Two Criticisms against Mathematical Realism

Abstract
Mathematical realism asserts that mathematical objects exist in the abstract world, and that a mathematical sentence is true or false, depending on whether the abstract world is as the mathematical sentence says it is. I raise two objections against mathematical realism. First, the abstract world is queer in that it allows for contradictory states of affairs. Second, mathematical realism does not have a theoretical resource to explain why a sentence about a tricle is true or false. A tricle is an object that changes its shape from a triangle to a circle, and then back to a triangle with every second.

Keywords
Abstract World, Concrete World, Mathematical Realism, Mathematical Object, Tricle

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1. Introduction

According to mathematical realism (Balaguer, 2016; Busch and Morrison, 2016: 436; Linnebo, 2017), mathematical entities, such as numbers, triangles, and functions, exist in the abstract world, and a mathematical sentence is true or false, depending on whether the abstract world is as the mathematical sentence says it is. This definition captures the views defended by mathematical realists in the rationalist vein, such as Gottlob Frege (1884) and Kurt Gödel (1947), and the views defended by mathematical realists in the empiricist vein, such as Willard Quine (1948, 1980, 1992), Hilary Putnam (1971), Michael Resnik (1997), Mark Colyvan (2001), and Alan Baker (2005, 2009, 2012, 2016). Both rational and empirical mathematical realists hold the same metaphysical and semantic theses of the preceding definition of mathematical realism.

They, however, diverge on the issue of how we can acquire knowledge about mathematical objects. Rational mathematical realists contend that we can acquire mathematical knowledge with the use of the cognitive faculty, mathematical intuition. On their account, our mind can somehow grasp mathematical objects existing in the abstract world. By contrast, empirical mathematical realists contend that we can acquire mathematical knowledge via confirming mathematical statements with empirical evidence. In order to defend this epistemological thesis, they have constructed sophisticated arguments called the Quine-Putnam indispensability argument (Quine, 1948, 1980, 1992; Putnam, 1971; Resnik, 1997; Colyvan, 2001) and the enhanced indispensability argument (Baker, 2006, 2009, 2012, 2016).

This paper focuses not on the epistemological issue of how we can acquire mathematical knowledge but on the metaphysical issue of whether mathematical objects exist in the abstract world or not, arguing that the abstract world does not exist. If the abstract world does not exist, it is problematic for mathematical realists to say that mathematical sentences are rendered true or false by the abstract world, and that we can acquire mathematical knowledge via confirming mathematical sentences with observational evidence. In short, mathematical realism collapses along with the metaphysical thesis.

The outline of this paper is as follows. In Section 2, I display the properties of mathematical objects, comparing them with concrete objects, under the framework of mathematical realism. In Section 3, I unfold two objections to mathematical realism. In Section 4, I introduce an alternative position that competes with mathematical realism. This paper is intended to be helpful to those who defend alternative positions, such as mathematical inferentialism (Park, 2017) and mathematical fictionalism (Field, 1980, 1989; Balaguer, 1996, 1998, 2001, 2009; Rosen, 2001; Yablo, 2002; Leng, 2005a, 2005b, 2010).

2. Mathematical Objects

In order to understand what mathematical realism asserts, we first need to understand what the abstract world is in comparison to the concrete world. The concrete world is inhabited by concrete entities, such as cats, trees, and electrons. Concrete entities are temporal and spatial, so it is adequate to attribute spatial and temporal predicates to them. It is not a category mistake, for example, to say that a cat is <u>here</u>, and that it <u>fought</u> with a dog <u>yesterday</u>. In contrast, the abstract world is inhabited by abstract entities, such as mathematical objects and propositions. Abstract entities are atemporal and aspatial, so it is inadequate to attribute spatial and temporal predicates to them. It is a category mistake, for example, to say that number one exists <u>here</u>, or that one plus one <u>was</u> two <u>yesterday</u>.

Concrete and abstract entities differ from each other in another important respect. Concrete entities are causally efficacious, so they can interact with one another, as the aforementioned example of the cat and the dog illustrates. By contrast, abstract entities are causally inert, so they can interact neither with one another nor with concrete entities. After all, an interaction requires space and time. To say that two objects interact with each other in the abstract world implies that they interact with each other in no place and in no time, which in turn implies that they do not interact with each other. Therefore, it is a contradiction to say that abstract entities interact with one another.

Benjamin Callard (2007), however, argues that mathematical objects, although abstract, can be causally efficacious in that it can produce mathematical knowledge in human beings.[1] This paper rejects Callard's characterization of mathematical objects, and operates under other philosophers' characterization of them. Daniele Molinini, Fabrice Pataut, and Andrea Sereni take mathematical realism as asserting that mathematical objects are "thought to be abstract, non spatial, non temporal and non causal" (2016: 318). Øystein Linnebo also takes mathematical realism as asserting that mathematical objects are abstract, and "an object is said to be abstract just in case it is non-spatiotemporal and (therefore) causally inefficacious"

[1] See Jody Azzouni (2008) and Andrea Sereni (2016) for responses to Callard. It requires a separate paper to explore this debate.

(2017). All these philosophers take mathematical realism as maintaining that mathematical objects are not causal.

Mathematical realists posit the existence of the abstract world to secure the objectivity of mathematics. They argue that a mathematical sentence is true or false just as objectively as a concrete sentence is true or false. A concrete sentence, 'The Earth is round,' is true, not because we think that it is true, but because the concrete world is as the sentence says it is. Likewise, a mathematical sentence, '1+1=2,' is true not because we think that it is true but because the abstract world is as the mathematical sentence says it is. If you believe that '1+1=3,' you have a false belief about the abstract world.

3. Two Criticisms

3.1. The Queer World

Let me now raise an objection against mathematical realism. The abstract world is queer, to say the least. Consider a line and a point not on it. Euclidean geometry claims that only one line can be drawn that goes through the point without intersecting the original line. In contrast, Lobachevskian geometry claims that an infinite number of such lines can be drawn, while Riemannian geometry claims that no such line can be drawn. An interesting question arises. How many such lines can be drawn in the abstract world? Mathematical realists would say that no such line, one such line, and infinitely many such lines can be drawn in the abstract world, given that they believe that all the three geometries are true of the abstract world. To say so, however, is to admit that the abstract world allows for contradictory states of affairs.

Mathematical realists might reply that the abstract world has three distinct regions: the Euclidean region, the Lobachevskian region, and the Riemannian region. The three regions obey the rules of Euclidean, Lobachevskian, and Riemannian geometries, respectively. Therefore, it is not problematic to say that the three geometries are all true of the abstract world.

The preceding reply, however, has two problems. First, it is *ad hoc* to claim that the abstract world has the three distinct regions. This claim is advanced purely for the sake of diverting my objection that the abstract world allows for contradictory states of affairs. Mathematical realists have to offer an independent account of how many regions there are in the abstract world. In other words, they should set the three geometries aside and give a story of how many regions there are in the abstract world. Second, given that there is no space in the abstract world, it is not clear whether it makes sense to say that the aspatial world has the three distinct regions. It appears contradictory to say that something does not have width, length, and height, but it is composed of different regions.

Mathematical realists might argue that my objection to mathematical realism confuses abstract and concrete regions. There are no concrete regions in the abstract world, but there are abstract regions in the abstract world. Abstract regions are different from concrete regions in that abstract regions do not have width, length, and height, but concrete regions have width, length, and height. There are abstract regions in the abstract world, just as there are concrete regions in the concrete world.

This reply, however, can be reduced to absurdity. If the notion of abstract region is coherent, so should be the notion of abstract blood. So it should also be legitimate to say, for example, that a triangle and a square collided with each other, and as a result they shed blood, but that their blood is not concrete but abstract, meaning that it does not occupy space and time, and hence does not have width, length, height, and color. Our intuition says, however, that this talk of the abstract blood is absurd. If it is absurd, however, so is the talk of the abstract region. Both 'abstract blood' and 'abstract region' are empty expressions referring to nothing.

Mathematical realists might retort that my preceding objection to mathematical realism is built upon the assumption that geometrical objects are mathematical objects. If geometrical objects are not mathematical objects, my objection loses its bite (referee).

I am, however, following mathematical realists when I treat geometrical objects as mathematical objects. Fabrice Pataut (2016: 352) takes Baker's (2005, 2009) enhanced indispensability argument for mathematical realism as implying that numbers and *polygons* exist. According to Balaguer, mathematical realism holds "that the argument for the existence of mathematical objects is entirely general, covering all branches of mathematics, including geometry, so that on this view, we already have reason to believe in lines and shapes, as well as numbers" (Balaguer, 2016). Note that these philosophers treat geometrical objects as mathematical objects. In my view, all geometrical objects are mathematical objects. No geometrical object can be a concrete object. Of course, there are concrete objects that resemble geometrical objects. Strictly speaking, however, they are not geometrical objects. For example, a string resembles a line, but it is not a line. After all, a string occupies space, whereas a line, by definition, does not.

Mathematical realists might now give up the contention that the abstract world consists of different regions, and contend that there are three abstract worlds. One of them makes Euclidean geometry true, another makes Lobachevskian geometry true, and the last one makes Riemannian geometry true. On this account, mathematical realism is not saddled with the view that there is only one abstract world which makes all the true mathematical sentences true.

This move, however, is also ad hoc, multiplying abstract worlds solely for the sake of diverting my objection that the abstract world allows for contradictory states of affairs. If such a move were legitimate, a similar move should also be legitimate in empirical science. For example, Galileo observed the phases of Venus, thereby refuting the Ptolemaic theory. Imagine, however, that Ptolemaic scientists replied that the Ptolemaic theory is still true because there is another concrete world which makes the Ptolemaic theory true. We would say that it is absurd to multiply concrete worlds to save the Ptolemaic theory. Analogously, we would say that it is absurd to multiply abstract worlds to save mathematical realism.

Mathematical realists might argue that there are infinitely many abstract worlds, just as there are infinitely many concrete worlds (referee). Some physicists today seriously consider the multiverse hypothesis according to which there are infinitely many universes. The support for the multiverse hypothesis came from three independent researches: the researches on eternal inflation, dark energy, and string theory (Greene, 2011). If it is reasonable to suppose that there are infinitely many concrete worlds, it would also be reasonable to suppose that there are infinitely many abstract worlds. If there are infinitely many abstract worlds, Euclidean geometry would be true of some of those abstract worlds. So would be Lobachevskian geometry and Riemannian geometry. Hence, it is not ad hoc to suggest that the three geometries are true of different abstract worlds.

No scientist has yet presented direct experimental evidence for the multiverse hypothesis, and the hypothesis is contentious in the physics community (Greene, 2011). Set this scientific issue aside, however, and suppose for the sake of argument that there are infinitely many concrete and abstract worlds. If mathematical realists appeal to the existence of infinitely many abstract worlds to divert my preceding objection, it is not clear on what grounds they can say that it is false that 1+1=3. After all, there might be some abstract worlds that make '1+1=3' true. What is the guarantee that there are no such abstract worlds?

Recall that mathematical realists postulate the existence of the abstract world to ensure that a mathematical sentence is objectively true or false. They have achieved the objectivity of mathematics at the cost of obfuscating the distinction between true and false mathematical sentences. This obfuscation comes with an enormous practical disadvantage. Imagine that some students state in their mid-term exam that 1+1=3, and as a result, teachers give them failing grades. The students protest that it is true that 1+1=3 because some of the infinitely many abstract worlds make it true. It is not clear how teachers, if they are mathematical realists believing in the existence of infinitely many abstract worlds, can persuade their students that their mathematical beliefs are false.

3.2. Tricle

Let me now turn to a different sort of objection to mathematical realism. Imagine that there is an object that changes its shape from a triangle to a circle, and then back to a triangle with every second. Let me call it a 'tricle' (triangle + circle). It is true that a tricle does not have four straight edges. But why is that true? What makes the sentence, 'A tricle does not have four straight edges,' true? Mathematical realists owe us an answer to this perplexing question.

Mathematical realists are in a dilemma. They can say either that a tricle is a mathematical object, or that it is not a mathematical object. On the one hand, if they say that it is a mathematical object, they owe us an answer to the question of how many straight edges it has in the abstract world. They cannot say that it has three straight edges at some times and none at other times, for time does not pass in the abstract world. On the other hand, if they say that it is not a mathematical object on the grounds that it is temporal, they face a disconcerting issue of what counts as a mathematical object. Recall that Callard (2007) claims that an object is a mathematical object, even if it can be causally efficacious. Why is it that an object is not a mathematical object if it is temporal, but it is a mathematical object even if it can be causal?

Mathematical realists might suggest that a tricle is a mathematical object existing in the abstract world, and that my objection to mathematical realism confuses mathematical time with concrete time. Mathematical objects exist outside of concrete time, but they exist inside of mathematical time. So it makes sense to say that a tricle changes its shape with the flow of mathematical time, and that it has three straight edges at some mathematical times, but none at other mathematical times, in the abstract world. This reply would be tempting to those who invoke the abstract world and the abstract regions to argue that a mathematical sentence is objectively true or false.

The reply, however, can be reduced to absurdity just like the previous reply that concerns abstract regions. If the notion of mathematical time is coherent, it should also be legitimate to say, for example, that a triangle and a square collided with each other yesterday, but that the collision is not a concrete collision but an abstract collision. The difference between them is that the former involves deformation while the latter does not. Also, the term 'yesterday' does not refer to a point in concrete time, but it refers to a point in mathematical time. Our intuition says, however, that this talk of abstract collision is absurd. If it is absurd, however, so is the talk of the mathematical time. Both 'abstract collision' and 'mathematical time' are empty expressions referring to nothing.

Mathematical realists might now suggest that the aforementioned sentence, 'A tricle does not have four straight edges,' is analytically true, i.e., it is true solely by virtue of the definitions of the words in it. This suggestion, however, is not available to mathematical realists, for it undermines mathematical realism. Mathematical antirealists can make the same suggestion about a mathematical sentence like '1+1=2.' That is, this mathematical sentence is true not in virtue of the way the abstract world is but solely in virtue of the definitions of the words in the sentence. Thus, it is otiose to posit the existence of the abstract world.[2]

[2] A referee brings my attention to the Fregean tradition according to which a mathematical statement is both analytic and true of the abstract world. An examination of this tradition, however, has to await another occasion.

The only way for mathematical realists to defuse my preceding objection is to present a relevant difference between ‘A tricle does not have four straight edges’ and ‘1+1=2.’ Why is it that the former sentence is a true solely by virtue of the definitions of its words, whereas the latter is true in part by virtue of the way the abstract world is? I leave the task of answering this question to mathematical realists.

4. Mathematical Inferentialism

My two objections to mathematical realism sketched in the previous section surround the thesis that mathematical objects exist in the abstract world. As mentioned earlier, mathematical realism posits the existence of the abstract world in order to ensure that a mathematical sentence is objectively true or false. This section introduces an alternative position that secures the objectivity of mathematics without postulating the existence of the abstract world.

Mathematical inferentialism (Park, 2017: 71-74) holds that the concrete world exists whereas the abstract world does not, a mathematical sentence does not even purport to describe the abstract world, it facilitates deductive inferences from some concrete sentences to other concrete sentences,[3] it is true if and only if only true concrete sentences can be derived from it along with other concrete sentences, and it is false if and only if a false concrete sentence can be derived from it along with other concrete sentences.[4]

To take an example, the mathematical sentence, ‘1+1=2,’ facilitates the deductive inference from the concrete sentences, ‘Alice gave me an orange’ and ‘Bob gave me an orange,’ to another concrete sentence, ‘Alice and Bob gave me two oranges in total.’ It is true that 1+1=2 because the conjunction of the mathematical sentence and true concrete sentences entails only true concrete sentences, and never entails a false concrete sentence. It is false that 1+1=3 because the conjunction of the mathematical sentence and true concrete sentences entails false concrete sentences like ‘Alice and Bob gave me three oranges in total.’

Given that a concrete sentence is true or false in virtue of the way the concrete world is, it is ultimately the concrete world that makes ‘1+1=2’ true and ‘1+1=3’ false. A mathematical sentence is true not because we think that it is true but because the conjunction of it and true concrete sentences issues only true concrete sentences. Thus, mathematical inferentialism ensures that a mathematical sentence is objectively true or false without positing the existence of the abstract world.

Moreover, given that it is ultimately the concrete world that makes a mathematical sentence true or false, we can know whether a mathematical sentence is true or false by observing the concrete world. Recall that it is true that 1+1=2, and false that 1+1=3 because when conjoined with the concrete sentences, ‘Bob gave me an orange’ and ‘Alice gave me an orange,’ the former entails the true concrete sentence, ‘Alice and Bob gave me two oranges in total,’ while the latter entails the false concrete sentence, ‘Alice and Bob gave me three oranges in total.’ We can know whether these concrete sentences are true or false by observing the concrete world. So we can know whether they entail true or false concrete sentences or not, i.e., whether they are true or false. Under the framework of mathematical inferentialism, there is no epistemological puzzle over how we acquire mathematical knowledge.[5]

5. Conclusion

[3] The referee observes that Field (1980, 1989) also stresses that mathematics facilitates deductive inferences.
[4] Thus, mathematical inferentialism is different from if-thenism explicated in Balaguer (2015).
[5] The referee points out that several issues can be raised against mathematical inferentialism.

This paper has raised two objections to mathematical realism. First, the abstract world is queer in that it allows for contradictory states of affairs, so it does not exist, and it is problematic to suggest that the abstract world makes mathematical sentences true or false. Second, mathematical realism does not have a theoretical resource to explain why a sentence about a tricle is true or false. Mathematical realism is vulnerable to these objections because it invokes the abstract world which allegedly makes mathematical sentences true or false. There is, however, an alternative position, viz., mathematical inferentialism, that secures the objectivity of mathematics without positing the existence of the abstract world.

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