} y = x$, a contradiction. $\square$

Let $FI(FO)$ denote the extension of FI where any first-order θ is allowed to occur in $/\!\!/_\theta$. Lemma 2 implies that $FI(FO)$ contains all of first-order logic. Let $FI(IF)$ denote the extension of FI where any θ from IF-logic is allowed to occur in $/\!\!/_\theta$. We know that non-well-foundedness can be expressed in the IF-logic. Lemma 2 implies that $FI(IF)$ can express also well-foundedness. Thus $FI(IF)$ is not included in IF-logic.

The FI as we have defined it turns out to be translatable into first-order logic:

Theorem 11 *Every FI-sentence is equivalent to an $\exists\forall$-sentence, and vice versa, every $\exists\forall$-sentence is equivalent to an FI-sentence.*

Proof. One direction follows from Theorem 8. For the converse implication it suffices to notice that following are equivalent:

$$\begin{aligned} &\mathfrak{A} \models (\exists x_1)\ldots(\exists x_n)(\forall y_1)\ldots(\forall y_m)\varphi \\ &\mathfrak{A} \models_{\{\emptyset\}} (\exists x_1/\!\!/)\cdots(\exists x_n/\!\!/)(\forall y_1/\!\!/)\ldots(\forall y_m/\!\!/)\varphi', \end{aligned}$$

where φ is obtained from φ by replacing each disjunction $\theta(\vec{x}) \vee \psi(\vec{x})$ by $\theta(\vec{x}) \vee /\!\!/_{\theta(\vec{x}),\psi(\vec{x})}\psi(\vec{x})$. Note that ϕ is quantifier-free, so its subformulas can occur in connection with $/\!\!/$. We assume that $\exists$ knows her own strategy. □

Theorem 12 *FI has an exponential compression relative to first order $\exists\forall$ logic.*

Proof. Consider the structure $\mathfrak{A}$ whose domain consists of all binary numerals. The predicate $C(x, y)$ means that $y = x + 1 \ mod \ 2^n$. Of course $0 \leq y < 2^n$. The predicate $P_i(x)$ for $i \leq n$ means that the i-th digit of x from the right is 1. Consider the formula $\theta = (\forall x)(\exists y/\!\!/P_1(x), ..., P_n(x))C(x, y)$. The formula says that $\exists$ can choose y knowing only the truth values of $P_i(x) : i \leq n$. θ is true in $\mathfrak{A}$, and remains true if we only take integers $< 2^n$. But it is not true in any sub-structure of size $< 2^n$. Thus any $\exists\forall$ formula which was equivalent to θ would have to have at least 2^n quantifiers. □

However, note that if we use full first order logic to express θ we do not need exponential growth. For the formula $\phi = (\forall x)(\exists y)(\forall z)([\bigwedge_{i\leq n} P_i(x) \leftrightarrow$

$P_i(z)] \to C(z, y))$ *is* equivalent to θ. If $\forall$ is allowed to change his move *after* $\exists$ has played hers then she is in effect restricted to what she could have done had she known *only* the values of the booleans.

We now show that every model of a FI-formula has a finite submodel of at most exponential size.

Theorem 13 *Let $\mathfrak{A} \models \varphi$ where the logical complexity of φ is n. Then $\mathfrak{A}$ has a submodel $\mathfrak{B}$ of φ of size at most $n2^n$.*

Proof.: Assume that φ is written so that all negations apply only to atoms, so that φ is constructed from literals using $\exists, \forall, \vee, \wedge$ only. Eloise has a winning strategy for the game corresponding to φ. For each move $\exists y$ of Eloise, consider the moves $\forall x /\!\!/ P(x)$ in whose scope y lies. There are at most n of such predicates $P(x)$ and the value of y is determined by the truth values of these $P(x)$. (y may be determined also by previous moves y' of Eloise, but these are also determined by these booleans P and therefore by *all* booleans, whether y is in their scope or not.) So consider the set $\mathcal{V}$ of all boolean vectors governing *any* move of Eloise. The cardinality of $\mathcal{V}$ is at most 2^n. For each move $\exists y_i$ of Eloise, her strategy gives a function f_i from $\mathcal{V}$ into A, the domain of $\mathfrak{A}$. Since Eloise has at most n moves, there are at most n functions, and the range of all these functions gives us a subset of A of size at most $n2^n$. Let this subset be B.

Consider the modified game where Abelard is allowed to move in A but Eloise is restricted to move in B. Clearly Eloise is free to use her former winning strategy and wins. Consider now a further restriction where Abelard is also restricted to B. Surely this does not harm Eloise and she still wins. But that means that if $\mathfrak{B}$ is the submodel corresponding to B, its size is at most $n2^n$ and $\mathfrak{B} \models \varphi$. □

This result does not imply an exponential translation of FI logic into $\exists\forall$ logic, but makes it highly likely. Consider an arbitrary formula $\theta \vee \exists x \forall y (y = x)$. Assuming that θ is consistent, consider any of its models M. Then M is also a model of $\theta \vee \exists x \forall y (y = x)$. Now if we take any 1-element submodel of M, it is a model of $\theta \vee \exists x \forall y (y = x)$, but we would not thereby expect $\theta \vee \exists x \forall y (y = x)$ to have a translation into an $\exists\forall$ formula.

Theorem 14 *$FI(FO) = FO$.*

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