Lakatos and the Euclidean Programme[1]

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Abstract

Euclid's *Elements* inspired a number of foundationalist accounts of mathematics, which dominated the epistemology of the discipline for many centuries in the West. Yet surprisingly little has been written by recent philosophers about this conception of mathematical knowledge. The great exception is Imre Lakatos, whose characterisation of the Euclidean Programme in the philosophy of mathematics counts as one of his central contributions. In this essay, we examine Lakatos's account of the Euclidean Programme with a critical eye, and suggest an alternative picture which builds on his work, but differs in a number of important respects.

In the *Elements*, Euclid starts from some definitions, postulates and common notions, from which he derives the geometry of his day theorem by theorem, in a cumulative manner over the course of 13 books. Euclid himself offers no philosophical gloss on his method; that is to be expected, as he is a mathematician rather than a philosopher. Others, however, have not shied away from doing so. They have read into the *Elements* a methodological ideal to be emulated throughout mathematics and elsewhere. The *Elements* has inspired a foundationalist vision of mathematical knowledge, even of knowledge in general.

Although Euclidean foundationalism has been much praised over the centuries, it has been little analysed by recent philosophers. An important exception is Imre Lakatos, whom the present volume honours. Lakatos was no Euclidean; quite the contrary. But he believed in knowing his enemy, so was careful to describe the Euclidean picture in some detail. With Euclideanism as a foil, he developed his own 'quasi-empiricist' and fallibilist epistemology of mathematics.

We aim to follow in Lakatos's footsteps and take a closer look at Euclideanism. Our main motivation is that although the picture is commonly referred to, it is not entirely clear what it is. Contemporary philosophers are superficially familiar with 'Euclidean foundationalism' in the philosophy of mathematics; but dig down, and the details are fuzzy. Euclidean foundationalism is like a great-aunt who has always been around and seems very familiar, though you have never bothered to get to know her. When you finally have a long conversation with her, you realise quite how fascinating she is, even if you don't necessarily agree with her. Lakatos would have concurred: he wrote, 'The fascinating story of the Euclidean programme and of its breakdown has not yet been written' (1962, p. 6). The first half of that story, before the breakdown, must start with what the programme actually is.

Our essay is devoted to drilling down into the details of Euclideanism, with Lakatos as our guide. It falls into two parts. The first and principal part (§1) outlines Lakatos's views about Euclidean foundationalism, which we follow him in calling the Euclidean Programme, or EP

[1] We are grateful to audience members at the Imre Lakatos Centenary Conference in November 2022 for helpful comments and suggestions, and to Brendan Larvor and an anonymous referee for comments on a later draft.

for short. Along the way, we analyse his account of it, noting where we part company with him. In §2, using Lakatos's discussion as a springboard but moving beyond it, we characterise the EP in our preferred way by means of seven principles. Our own assessment of where the EP stands today is too lengthy and unrelated to Lakatos for inclusion in this volume; it may be found in our forthcoming book *The Euclidean Programme* (Paseau and Wrigley 2024). In the present essay, our focus is on comparing and contrasting Lakatos's account of the EP to our own.

1. Lakatos on the EP

Lakatos wrote about the EP in several places. It crops up in his writings as something to be opposed, attacked and rejected, sometimes head-on, sometimes glancingly. An article in which the focus is squarely on the EP is the relatively early piece 'Infinite Regress and Foundations of Mathematics' (Lakatos 1962). In this article, Lakatos considers several ways of organising knowledge in a deductive system. The Euclidean way he characterises as follows:

> I call a deductive system a 'Euclidean theory' if the propositions at the top (axioms) consist of perfectly well-known terms (primitive terms), and if there are infallible truth-value-injections at this top of the truth-value True, which flows downwards through the deductive channels of truth-transmission (proofs) and inundates the whole system. (If the truth-value at the top was False, there would of course be no current of truth-value in the system.) Since the Euclidean programme implies that all knowledge can be deduced from a finite set of trivially true propositions consisting only of terms with a trivial meaning-load, I shall call it also the *Programme of Trivialization of Knowledge*. Since a Euclidean theory contains only indubitably true propositions, it operates neither with conjectures nor with refutations. In a fully-fledged Euclidean theory meaning, like truth, is injected at the top and it flows down safely through meaning-preserving channels of nominal definitions from the primitive terms to the (abbreviatory and therefore theoretically superfluous) defined terms. A Euclidean theory is *eo ipso* consistent, for all the propositions occurring in it are true, and a set of true propositions is certainly consistent (1962, pp. 4–5).[2]

In the rest of this section, we'll dissect this passage, and others, to extract some of the key features that Lakatos ascribes to the EP. We also assess them and highlight where we part ways with Lakatos. In §2, we shall present our own preferred codification of the EP by means of seven principles.

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[2] There is a footnote accompanying a sentence in this passage (ending with the words '*Programme of Trivialization of Knowledge*'). The footnote refers to Pascal's *De l'esprit géométrique* ('On the Geometrical Mind'), which Lakatos calls the EP's *locus classicus*. Never published in Pascal's lifetime, the *Esprit* is a short work that influenced the *Port-Royal* Logic. In his one-sentence footnote, Lakatos refers to it as 'Pascal [1657–8]', but more recent scholarship has tended to settle on 1655 as the date of its composition, following Jean Mesnard, the editor of Pascal's works. Lakatos's reference to it is interesting in that the work is largely unknown to English-speaking philosophers; indeed, the philosophical literature on the *Esprit* in the analytic tradition is virtually non-existent, even today. For discussion of the *Esprit* by us, see §5.2 of our book (Paseau and Wrigley 2024).

We should clarify at the outset that we aren't concerned with Lakatos's criticisms of the EP, or with his criticism of dogmatic epistemology more generally or formalism more specifically. We will not, for example, consider how Lakatos's criticism of formalism stands up in today's age of computer proof. Our chief concern is his *description* of the EP, and our question is whether he got this target right. Of course, the EP is a rational reconstruction, not a historically attested manifesto, so there is some leeway in how to describe it. Nevertheless, given its history, there is something to get right here, and in our opinion Lakatos manages to do so for some aspects of the EP, but not others. This will be our concern in the rest of this section.

Truth

Lakatos emphasises, both in the core passage above and elsewhere, that the axioms of a Euclidean theory are true, or at least aspire to be true. He is clearly right that the axioms of a Euclidean theory are (supposed to be) true, and that this is an essential aspect of the EP.

Lakatos is not particularly clear about precisely *why* the truth of the axioms is an essential feature of the EP, and appears to take this for granted. An obvious point is that the EP is a foundationalist account of mathematical knowledge, and knowledge implies truth. It also chimes with how mathematicians down the ages have thought of, say, geometry, unhesitatingly taking its axioms to be correct.

A further point is that the major figures in the history of mathematics and its philosophy that one would want to identify as Lakatos's targets are all explicit that mathematics is a body of truths, starting from true axioms or first principles. For example, Aristotle gives an account of how *episteme* ('understanding' or 'scientific knowledge', which he identifies the highest epistemic state) can be gained from demonstrations, in terms that resemble Lakatos's characterisation of proof in the EP. And Aristotelian demonstrations all start from true principles (*Posterior Analytics* I.2 71b17-25).[3] Centuries later, Descartes gives a Euclidean account of *scientia* (the epistemic ideal of the scholastic and early modern periods) where first principles are truths which are understood so clearly and distinctly as to be rationally indubitable (1637/2001: 16–17; AT 6: 19).[4] And Pascal, who Lakatos paints as the arch-Euclidean (see footnote 2, above) unambiguously describes the axioms of geometry as 'vérités' ('truths').

So on the truth of the axioms, we are in complete agreement with Lakatos. No commitment is thereby made to any particular analysis or philosophical account of truth. As we shall see below, however, we do not entirely share Lakatos's view about the role that truth plays in a Euclidean theory.

Flow

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[3] Although Aristotle predates Euclid by a few decades, one way to read him is as an early exponent of the EP. There are several important commonalities in between Aristotle's account of method and the EP as described by Lakatos. And naturally, Aristotle's account of geometric method was influenced by the geometry of his time, which was likely very similar to that of Euclid's time. For more details, see §4 of our book, (Paseau and Wrigley 2024).
[4] The first citation is to Olscamp's English translation of the *Discourse*. The second is to Adam and Tannery's *Oeuvres de Descartes* in volume: page format, for readers who wish to consult the original texts.

One of the most interesting features of the passage above is Lakatos's metaphor about the *flow* of truth in a Euclidean system, downward from axioms at the 'top' of the theory to theorems at the bottom. To illustrate the significance of this point, Lakatos contrasts the Euclidean Programme with the 'Empiricist Programme'.

> The Euclidean programme proposes to build up Euclidean theories with foundations in meaning and truth-value at the top, lit by the *natural light of Reason*, specifically by arithmetical, geometrical, metaphysical, moral, etc. intuition. The Empiricist programme proposes to build up Empiricist theories with foundations in meaning and truth-value at the bottom, lit by the *natural light of Experience*. Both programmes however rely on Reason (specifically on logical intuition) for the safe transmission of meaning and truth-value. (1962, p. 5)

It is important to appreciate that empiricist theories do not need to be strictly empirical. In particular, one could have an empiricist account of mathematics, in Lakatos's sense. The salient epistemological point is that in an empiricist theory, the relevant flow is not *downward*, from axioms to theorems, but rather *upward* from 'basic statements' (perhaps observations, or elementary arithmetical sentences) to higher-level statements (perhaps theoretical scientific principles, but perhaps mathematical axioms).

Lakatos returns to Euclideanism in a later work, 'A Renaissance of Empiricism in the Recent Philosophy of Mathematics'.[5] In this article, he clarifies that empiricist theories are *quasi-empirical*. The relevant direction of flow is upward, and what is transmitted is typically falsity, rather than truth. If a theory, empirical or otherwise, implies a false basic statement, this 'inundates' the system, which is refuted. To think that truth, in addition to falsity, can be transmitted upward is to indulge in what Lakatos refers to in Popperian vein as the *inductivist delusion* (1976a, p. 41).

We do not wish to dwell on this point, since we are not concerned here with quasi-empirical theories. We simply note, against Lakatos, that the inductivist idea of basic statements retransmitting truth to the axioms which imply them is not obviously a delusion, as he characterises it. It is common enough, of course, to take a scientific theory to be confirmed to some degree when its observational predictions are correct. And in mathematics as well, many philosophers have thought that axioms are given some degree of confirmation when they imply elementary truths which we already take ourselves to know (such as that 2+2=4, for instance).[6] But Lakatos is clearly correct about the direction of flow in the Euclidean account of mathematics. Indeed, so deeply embedded is this idea that it has seemed obvious to many that the direction of flow is top-down in mathematics, an idea just as obvious as that the direction of flow in the empirical sciences is bottom-up.

But Lakatos does more than just identify this commitment of the EP. In the above passages, and in others to be quoted later, he consistently talks of truth-value, and of meaning, as flowing through the channels of the system, a point to which we return to with a more critical eye later in this section. His focus on meaning and truth notwithstanding, he also offers what

[5] This essay, posthumously published as Lakatos (1976a), is an expanded version of an earlier 1967 paper.
[6] For an influential manifestation of this idea, see Russell (1907).

we see as an absolutely crucial insight. This is the observation that the EP is less about *what* flows from axioms to theorems and more about *how* it flows. He puts the point as follows:

> We can get a long way merely by discussing *how* anything flows in a deductive system without discussing the problem of *what in fact flows* there, infallible truth or only, say, Russellian 'psychologically incorrigible' truth, Braithwaitian 'logically incorrigible' truth, Wittgensteinian 'linguistically incorrigible' truth or Popperian corrigible falsity and 'verisimilitude', Carnapian probability. (1962, p. 6)

In other words, what is distinctive of the EP as a methodological account of mathematics is that the mathematician begins with prior access to the axioms, and by means of proof, establishes the theorems. Different Euclideans could mean any number of things by 'access' and 'establish'. With his key observation that what flows in the EP is of lesser interest than how it flows, Lakatos is a locksmith who has opened the way to its proper understanding.

Fourteen years later, in the 'Renaissance' article, Lakatos maintained his insistence on the flow-idea: truth is injected at the top and flows down to the bottom. Indeed, he draws the very distinction between Euclidean and quasi-empirical theories in these terms; as he says, '[i]t is the *how* of the flow that is decisive' (1976a, p. 29).

The insight we extract from Lakatos is, put succinctly, that the EP is all about *Euclidean hydraulics*. An analogy: think of the Phillips machine, a post-war hydraulic model of the economy. Its inventor, Bill Philips, used it to demonstrate how money moves through an economy by letting coloured water flow through clear pipes. In our analogue, the coloured water corresponds to some theoretical good. This for Lakatos is truth, but for us (see §2) it will be something epistemic, such as certainty, knowledge, or justification. But whatever it is, it is injected at the top, where the axioms lie, and thence flows down to the theorems.

What is injected?

Although we agree with Lakatos that the axioms of a Euclidean theory are supposed to be true, we part ways with him in a crucial respect on the point of truth. In the passage cited above from the 'Foundations' paper, Lakatos speaks of a truth and meaning injection. This idea that truth is injected into the theory via the axioms persists into the 'Renaissance' paper, where he writes:

> Classical epistemology has for two thousand years modelled its ideal of a theory, whether scientific or mathematical, on its conception of Euclidean geometry. The ideal theory is a deductive system with an indubitable truth-injection at the top (a finite conjunction of axioms)—so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system (1976a, p. 28)

We find this talk at best misleading, at worst confused. What would it even mean for truth itself to flow from axioms to theorems? In mathematics at least, truth is not tensed: mathematical propositions are either eternally true or false. The theorems of geometry are all eternally true, and there is no literal sense in which the truth of one theorem or axiom is transmitted to that of a theorem. Of course, logicians like to speak of rules being 'truth-preserving', but that image is more easily literalised than the flow or transmission idea: it simply means that if the rule's premises are true then so is the conclusion. It's possible, of course, that Lakatos meant no more than this. A similar point applies to meaning: the meanings of the theorems do not depend on the axioms' meaning. Although perhaps that view is more tenable than the analogous one about truth, especially if the axioms are consciously stipulated at the start of the practice rather than extracted from it.

We highlighted above that the crucial point about flow in a Euclidean theory is its direction. We think the flow metaphor is best construed as transmission of an *epistemic* good of some sort. What this good is exactly will vary from Euclidean theorist to theorist.[7] A modern approach is to think of the epistemic good as justification. The resulting picture is then a foundationalist one in which one gains justification for axioms first and this justification is transferred to the theorems when they are inferred from the axioms. More generally, the EP as we see it represents an epistemological conception. The hierarchical path from axioms to theorems is an epistemic path the mathematician follows, or could follow.[8]

As mentioned, it's quite possible that Lakatos appreciated this point but wrote misleadingly. (Or to be fairer to him, that he wrote in a way that two philosophers in the 2020s taking him very literally find misleading.) He seems to recognise it in passages such as the following:

> Whether a deductive system is Euclidean or quasi-empirical is decided by the pattern of truth value flow in the system. The system is Euclidean if the characteristic flow is the transmission of truth from the set of axioms 'downwards' to the rest of the system—logic here is an *organon of proof*; it is quasi-empirical if the characteristic flow is retransmission of falsity from the false basic statements 'upwards' towards the 'hypothesis'—logic here is an *organon of criticism*. (1976a, p. 29)

The focus on the role of proof in Euclidean theories and criticism in quasi-empirical ones is most welcome. But despite that, he seems to think epistemic facts enable—in the best case, guarantee—truth-injection, rather than constitute the injection itself. This is plain in the talk of truth value being transmitted from axioms to theorems. And as he put it much earlier, empiricists 'criticized the guarantee of the intuitive Euclidean truth-injection: self-evidence' (1962, p. 9). We shall clarify our way putting things in the next section. For now, let's just say that the transmission of an epistemic good from axioms to theorems is the right way to characterise the EP if we are to maintain generality and to take in historical Euclideans.

Finitude

In the long quotation from 'Foundations' at the start of §1, Lakatos describes the axioms of a Euclidean theory as a 'finite set of trivially true propositions'. In the 'Renaissance' passage cited on p.4, he characterises the 'top' of a Euclidean theory as 'a finite conjunction of axioms'. No clear justification for this is given by Lakatos; he merely claims that it is 'implied' by the Euclidean Programme.

That said, this point about finitude is broadly, but not entirely, correct in historical terms. Axiomatic theories prior to the 20th century, including Euclid's geometry, are finite. Even today, most axiomatic theories of mainstream mathematical interest are finitary, in an important sense. This makes Lakatos's inclusion of this point defensible, though it needs to

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7 We note, however, that not every epistemic good possessed by the axioms will flow down the relevant channels; in particular if the axioms are self-evident, it is evidence rather than self-evidence that is transferred to the theorems via deduction.

8 Since deduction is truth-preserving, and the axioms of a Euclidean theory are true, then so, are the theorems. This point stands even if we think of *Flow* as a primarily epistemic, rather than alethic or semantic, principle.

be made more precise. In particular, we must take due notice of axiom schemata. First-order Peano Arithmetic (PA), for example, cannot be finitely axiomatised, and hence cannot be presented as a finite conjunction. But PA *can* be finitely formulated as long as schemata are allowed, the usual way of doing so being to adopt a schematic form of the induction axiom. Euclideans should allow this sort of latitude.

Moreover, there are important Euclidean thinkers who contradict this point, or remain silent on it. For example, Aristotle, who can be identified as a forerunner of the EP, is explicit in the *Posterior Analytics* that science as a whole requires an infinity of axioms (I 32 88b6). Prominent advocates of the EP in the seventeenth century do not share this commitment as far as we know, but nor are we aware of a proactive commitment to the finitude of the axioms in these writers. No clear endorsement is discernable in the relevant works of Descartes (*The Dsicourse on Method,* including *The Geometry*) or Pascal (*On the Geometric Mind*), for instance.

So on this point, we broadly agree with Lakatos, but insist that more care be taken over its formulation, and that the principle is not central to the EP. We return to it in the next section.

Triviality

As we saw in the quotation from ‘Foundations’, Lakatos takes the axioms of a Euclidean theory to be ‘trivially true’ and says they bear a ‘trivial meaning-load’ (1962, pp. 4–5). What does Lakatos mean by ‘trivial’? We confess we’re not entirely sure.

One understanding of ‘trivial’ is ‘logical’. But if the axioms were so trivial as to be logical then they would be unnecessary, as they would be delivered by the logic. So we take it that Lakatos has a broader sense of triviality in mind.

Another way to understand triviality is as the broadly empiricist idea, favoured by Hume and the logical empiricists: mathematical statements are true in virtue of meaning and therefore empty of content. If this is what Lakatos intends, we part ways with him: it is entirely compatible with the EP that axioms are not ‘trivial’ in this sense, but substantive. For example, recognition of the axioms’ truth could be the product of mathematical intuition, a faculty distinct from any that informs us of the trivial truth of statements such as ‘bachelors are unmarried’. But given Lakatos’s general disparagement of logical empiricism, and his explicit mention of intuition in ‘Foundations’, it is unlikely that this is his intended sense.

A third possibility is that triviality in ‘Foundations’ is related to non-explanatoriness in ‘Renaissance’, where the term ‘trivial’ does not appear. The early Lakatos describes proof as *giving way* to explanation (1962, p. 14), as Euclidean theories are replaced by their Empiricist successors. And the later Lakatos draws the contrast in the following way:

> [I]n a Euclidean theory the true basic statements at the ‘top’ of the deductive system (usually called ‘axioms’) *prove*, as it were, the rest of the system; in a quasi-empirical theory the (true) basic statements are *explained* by the rest of the system. (1976a, p. 29)

So, perhaps the intended sense of *trivial* in the early paper is simply that the axioms are non-explanatory. They represent fossilised truisms rather than theoretically hard-working explanatory principles that have a role to play in making bold theoretical conjectures.

Lakatos believes that in quasi-empirical theories of the type he favours, basic statements are explained by the rest of the system, but (by implicit contrast) that this is not the case in a Euclidean theory. We agree on one point: that axioms are explanatory of theorems need not be built into a Euclidean theory. But we do not see why explanatoriness has to be ruled out either. Perhaps a Euclidean could think of the axioms as explaining the theorems. Indeed, this seems to be Aristotle's position (*Posterior Analytics* I.2 71b17–25), and his thought bears a strong resemblance to the EP according to at least one major interpretative school (see our (2024) for more details). So the thought that axioms explain theorems is not *per se* un-Euclidean or un-foundational. Perhaps there is good reason to think that the axioms cannot *in fact* explain the theorems in a Euclidean theory; but in so far as we are undertaking a rational reconstruction of the EP, we see no reason to include non-explanatoriness as one of its features.

There is a fourth way to interpret the word 'trivial'. This is the idea that axioms are self-evident; that their discovery, as opposed to their content, is trivial. This reading is suggested by Lakatos' insistence that the injection of truth value in a Euclidean theory is supposed to be infallible. In the development of his quasi-empirical account of mathematics, Lakatos sets himself fiercely against the axioms' alleged self-evidence and the indubitability of mathematics, so he clearly meant to bake this idea into the EP. If this is the intended sense of 'trivial', then we agree. Indeed, we take one of the ideas at the heart of the EP to be that axioms are self-evident (more on this in §2).

Terms

Related to the issue of triviality, Lakatos requires that the primitive terms of a Euclidean theory be *perfectly* well-known (1962, pp. 4–5). What is significantly less clear to us, however, is why he insists on this.

In the context of the 1962 paper, the reason seems to be to do with scepticism. Here, Lakatos is concerned with two regressive sceptical arguments that aim to show that meaning and truth cannot be conclusively established (1962, p. 3). Meaning cannot be established because when defining an expression *E*, for instance, one must use at least one expression, call it *F*. Presumably *F* itself requires a definition, and if circularity is to be avoided, this definition will use expressions that are not defined in terms of *E* or *F*. Rather, a new term, *G*, must be introduced. *G* apparently needs its own non-circular definition, and so the regress goes on *ad infinitum.* There is also the more familiar regress in terms of proof and knowledge. If one claims to know some mathematical theorem by giving a proof, that proof will have premises, which in turn require their own proofs, and so on *ad infinitum* again.

Lakatos paints the EP as a response to *both* problems. The regress in proof is blocked by the axioms; these truths are known indubitably and are not in need of proof at all. If the EP is to block the semantic regress also, it is natural to think of the Euclidean theorist as making a similar pronouncement on the primitive terms of the theory; they are understood *perfectly*, and so are not in need of a definition or elucidation, and the regress is blocked. In short, Lakatos sees the EP as both an epistemological and a semantic manifestation of foundationalism.

It is questionable, however, whether this requirement is justifiable in historical terms. It is not clear (to us, at any rate) that the history of the Euclidean programme is so closely connected to semantic scepticism. In the *Posterior Analytics,* Aristotle discusses a relevant issue. He claims that 'all teaching an all learning of an intellectual kind proceed from pre-existent knowledge', clarifying that '[t]here are two ways in which we must already have knowledge: of some things we must already believe that they are, of others we must grasp what the items spoken about are (and of some things both).' The requirement that we *must grasp what the items spoken about are* is something like a requirement that the terms of a theory be previously understood, and Aristotle gives the example 'of the triangle, that it means *this*' as something we need to know in order to learn about triangles (I.1 71a1–16). Now of course one must know the meaning of the term 'triangle' *in some sense* to have knowledge of triangles. But there is nothing here to suggest that such understanding must be perfect. Rather, the use of the demonstrative seems to suggest that Aristotle requires simply that one be able to identify triangles when confronted with them, which on its own seems to fall far short of understanding 'triangle' perfectly. And, at least at this juncture, Aristotle is not even responding to the sceptical regress about meaning Lakatos had in mind. Rather, he is responding to 'the puzzle in the *Meno*' (I.1 81a29–31), that one cannot enquire after what one is ignorant of, since one will not recognize the correct answer to the query when one comes across it.

Reading the great Euclideans of the 17^{th} century also casts Lakatos's claims in a dubious light. Descartes, of course, is extremely concerned to respond to scepticism. But he is most naturally read as responding to *epistemological* scepticism about the possibility of knowledge (*scientia*) rather than to semantic scepticism about the meaning of terms in mathematics or elsewhere (see *The Meditations* (in 1984b) for instance).

The situation is even worse when we turn to Pascal, Lakatos's paradigm Euclidean. Pascal himself remarks that one would ideally like to define all one's terms and to prove absolutely everything. But since we cannot proceed back indefinitely, we must resort to primitive undefinable terms and principles so obvious that none more obvious might be used to prove them. So Pascal is clearly alive to both of Lakatos's regresses. But he goes on to assert that trying to further elucidate geometrical primitives would engender more confusion than enlightenment.[9] This makes it clear that Pascal also requires primitive terms to be understood, but suggests that the understanding may be imperfect. We might *ideally* define these terms, or provide an elucidation of them, but in practice, this does more harm than good. Pascal is also clear that geometric knowledge is in good standing when it is obtained by the method he outlines, making it clear that failing to live up to our initial ideals is not to the detriment of geometry, contrary to what Lakatos asserts about the Euclidean position on semantics.

In short, we see the primitive terms requirement as a convenience that Lakatos adds to his characterisation of the EP in order to present it as a broader form of foundationalism than the textual evidence allows for. Lakatos's thoughts on primitive terms and meaning in mathematics are interesting, but we do not find in practice that historical Euclideans address these semantic issues in the way Lakatos describes, if indeed they address them at all.

Formality

In the earlier 1962 article, Lakatos also comments parenthetically that deductions in a Euclidean system need not be formal. Changing the clause's italicisation to emphasise this aspect of it:

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[9] Pascal (1655/1991, p. 396).

> The basic definitional characteristic of a (*not necessarily formal*) deductive system is the principle of retransmission of falsity from the 'bottom' to the 'top'… (1962, p. 4)

Lakatos is of course famous for denigrating 'formalist' philosophies of mathematics, or at least insisting that there is a lot more to the philosophy of mathematics than 'formalist' approaches. As he explains in the introduction to *Proofs and Refutations* (1976b), formalists identify mathematics with its formalised axiomatic version; Carnap, Church, Peano, Russell and Whitehead and others are examples of formalists in this sense. As far as the EP goes, however, Lakatos builds no requirement of formality into it. Formalism, in the hands of some of its advocates, is a late 19th/early 20th-century incarnation of Euclideanism, but the two should not be identified more generally, on pain of being blind to all pre-19th-century forms of Euclideanism, which were non-formal.

Despite the contemporary logician's understanding of a theory as a deductively closed set of formal sentences in a formal logic, the language of a theory putatively instantiating the EP does not have to be formal. In this, Lakatos is surely correct. Euclidean theories could be, and until the late 19th century, were formulated in natural language. So it should be no part of the EP that an axiomatisation be formal, as that would be unfaithful to its history. Indeed, as Jonathan Barnes points out, the idea of a formal language was alien to ancient deductive thought.[10] So if we insist that Euclidean theories are formal, we thereby rule out huge swathes of mathematics, even reconstructions of mathematics, by definitional fiat.

Lakatos stresses a related point. Not only can entailment in a Euclidean theory be informal, it can also be non-logical (1962, p. 13). Subject-specific inferential rules and construction techniques may be a perfectly legitimate component of informal deductions. For example, Kantians might maintain that mathematics employs ineliminably mathematical modes of inference (say, spatial intuition in geometry); if so, the conclusion is implied by the premises but does not *logically* follow from them. So as not to restrict the EP's range of application too narrowly, this sort of implication should count as well. Moreover, which entailments one considers logical will be sensitive to the background logic, and the EP should not prescribe a particular background logic to be used. In short: a Euclidean theory is simply a collection of sentences about a subject matter, closed under a relation that need not be formal, or even logical.

Lakatos's insight here has not, we think, been taken sufficiently seriously by analytic philosophers. To illustrate this point, consider the Kneales' account of the geometric method in their classic text on the history of logic, *The Development of Logic*. Interestingly, though perhaps coincidentally, *The Development of Logic* was published in 1962, the same year as Lakatos's first major article on the EP. The Kneales approach the subject by singling out three ingredients in the 'customary presentation of geometry as a deductive science' (K&K 1962, p. 3). First, 'certain propositions of the science must be taken as true without demonstration'; second, 'all the other propositions of the science must be derived from these' (K&K 1962, p. 3). The last ingredient is at once the most distinctive and the most controversial of the three:

[10] '[N]either they [the Stoics] nor any other ancient logician ever considered inventing an artificial language for the use of logic' (Barnes 2005, p. 512).

> …the derivation must be made without any reliance on geometrical assertions other than those taken as primitive, i.e. it must be *formal* or independent of the special subject matter discussed in geometry…. [thus] elaboration of a deductive system involves consideration of the relation of logical consequence or entailment. (K&K 1962, pp. 3–4)

Kneale and Kneale do not clarify whether the 'or' in '*formal* or independent of the special subject matter' is supposed to present two alternatives (the second condition being different) or just one (the second spelling out the first). Whatever they intended, the idea that the deductive apparatus in any axiomatic presentation of geometry must be formal should be resisted. Even if, as we believe, logic is formal, it should be no requirement on a Euclidean account of geometry that its logic be formal. Indeed prior to the 19th century, it is hard to see any logic as formal judged by today's standards.

What is clear is that Kneale and Kneale insist on derivations being strictly logical. But their attention is not restricted to modern axiomatic presentations of the various branches of mathematics. They mean to take in any of the 'customary' such axiomatisations, including Euclid's. To stipulate that such an axiomatisation's rules *must* be strictly logical seems too stringent a requirement; it risks, for example, making the *Elements* not be a 'customary' axiomatisation of geometry, if Euclid's system is not strictly logical because it appeals to diagrams and geometric insight in various places. More generally, there is no strong historical precedent, prior to the late 19th century,[11] for thinking that the rules in a Euclidean axiomatisation may not be topic-specific. It is better, then, to characterise the rules more neutrally and to avoid decreeing that they must be formal.

Lakatos and the EP

Although we will not dwell on Lakatos's own assessment of the Euclidean picture, it is clear which side he is on. He thinks that '[f]rom the seventeenth to the twentieth century Euclideanism has been on a great retreat', and that rearguard attempts to 'break through beyond the hypotheses, towards the peaks of *first principles*' have all failed. The upshot: '[t]he fallible sophistication of the empiricist programme has won, the infallible triviality of Euclideans has lost'. That said, the 'four hundred years of retreat seems to have by-passed mathematics', and Lakatos clearly sees his own role as being to wield the axe in this subject too.[12] The point of his most famous work, *Proofs and Refutations*, is to show 'that informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems [the Euclidean picture] but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations' (1976b, p. 5). In the words of Pi, one of that great book's Greek-alphabet-named characters: 'Heuristic is concerned with language-dynamics, while logic is concerned with language-statics' (1976b, p. 99). The latter could equally well apply to the Euclidean picture, which is static rather than dynamic. To present mathematical knowledge in static fashion, as an unchanging pyramidal-shaped system of immutable truths, is to belie it. Towards the end of *Proofs and Refutations,* its author comments:

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[11] The slightly oblique reference here is to Frege, who believed that the rules were (as we would now put it) topic-neutral.

[12] The quotations in this paragraph so far are from his (1962, p. 10). The theme of Euclidean theories' decline, especially outside mathematics, is repeated in his (1976a, p. 30).

> In deductivist style, all propositions are true and all inferences are valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter. An authoritarian air is secured for the subject by beginning with disguised monster-barring and proof-generated definitions and with the fully-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility. (1976b, p. 142)

The present authors are also critical of the Euclidean Programme, though we do not have the space to discuss our criticisms of it here. Interested readers are directed toward our forthcoming monograph on the subject (Paseau and Wrigley 2024).

We conclude this section by raising a more general sort of worry. One may criticise Lakatos's ambition to even discuss the EP in the relatively ahistorical way that he does. Perhaps it is historically insensitive to throw a single critical blanket over a great swath of the past;[13] perhaps one should not try to capture the essence of Euclideanism in the way Lakatos tried to.

We have some sympathy for this complaint, but only up to a point. Clearly, different authors tempted by Euclideanism have stressed different points and added their individual imprint to its expression. Indeed, the body of work attributed to Euclid has varied across time and place, so that different Euclideans may have even drawn their inspiration from varied sources. But that said, we believe there is an identifiable body of doctrine reasonably called 'The Euclidean Programme' that runs through the ages, even if it is not precisely defined and differs from writer to writer. As philosophers, we see our role as trying to identify these doctrines and, once identified, to assess them. Euclideanism is not at bottom different from many other 'isms' philosophers blithely engage with in fairly ahistorical fashion—just within epistemology, think of coherentism, foundationalism, internalism, externalism, etc. If there is room for discussion of these 'isms' in a relatively abstract way, there should also be room for similar discussion of Euclideanism. A more sensitive approach might be to compare a rational reconstruction that tries to capture the centre of gravity of a body of thought—Euclideanism— within individual historical writers' conceptions. Although there is no space for the latter here, we have attempted it in our book, which compares the EP as an abstract methodological ideal with some historical authors: Aristotle, Euclid, Descartes, Pascal and other more recent ones. This leads on to the next section: having seen what Lakatos thinks the EP is, it is high time we say what we take it to be.

2. The Euclidean Programme in Seven Principles

Lakatos's discussion of the EP was instructive. Let's try to capture the general picture and the lessons learnt in §1 in a more structured way, and set aside the historical scruples mentioned at the end of §1.

We take the EP to be characterised by seven principles. Three of these are *core* principles, which we take to be present in any historical manifestation of the Euclidean Programme worthy of the name. The other four principles are peripheral to the programme, manifesting themselves in many, but not all, occurrences of the EP throughout history.

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[13] We owe this phrase to Brendan Larvor.

The first core principle is that the axioms of a Euclidean theory are supposed to be true. We agree with Lakatos that this is essential to the EP, and its inclusion as mandatory. All the major figures in the Euclidean tradition subscribe to some version of it: Aristotle (on one common interpretation), Descartes, and Pascal, to mention just a few.[14]

The second core principle of the EP in our reconstruction is that the truth of the axioms should be *self-evident*. While this is not something Lakatos focused on,[15] all the major Euclidean thinkers subscribe to this principle, or something very similar. As a distinctly foundationalist epistemology, the EP requires the truth of the axioms to be completely secure and unmediated by inference. Given that this aspect of the programme is so historically well-attested, we shall not dwell on it here. Suffice it to say that we agree with Lakatos' assessment that our assurance of the truth of the axioms should be infallible in the context of a Euclidean theory.

As we see it, the EP is primarily an epistemology of mathematical *propositions*, not terms, and hence we have no parallel to Lakatos' requirement that the primitive terms of a Euclidean theory be perfectly well-understood. Given the axioms' pride of place in the EP, our understanding of the primitive terms must be sufficiently clear to enable the mathematician to understand, and hence see the truth of, the axioms. But *perfect* understanding of the terms is not required for the axioms to be self-evident.

The easiest way to appreciate this point is to think of simple logical propositions involving imperfectly understood terms. For example, it should be as obvious as can be that 'all democracies are democracies' is true. It is equally obvious that if 'all horses are ungulates' and 'all ungulates are mammals' are both true, then 'all horses are mammals' is true. One can appreciate this even if the terms 'democracies' and 'ungulates' (or even 'horses' and 'mammals') are not perfectly clear. But we can go beyond logical truths. It is completely evident that anybody taller than a tall person is tall, even to someone with a less-than-perfect understanding of the extension of the predicate 'is tall'. Or, for a mathematical example, it would have been completely evident to an 18^{th}-century mathematician that the identity mapping on the reals was a function, even in the absence of a clear understanding of what real-valued functions, or even the reals, are.

So we require only that the axioms be self-evident to a mathematician who has grasped the meanings of the primitive terms to an extent which allows them to understand the axioms, whether or not their grasp of the primitive terms is perfect. To continue the hydraulic metaphor, we can tolerate some impurity in the water, so long as it does not affect the flow.

Talk of flow brings us to the third core Euclidean principle. In one respect, we take it to be Lakatos' greatest contribution to the study of the EP that he identifies the flow-idea as an essential and defining principle of it, a principle understood in distinct ways by distinct philosophers and mathematicians working in the tradition. Where we part company with

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[14] A difficult question is whether this principle is actually subscribed to by Euclid. Very briefly: we take the EP to be inspired by the methodology of the *Elements*, whether or not its author was a 'Euclidean' foundationalist; for a little more detail, see our forthcoming monograph (Paseau and Wrigley 2024).

[15] Unless self-evidence is how we are supposed to understand his idea of 'triviality'.

Lakatos, however, is in his focus on the flow of semantic content, such as truth and meaning, inherited, in his reconstruction of the EP, by theorems from the axioms. We think this is an unfaithful representation of the historical Euclidean ideal, where the focus has been significantly more skewed towards epistemological issues. In our reconstruction, the direction of flow is indeed downwards, from axioms to theorems, but what is inherited is an epistemological good, which one exactly varying from one manifestation of the EP to the next. In addition to an account of the relevant epistemic good, a particular manifestation of the EP must include a principle governing flow or transmission of said good. In a strong version, the epistemic good (such as justification) is perfectly preserved from premises to conclusion; in a weaker version, it is more or less preserved.

We may thus summarise the three tenets or core principles of the EP as follows:

EP-Truth	All axioms and theorems are true.
EP-Self-Evidence	All axioms are self-evident. If a subject clearly grasps a self-evident proposition then she bears the *Ep*-relation to it to the maximal degree.
EP-Flow	If a conclusion is deducible from some premises, and the subject clearly grasps this, and bears the *Ep*-relation to these premises to a high degree, she thereby bears the *Ep*- relation to the conclusion to a similarly high degree.

Here, the *Ep*-relation is a placeholder for some epistemic relation. It is crucial that it not be further specified, to make the EP an umbrella conception large enough to cover many and varied historical instances. Choosing a specific relation for *Ep* would rule out some paradigm examples of the EP and obscure deep commonalities.

This reconstruction of the EP permits a thoroughgoing comparison of diverse historical figures in the Euclidean tradition, and facilitates a comparison of their actual methodology to this reconstructed ideal. Of course, any relation between the two is bound to be loose and inexact; a perfect fit is *not* to be expected. The historian of philosophy, no less than the historian of ideas, must be careful to avoid attributing claims to past philosophers in terms they would not acquiesce to.[16] But we hope to have avoided the potential charge of caricature, which, as Lakatos warns us, is often levelled at these kinds of projects by '[r]espectable historians' (1962, p. 4). As explained, our aim, pursued more fully in our book (Paseau and Wrigley 2024) is to relate the EP, stated *in vacuo*, to real historical conceptions. Although the point of the exercise is to show that the EP does relate interestingly to various historical expressions of 'Euclideanism', we must be careful not to confuse an abstract prototype with historical expressions that suggest or approximate it in some interesting fashion. Having said that, something would be amiss with our rational reconstruction if it did *not* display important similarities with these historical expressions.

So much for the core principles of the Euclidean Programme. We also give the following four *peripheral* principles:

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[16] See Skinner (1969) for an influential expression of this point.

EP-Finite	The axioms and rules are each finitely many.
EP-General	All axioms are general propositions.
EP-Independence	Each axiom is independent of the others.
EP-Completeness	All truths of a certain kind can be deduced from the axioms.

Most Euclideans have historically subscribed to at least some of these, although we do not take them to characterise an essential aspect of Euclidean foundationalism. We explain these four principles in turn and justify their inclusion in the EP.

We are happy to follow Lakatos in including *EP-Finite* as part of the EP, but only as a peripheral principle and with the caveat mentioned in the previous section. We do not take the relevant sense of finitude quite so literally as Lakatos, since schematic theories can have a finite presentation, despite having an infinite number of axioms. Of course, prior to the rise of modern logic, there was no way for mathematicians to distinguish between, say, first-order axiom schemata and single second-order axioms when formulating a principle such as mathematical induction. But with the distinction in hand, we take it that the relevant epistemological features of a finite theory also accrue to theories employing finitely many axiom schemata. So we regard a theory as satisfying *EP-Finite* if it has a finite presentation, that is, the number of non-schematic axioms plus the number of schemata is finite.

EP-General is also very standard. Although it is not a feature of the Euclidean Programme as Lakatos reconstructs it, we include it in our characterisation due to its prevalence in the history of the EP. What exactly generality comes to is hard to state,[17] but often easy to recognise: Peano Arithmetic's axiom that distinct numbers have distinct successors is general, as are any of Euclid's common notions in the *Elements*. Four of the five postulates listed there are also recognisably general; they are about any lines and points with some given properties, or parts and wholes, etc. But the fourth, which states that all right angles are equal to one another, mentions angles of a particular type. So one might worry that this is *not* general in the intended sense. However, a right angle is easily defined in more general terms by exploiting the fact that two right angles make up a line. We may thus distinguish two ways in which an axiom may fail to be general. The first is by including terms for specific entities or kinds of entities; the second is by including terms for entities that are not definable using general vocabulary.[18] It is only the latter that falls foul of *EP-General*, for any occurrence of a term definable using general vocabulary is eliminable in favour of the definition.

EP-Independence requires of each axiom that there is no proof of it from the other axioms of the theory. Like *EP-General*, this principle is not one that Lakatos builds into his reconstruction of the EP, so it is worth saying something to justify its inclusion. The concept

[17] One might suppose that general statements are all and only those that begin with a universal quantifier. But as logicians well know, any statement is equivalent to a universally quantified one, e.g. p is logically equivalent to $\forall x(x = x \wedge p)$.

[18] This raises the question of what general vocabulary is. A rough elucidation is that it is vocabulary that applies to all or most entities in the domain, or is defined in terms of such vocabulary. Since the task before us is to describe *EP-General* rather than to define it in non-circular terms, we shall not dwell on this further.

of independence is most familiar from the history of the fifth postulate Euclid laid down in Book I of the *Elements.*[19] This postulate was long suspected by the mathematical community to be provable from the other axioms; indeed, the converse of the Parallel Postulate is proved by Euclid himself in Book I's Proposition 27. Interestingly, the mathematicians attempting throughout the ages to prove the Parallel Postulate were near-unanimous in their agreement that it was true. A prominent fifth-century philosopher, Proclus, acknowledges that the postulate is obviously true; the central problem is that 'its obvious character does not appear independently of demonstration but is turned by proof into a matter of knowledge' (1970, p.151). This suggests instead that on the Euclidean Programme, that which admits of proof requires it for the highest standard of knowledge—and this would imply a principle such as *EP-Independence*. There is indeed some evidence of a requirement such as this in Euclid's practice, where he gives proofs for propositions whose self-evidence seemingly outstrips that of the Parallel Postulate; an example is Proposition 20 of Book I, that the sum of any two sides of a triangle are greater than the third.[20] That such propositions are proved, rather than taken as redundant axioms, suggests, although not conclusively, the working of a principle such as *EP-Independence*. Much as with Lakatos's finitude requirement, independence is not discussed or endorsed by all the Euclidean theorists we consider, and it appears prominently in the writings of theorists outside the tradition too. Thus we take it to be a merely peripheral component of the programme.

EP-Completeness says that the rules and axioms are sufficient for the deduction of *all* truths in some important class. Although the issue is not prominent in Lakatos's characterisation of the EP, he is clearly aware of its presence in Euclidean thought generally; for example he highlights that the Euclidean believes 'all knowledge can be deduced from a finite set of trivially true propositions' (1962, p. 4). This characterisation is ambiguous (as we'll see below) and Lakatos does not return to it in the later 'Renaissance' article. There he writes only that the axioms prove 'the rest of the system' (1976a, p. 206), though he does discuss completeness in (what he sees as) some specific manifestations of the EP, such as logicist foundations for mathematics and Hilbert's finitist programme.

Completeness is properly seen as a schematic requirement. Its importance and plausibility depend greatly on the class of truths to be specified. The weakest such principle of any interest is with respect to the *known* truths of the domain. On this version of the principle, the axioms might be viewed as primarily 'an organization of our knowledge, making it more manageable and more interesting' (Russell 1907, p. 580). A more ambitious version of the principle is that the axioms must be complete with respect to the *knowable* truths of a particular science. It is not, of course, straightforward to say what knowability amounts to in this context, as the modality packed into it can be understood in different ways. The strongest

[19] As a reminder, this postulate is (in Heath's 1925 translation): That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
[20] Cited in this connection by Lewis (1920, p. 17).

version of the principle says that *any* truth of the relevant domain can be attained; in modern terminology, the axioms are negation-complete.[21]

Some version of *EP-Completeness* has traditionally been aspired to, often in one of its more ambitious forms, including amongst Euclideans. Despite its prevalence, we include it as a subsidiary principle because its different versions differ wildly in how demanding they are, and because it is not a distinctively *Euclidean* principle. Even its strongest version is subscribed to by figures whose classification as Euclideans is best resisted, for example Hilbert and Kant.[22]

A principle that we do not wish to build into the EP, either in its core or on its periphery, is the metaphysical dependency of theorems on axioms, and more generally of derived theorems on the earlier theorems they are derived from. It isn't clear to us whether Lakatos intended this principle to be included in his reconstruction of the EP, though something like it is perhaps suggested by his talk of *truth* flowing from the axioms to the theorems. And the idea is of course a historically prominent one. The dependence of theorems on axioms is very much part of Frege's conception of the foundational method, who wrote that the aim of proof was 'to afford us insight into the dependence of truths upon one another' (1884, §2). Frege believed that this dependence (*Abhängigkeit*) was an objective matter, and was not alone in this: he cites Leibniz as a precursor (1884, §17), and in fact the dependency of theorems on axioms dates back to Aristotle. But it is precisely because the relation of dependence is metaphysical[23] that we exclude it from consideration. Its inclusion in the EP would add a whole new metaphysical dimension to the EP, and as we've emphasised, we take the core ideas of Euclidean foundationalism to be essentially epistemological.

3. Conclusion

With Lakatos's help, we reconstructed the EP. His idea that what matters is *how* some theoretical good flows from axioms to theorems, not *what* flows, formulated in the principle *EP-Flow*, was key to this reconstruction. In other ways, we parted company with Lakatos, for the reasons given. The next thing to do is to assess the EP in light of developments in contemporary epistemology and contemporary mathematics. Our book takes that next step, and also compares and contrasts the ahistorical EP with some flesh-and-blood authors.

Lakatos, as we have had occasion to mention, was strongly opposed to the EP. But he was clear-sighted enough to recognise that it is a formidable opponent. Well-versed in Popper's philosophy, he knew how hard existential claims are to refute. We will let him have the last word:

> A Euclidean never *has* to admit defeat: his programme is irrefutable. One can never refute the pure existential statement that there exists a set of trivial first principles from which all truth follows. Thus science may be haunted for ever by the Euclidean programme as a regulative principle, 'influential metaphysics'. A Euclidean can

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[21] That is, for every sentence ϕ of the relevant language, either ϕ or not-ϕ is entailed by the axioms.

[22] See, for example, (1787: A476/B504) for remarks by Kant, and (1902, p. 445) for remarks by Hilbert.

[23] As Shapiro (2009, p. 183) and others have noted.

always deny that the Euclidean programme as a whole has broken down when a particular candidate for a Euclidean theory is tottering. (1962, pp. 6 –7)

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