Preposcription semantics and KDDc4[*]

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1 KDDc4

In "Command and Consequence", I gave a two-dimensional semantics for a language containing an imperativising operator !, in which ! was a Stalnaker dagger-like operator. I suggested (and in this paper will prove) that the resulting logic is the normal modal logic KDDc4, that is, the normal propositional modal logic K with the following axioms:

(N) if φ is a theorem than !φ is a theorem

(K) !(p→q)→(!p→!q)

(D) !¬p→¬!p

(S4) !p→!!p

(Dc) ¬!p→!¬p

This logic has a standard one-dimensional possible worlds semantics with an accessibility relation (I will call this, for short, the *accessibility semantics* for KDDc4, contrasting with the *preposcription semantics* given in "Command and consequence"). In the accessibility semantics, the semantic value of a sentence is a world (rather than a pair of worlds).

The imperativising operator ! is the box operator of KDDc4, and it is defined in terms of an accessibility relation:

w ∈ V'(⌜!φ⌝) iff for all w' such that wAw', w' ∈ V'(φ)

In order that the axioms (D), (S4), and (Dc) be logical truths on this semantics, the accessibility relation must have the following three features, corresponding to the three axioms, respectively:

(η)	A is *extensible*:	for all w, there is a w' such that wAw'	(D)
(τ)	A is *transitive*:	if wAw' and w'Aw'', then wAw''	(S4)
(φ)	A is *functional*:	if wAw' and wAw'', then w'=w''	(Dc)

I won't rehearse here soundness and completeness proofs for KDDc4 with respect to this semantics – these proofs would resemble textbook soundness and completeness proofs for more familiar modal logics. The only unusual feature of KDDc4 is the presence of the axiom (Dc) – so named because it is the converse of (D). It's easy to see that if A is functional, then (Dc) is valid: suppose that w ∈ V'(¬!p); then there must be some w' accessible from w such that w' ∈ V'(¬p); by functionality, w' is the only world accessible from w, so every world accessible from w is in V'(¬p); so w ∈ V'(!¬p). Conversely, if A is not functional, then (Dc) is invalid: let there be three worlds w, w' and w'' such that wAw' and wAw'', with w' ∈ V'(p) and w'' ∈ V'(¬p); this is a countermodel to (Dc). Armed with those facts, it is easy to adapt standard soundness and completeness proofs for KD4 to KDDc4.

The three features of the accessibility relation work together in an interesting way: notice that they don't require either that worlds are accessible from themselves, or that they are not. For reasons that

* Thanks to David Ripley for pointing out that KDDc4 is the logic generated by preposcription semantics to me. Any errors in the proof below, however, are my own.

will become clear shortly, call a world that is accessible to itself an *ideal* world, and a world that is not accessible to itself a *non-ideal* world. If a world is ideal, then it accesses itself and itself only (by functionality); so, where w is an ideal world, $w \in V'(!\varphi)$ iff $w \in V'(\varphi)$. If a world is non-ideal, then it accesses some other world (by extensibility) and that other world is ideal (by functionality and transitivity). (A proof of this last point: suppose a non-ideal world w accesses another non-ideal world w'; since w' is non-ideal, it must access some world w" such that $w' \neq w''$, then by transitivity, w accesses both w' and w", but that is contrary to functionality; therefore no non-ideal world accesses any other non-ideal world). So the accessibility relation in this semantics partitions the worlds into families each consisting of a single ideal world together with the non-ideal worlds from which it is accessible.

Now that I have described the formal features of KDDc4, I can say what it has to do with imperatives. Think of a world as a specification both of what happens at that world, and of what commands are "in force" at that world (perhaps this latter information is already implicit in "what happens"). In an ideal world, everyone "does as they are told" – every command that is in force at that world is complied with at that world. A non-ideal world is one in which some of the commands that are in force at that world are not complied with. A world w' is accessible from a world w iff exactly the same commands that are in force at w are in force and complied with in w'. This accessibility relation satisfies the formal requirements on accessibility: every non-ideal world accesses exactly one ideal world, and every ideal world accesses itself and itself only.

Here is a way to see the connection between the accessibility semantics and the preposcription semantics: think of the "worlds" of the accessibility semantics as pairs of worlds drawn from the preposcription semantics (or equivalently, as cells of a preposcription matrix). The first member of each pair represents "what happens" at that pair; the second member represents "what commands are in force". Accessibility is as follows: a pair (x,y) accesses the pair (y,y) (and only that pair); to put it in terms of cells, a cell accesses the diagonal cell on its column (and only that cell). An ideal world is a pair (x,x); or, in terms of cells, an ideal world is a diagonal cell. A non-ideal world is any other pair (resp. any non-diagonal cell).

A proof of the soundness and completeness of KDDc4 with respect to the preposcription semantics given in "Command and consequence" follows.

1.1 Preposcription semantics

A *preposcription-interpretation* is any pair (W,V) where:

- W is a set
- V a function from sentences to subsets of W×W; W×W being the set of all pairs (x,y) where x and y are members of W

and where V satisfies the following *valuation clauses*:

$p \in V(\ulcorner \varphi \wedge \psi \urcorner)$ iff $p \in V(\varphi)$ and $p \in V(\psi)$

$p \in V(\ulcorner \varphi \vee \psi \urcorner)$ iff $p \in V(\varphi)$ or $p \in V(\psi)$

$p \in V(\ulcorner \neg \varphi \urcorner)$ iff $p \notin V(\varphi)$

$p \in V(\ulcorner \varphi \rightarrow \psi \urcorner)$ iff $p \notin V(\varphi)$ or $p \in V(\psi)$

$(x,y) \in V(\ulcorner !\varphi \urcorner)$ iff $(y,y) \in V(\varphi)$

A sentence θ is *designated* at p in (W,V) iff $p \in V(\theta)$.

A sentence θ is a *preposcription-consequence* of a set of sentences Γ – $\Gamma \vDash \theta$ – iff, for every preposcription-interpretation (W,V), and for every $p \in W \times W$, if every member of Γ is designated at p in (W,V), then θ is designated at p in (W,V).

1.2 Accessibility semantics

An *accessibility-interpretation* is any triple (W,A,V) where:

- W is a set
- A is a extensible, transitive, functional relation over W
- V a function from sentences to subsets of W

and where V satisfies the following *valuation clauses*:

$w \in V(\ulcorner \varphi \wedge \psi \urcorner)$ iff $w \in V(\varphi)$ and $w \in V(\psi)$

$w \in V(\ulcorner \varphi \vee \psi \urcorner)$ iff $w \in V(\varphi)$ or $w \in V(\psi)$

$w \in V(\ulcorner \neg \varphi \urcorner)$ iff $w \notin V(\varphi)$

$w \in V(\ulcorner \varphi \rightarrow \psi \urcorner)$ iff $w \notin V(\varphi)$ or $w \in V(\psi)$

$w \in V(\ulcorner ! \varphi \urcorner)$ iff for all w' such that wAw', $w' \in V(\varphi)$

A sentence θ is *designated* at w in (W,A,V) iff $w \in V(\theta)$.

A sentence θ is an *accessibility-consequence* of a set of sentences Γ – $\Gamma \vDash' \theta$ – iff, for every accessibility-interpretation (W,A,V), and for every $w \in W$, if every member of Γ is designated at w in (W,A,V), then θ is designated at w in (W,A,V).

1.3 Soundness

We want to prove that KDDc4 is sound with respect to preposcription semantics; that is, that if $\Gamma \vdash \theta$, then $\Gamma \vDash \theta$. To do this, we exploit the fact that KDDc4 is sound and complete with respect to accessibility semantics – that $\Gamma \vdash \theta$ iff $\Gamma \vDash' \theta$. So what we need to show that is that if $\Gamma \vDash' \theta$, then $\Gamma \vDash \theta$. We will prove this by proving its contrapositive: if $\Gamma \nvDash \theta$, then $\Gamma \nvDash' \theta$.

Suppose that $\Gamma \nvDash \theta$. Then there is a preposcription-interpretation (W,V) and a $p \in W \times W$, on which all members of Γ are designated but θ is not designated. We will construct an *accessibility*-interpretation (W',A,V') on which, for some $w \in W'$, all members of Γ are designated at p but θ is not designated at w. That will show that $\Gamma \nvDash' \theta$. This is particularly easy because we can make the "worlds" of the accessibility-interpretation be the pairs of worlds from the preposcription-interpretation; then V is already a suitable valuation function for an accessibility-interpretation.

Let W' be $W \times W$.

Let (x,y)A(x',y') iff y=x'=y'. I.e A is the smallest relation such that for all $x,y \in W$, (x,y)A(y,y).

Let V' be identical to V.

It is easily checked that A is extensible, transitive and functional. Since each of the first four valuation clauses in the definition of an accessibility-interpretation must be satisfied by the valuation function of a preposcription-interpretation, and V' is such a function, each of those clauses is satisfied by V'. Because of the way A was defined above, the fifth clause is equivalent to:

$(x,y) \in V(\ulcorner !\varphi \urcorner)$ iff for all (x',y') such that $y=x'=y'$, $(x',y') \in V(\varphi)$

which is equivalent to the fifth valuation clause in the definition of a preposcription-interpretation. So (W',A,V') satisfies the definition of an accessibility-interpretation.

Since V' is identical to V, all members of Γ are designated and θ not designated at p on the interpretation (W',A,V'). Therefore $\Gamma \nvDash' \theta$. So, by conditional proof, if $\Gamma \nvDash \theta$, then $\Gamma \nvDash' \theta$ – KDDc4 is sound with respect to preposcription semantics.

1.4 Completeness

Now we want to prove that KDDc4 is complete with respect to preposcription semantics; that is, that if $\Gamma \vDash \theta$, then $\Gamma \vdash \theta$. Again, we assume that KDDc4 is sound and complete with respect to accessibility semantics. So we need to show that if $\Gamma \vDash \theta$, then $\Gamma \vDash' \theta$; again, we prove the contrapositive: if $\Gamma \nvDash' \theta$, then $\Gamma \nvDash \theta$.

Suppose that $\Gamma \nvDash' \theta$. Then there is an accessibility-interpretation (W',A,V') and a w∈W' on which which all members of Γ are designated but θ is not designated. We will construct an preposcription-interpretation (W,V) on which, for some w∈W', all members of Γ are designated at w but θ is not designated at w; so that $\Gamma \nvDash \theta$.

Let W be the set containing w and any worlds w' such that wAw'.

Because of the features of the accessibility relation A, W contains either one or two worlds, one of which is ideal (recall, an ideal world is any world that accesses itself).

Let V be the function defined as follows:

$(x,y) \in V(\varphi)$ iff $f(x,y) \in V'(\varphi)$

where

$f(x,y)$ = the ideal world in W (call it w_i), if $x=y$;

or the non-ideal world in W (call it w_n), if $x \neq y$.

A way of seeing what is going on here is to think of the definition of V above as a being a way of populating a preposcription matrix. Where W has two members (one ideal, one non-ideal), the matrix created will be as follows (where a cell is ticked iff the formula in it is true):

φ	w_n	w_i
w_n	$w_i \in V'(\varphi)$	$w_n \in V'(\varphi)$
w_i	$w_n \in V'(\varphi)$	$w_i \in V'(\varphi)$

Where W has one member, the matrix is as follows:

φ	w_i
w_i	$w_i \in V'(\varphi)$

So defined, V satisfies the valuation clauses in the definition of a preposcription-interpretation. It is easy to see this for the first four clauses – take negation for example, for which the relevant clause is:

$(x,y) \in V(\ulcorner \neg\varphi \urcorner)$ iff $(x,y) \notin V(\varphi)$

If we substitute in the definition of V in terms of V' given above we get:

$f(x,y) \in V'(\ulcorner \neg\varphi \urcorner)$ iff $f(x,y) \notin V'(\varphi)$

which is a consequence of the corresponding clause for negation in the accessibility semantics.

V also satisfies the fifth valuation clause:

$(x,y) \in V(\ulcorner !\varphi \urcorner)$ iff $(y,y) \in V(\varphi)$

Again, substituting in the definition of V in terms of V':

$f(x,y) \in V'(\ulcorner !\varphi \urcorner)$ iff $f(y,y) \notin V'(\varphi)$

This, in turn, will always be true, because f(y,y) will always be the ideal world w_i, and f(x,y) will always be a world from which w_i, and w_i alone is accessible. So V satisfies the valuation clauses in the definition of a preposcription-interpretation – so (W,V) is a preposcription-interpretation.

Recall that w is the member of W' at which all members of Γ are designated but θ is not designated in (W',A,V'). w is guaranteed to be in W (by the way W was constructed) and is guaranteed to be the value of f(x,y) for some x,y in W. So there is some pair (x,y) such that

$f(x,y) \in V'(\varphi)$ for each $\varphi \in \Gamma$ and $f(x,y) \notin V'(\theta)$

Which is to say, using the definition of V above:

$(x,y) \in V(\varphi)$ for each $\varphi \in \Gamma$ and $(x,y) \notin V(\theta)$

Which is to say that $\Gamma \not\models \theta$. Therefore, by conditional proof, if $\Gamma \not\models' \theta$, then $\Gamma \not\models \theta$ – KDDc4 is complete with respect to preposcription semantics.