Time and Quantum Theory: A History and Prospectus

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1 Introduction

The conventional wisdom regarding the role of time in quantum theory is this: "time is a parameter in quantum mechanics and not an operator" (Duncan & Janssen, 2012, p. 53). The reason for this is 'Pauli's theorem,' a collection of results that show that (subject to a mild restriction on the Hamiltonian operator) conventional quantum mechanics does not permit the definition of a time observable, i.e. a self-adjoint operator canonically conjugate to energy.[1] If one wishes to have time appear as a genuine observable of the theory, then this is obviously a problem, called by some "the problem of time in quantum mechanics" (Hilgevoord & Atkinson, 2011; Olkhovsky, 2011). Hilgevoord's (2005) attempted resolution of the problem rests on his rejection of a particular motivation that one might have for wishing to regard time as a genuine observable. Hilgevoord's argument is essentially this: there is nothing problematic about time being represented by a parameter rather than an operator since *space* is represented by a parameter rather than an operator as well.

In a recent historical survey, Hilgevoord (2005) contends that the demand that time be an observable can be traced back to a conceptual confusion common among the progenitors of quantum mechanics, in particular Dirac, Heisenberg, Schrödinger, and von Neumann. Hilgevoord claims that the expectation of the authors of quantum mechanics that time should be an observable was due to this confusion between space and position: led by the role of position in the theory as an observable, they were mistakenly led to the idea that time should be observable too. He traces the source of the

[1] See Srinivas & Vijayalakshmi (1981) for a rigorous derivation of this result.

confusion to the frequent use of the spatial co-ordinates (x, y, z) to denote the spectral values of the position observable of a single particle (q_x, q_y, q_z).

When presented with an operator whose spectral values appear to correspond to points of space, it is natural to expect also an operator whose spectral values correspond to instants of time. And given the expectation of these authors that quantum mechanics would ultimately be a relativistic theory, it is reasonable to demand of a theory set in space-time that time and space should appear on the same footing. However, as Hilgevoord points out, the spectral values of position are *not* identical with spatial points—this correspondence is only valid for a system comprising a single particle since in general the dimension of configuration space (and so the spectrum of the position observable) is $3N$, where N is the number of particles. Once this confusion is made manifest and it is realized that time t (a parameter) is to be contrasted with space (x, y, z) (also parameters) the apparent asymmetry is removed and so the justification for regarding time as an operator (i.e. an observable) is removed, or so Hilgevood claims. This leads him to dismiss later developments, such as the more recent use of POVMs (Positive Operator Valued Measures) to define (generalized) time observables, as conceptually confused for the same reason.

Now, with regard to this particular justification for regarding time as an operator, I agree that Hilgevoord offers an apt diagnosis: what is being contrasted here is not time and space, but time and position. But while I agree wholeheartedly that it would be a mistake to confuse space, time and position in this way, I am not convinced that this was a confusion to which many (or perhaps any) of the authors of quantum theory were prone. Indeed, Hilgevoord acknowledges that there were other reasons which led to the expectation that temporal properties were apt for representation by operators. It is my view that these other reasons for defining time operators were more important to those authors—I will claim that some remain compelling today—and thus are not so easily dismissed as resulting from a simple conceptual error.

In this paper I will be concerned with analyzing in more detail how ideas and expectations regarding the role of time in the theory arose and evolved in the early years of quantum mechanics (from 1925-27). The general theme will be that expectations which seemed reasonable from the point of view of matrix mechanics and Dirac's q-number formalism became implausible in light of Dirac-Jordan transformation theory, and were dashed by von Neumann's Hilbert space formalism which came to replace it. Nonetheless, I will identify two concerns that remain relevant today, and which blunt the force of Hilgevoord's main claim.

First, I point out Dirac's use of an 'extended' classical configuration space, which includes time and energy as conjugate variables from the get-go, is not subject to Pauli's theorem, and, moreover, his motivations for using this extended configuration space are not merely relativistic, as Hilgevoord claims. This indicates another way to avoid this 'problem of time:' by defining an 'extended Schrödinger equation' for functions of space *and* time we can have a quantum theory in which time and energy are represented by canonically conjugate observables, as Dirac had originally intended. Second, I contend that the temporal quantities under consideration concerned not time in the abstract, but the time of particular events (initially so-called 'quantum jumps'). If 'the' time operator concerns the location of an event in time, then it is no false contrast to draw an analogy with the position observable, which concerns the location of an event in space (the event in question being something like 'the particle's being here'). I will also attempt to show how these considerations are related, in that exploring the second claim (that the time of an event is an observable quantity) leads naturally to the first (that quantum theory should be defined on extended configuration space).

2 A Brief History of Time in Quantum Mechanics

2.1 Time in Matrix Mechanics

The expectation that energy and time must form a canonically conjugate pair arose from the close relation of the new quantum mechanics to the action-angle form of classical (Hamiltonian) mechanics that inspired it. In Heisenberg's famous *Umdeütung* paper of 1925, time plays an almost identical role in the description of the new quantum variables as it did in the specific classical cases he sought to re-interpret. The classical equation of motion, Hamilton's equation in action-angle co-ordinates (J, w), reads

$$\frac{dw}{dt} = \frac{\partial H}{\partial J} \quad ; \quad \frac{dJ}{dt} = -\frac{\partial H}{\partial w}.$$

The time dependence in action-angle form is particularly simple since the canonical transformation into these co-ordinates is chosen such that $\dot{J} = 0$ and $\dot{w} = v_0$, a constant. Thus J is time independent and w is linear in t. This being the case, a general solution $x(t)$ of these equations (for periodic systems) may be Fourier decomposed into a sum over components labeled by amplitude and phase:

$$x(t) = \sum_{n=-\infty}^{\infty} \sum_{\tau \pm 1} A_\tau(J_n) e^{2\pi i \tau v_n t}.$$

So it was this special form of classical Hamiltonian mechanics, one in which time dependence takes an especially simple form, that provided the basis of the emerging quantum kinematics. The time evolution of these solutions was entirely confined to a complex phase, and so it was to be in the new quantum theory.

In the matrix mechanics of Born & Jordan (1925) kinematical quantities are represented by Hermitian matrices whose time dependence takes the same form,

$$\boldsymbol{p}(t) = p(nm)e^{2\pi i v(nm)t}; \boldsymbol{q}(t) = q(nm)e^{2\pi i v(nm)t}.$$

Having obtained a matrix representation of these kinematical quantities, it follows from the relation

$$v(nm) = W_n - W_m$$

that the time derivative of an arbitrary matrix function $\boldsymbol{g(pq)}$ may be written

$$\dot{\boldsymbol{g}} = \frac{i}{\hbar}[(W_n - W_m)g(mn)] = \frac{i}{\hbar}(\boldsymbol{Wg} - \boldsymbol{gW}), \tag{1}$$

where $\boldsymbol{W} = \delta_{mn} W_n$ is a diagonal matrix (pp. 288-9). Since the diagonal form of $\boldsymbol{W}$ was critical to the validity of this relation, the major practical difficulty of applying the new quantum mechanics to a particular system with a classical Hamiltonian of known functional form became essentially that of finding a representation in which the quantum mechanical energy took diagonal form.

By writing the Hamiltonian matrix $\boldsymbol{H}$ as a function of $\boldsymbol{p}$ and $\boldsymbol{q}$ Born and Jordan derived the following dynamical equations for quantum variables in the same form as Hamilton's equations in classical mechanics,

$$\left(\frac{\partial H}{\partial q} =\right) \dot{\boldsymbol{p}} = \frac{i}{\hbar}[\boldsymbol{Hp} - \boldsymbol{pH}] \quad ; \left(-\frac{\partial H}{\partial p} =\right) \dot{\boldsymbol{q}} = \frac{i}{\hbar}[\boldsymbol{Hq} - \boldsymbol{qH}].$$

They argued that this same relation holds true of a general function $\boldsymbol{g}(\boldsymbol{pq})$ as well, yielding the so-called Heisenberg equation of motion,

$$\dot{\boldsymbol{g}} = \frac{i}{\hbar}[\boldsymbol{Hg} - \boldsymbol{gH}], \tag{2}$$

which immediately gave the result that $\dot{\boldsymbol{H}} = 0$, i.e. that energy is conserved.

There is in this formalism no reason to suppose that time could not be represented by a matrix, and the fact that in classical mechanics w behaves very much like a time parameter is suggestive of the idea that there should be a matrix $\boldsymbol{t}(\boldsymbol{qp})$ canonically conjugate to $\boldsymbol{H}$. Indeed, if one demands that this matrix $\boldsymbol{t}$ vary linearly with time then (2) appears to imply that it is canonically conjugate to energy $\boldsymbol{H}$ since

$$\dot{\boldsymbol{t}} = 1 \Rightarrow [\boldsymbol{Ht} - \boldsymbol{tH}] = i/\hbar.$$

2.2 Time as a q-number: Dirac's Classical Analogy

Dirac, working in relative isolation in Cambridge, was led to the same dynamical equations by pursuing a structurally richer classical analogy. Like Born and Jordan he recognized the non-commutativity of the multiplicative operation as the key feature of Heisenberg's quantum variables, but rather than focusing on a particular representation of the variables, Dirac's approach led him to identify shared algebraic structures of the classical and quantum theories. We will briefly follow his development of the theory in his initial paper 'The Fundamental Equations of Quantum Mechanics' (Dirac, 1925).

Whereas Born and Jordan's derivative operation came for free from their use of matrix multiplication, Dirac sought to define his operation algebraically from the two basic conditions such an operation much satisfy: distributivity and the Leibniz law. He shows that the operation $ax - xa$, which is to say the commutator of two 'q-numbers' (a nomenclature introduced in a subsequent paper) satisfies these two conditions and so can be interpreted as a differentiation of x with respect to some parameter v. For a special case, Dirac let a be the diagonal matrix representing the energies of the allowed transitions, then v is the time and this returns $\dot{x}$, just as Born and Jordan had found.

But in contrast to Born and Jordan, who built up their dynamical equations from matrix operations acting according to the quantum condition, Dirac instead sought to establish a correspondence between classical and quantum *operations* by setting up a structural analogy between the two theories. He argued that as the quantum numbers become large the quantum commutator corresponds to the classical Poisson bracket (multiplied by a factor of $-i\hbar$).

$$\{x, y\} = \sum_r \left\{ \frac{\partial x}{\partial q_r}\frac{\partial y}{\partial p_r} - \frac{\partial y}{\partial q_r}\frac{\partial x}{\partial p_r} \right\}$$

The Poisson bracket is a canonical invariant, meaning that it takes the same value evaluated in any canonical co-ordinates. Moreover, the Poisson bracket expressions satisfy the two demands he placed on an operation of differentiation. This suggested to Dirac that the quantum commutator represented the same operation, valid for non-commuting 'q-numbers' — his own version of the correspondence principle.

Once this correspondence was established, the quantum equation of motion (2) followed immediately from the corresponding classical Poisson bracket by mere transcription according to the new quantum schema. The difference in Dirac's approach was manifest in his ability to import results from classical mechanics directly into his theory (although he was soon to see that his translation procedure led to ordering ambiguities). Since action-angle variables are classical conjugates with $\{w, J\} = 1$ (having been reached by a canonical transformation) the suggestion is very strong indeed that w and J, considered as q-numbers, must also be a canonical pair. Indeed, obtaining numerical results from Dirac's theory required transcription of the results of the corresponding classical problem, expressed in action-angle form.

2.2.1 'Relativity Quantum Mechanics'

When Dirac came to consider relativistic quantum physics his approach was, naturally enough, to define a suitable relativistic classical description in terms of Poisson Brackets, and then apply his quantum translation prescription (Dirac, 1926b).[2] It is worth quoting in full Dirac's description of this procedure and his view of the significance of defining suitable classical canonical variables.

> It will be observed that the notion of canonical variables plays a very fundamental part in the theory. Any attempt to extend the domain of the present quantum mechanics must be preceded by the introduction of canonical variables into the corresponding classical theory, with a reformulation of the classical theory with P.B.'s [Poisson Brackets] instead of differential coefficients. The object of the present paper is to obtain in this way the extension of the quantum mechanics to systems for which the Hamiltonian involves the time explicitly and to relativity mechanics. (pp. 406-7)

[2]Note that this work was done before he was aware of Schrödinger's wave mechanics.

In the following section, entitled "Quantum Time," Dirac immediately claims that "[t]he principle of relativity demands that the time shall be treated on the same footing as the other variables and so it must therefore be a q-number" (p. 407). In order to do so, Dirac defines a classical Poisson bracket that includes time t as a variable along with it's canonical conjugate $-W$ (i.e. , minus the total energy).

$$\{x, y\} = \sum_r \left\{ \frac{\partial x}{\partial q_r} \frac{\partial y}{\partial p_r} - \frac{\partial y}{\partial q_r} \frac{\partial x}{\partial p_r} \right\} - \frac{\partial x}{\partial W} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial W} \frac{\partial x}{\partial t}$$

In defining this Poisson Bracket, the set of canonical variables is extended by two to include t and $-W$, and so the dynamics of the system now takes place in this *extended* phase space. The physical solutions are defined by the demand that the Hamiltonian (defined on the extended phase space) vanishes with the total energy W, what is called today a constraint equation,

$$H - W = 0. \tag{3}$$

So while t and $-W$ are variables conjugate on the extended phase space (leading to the quantum commutators detailed in Dirac's equation (7)) the dynamics of the system are confined to a subspace of the phase space defined by this constraint (the constraint surface).[3] As we have seen, Dirac is explicit here that relativistic considerations motivate his introduction of time as a q-number, but also he cites the time dependence of the Hamiltonian as a motivation. There is nothing about the use of extended phase space which implies that the system in question is relativistic, as it is just the fact that the Poisson Bracket is defined on the extended phase space which implies time and energy are conjugate variables, not the fact that the Hamiltonian is relativistic. Hilgevoord (2005) regards the use of relativistic arguments to motivate the demand that energy and time be canonical conjugates in quantum mechanics as misguided due to the limited role that relativistic particle mechanics plays in classical and quantum physics. Be that as it may (and Dirac's rhetoric here notwithstanding) it remains the case that the conjugacy of energy and time has little to do with the fact that a system

[3]In fact, Dirac notes that the conjugacy relations may be inconsistent with the constraint, but this is just to say that the relations don't necessarily hold for functions on the constraint surface. Without the Schrödinger equation in hand, it is not yet clear that the dynamical equation is in fact a wave equation satisfied by certain functions, whereas the relations of conjuagacy hold generally for functions of the extended configuration space. It is also worth emphasizing that, whereas position and momentum are conjugate variables on both the unextended and extended phase space, energy and time are conjugate variables on the extended phase space alone.

is relativistic, and everything to do with the fact that the system's dynamics are defined in the extended phase space. Although in the case of a relativistic system the use of extended phase space is unavoidable, it is clear that Dirac also viewed the (non-relativistic) Schrödinger equation as a wave equation defined on functions of space and time, i.e., as an equation in the extended configuration space.

2.2.2 The Time-Dependent Schrödinger Equation

Introduced by means of an optical-mechanical analogy, Schrödinger's wave mechanics was initially met with hostility from the matrix mechanics camp. However, as we have seen, Dirac's q-number approach was more general, and so more flexible for extension in other directions. Following Heisenberg's discovery of a connection between the solutions of Schrödinger's wave equation and the energy values that appeared in the time dependence of the matrix values for a dynamical variable, Dirac seized on Schrödinger's new approach, recognizing the means to free himself from the overly restrictive reliance on classical methods, and overcome the problems introduced by the degeneracy of energy levels arising in systems of many particles (Darrigol, 1993, pp. 329–333). Unencumbered by the conceptual baggage of Schrödinger's painstaking path through classical physics, Dirac's starting point was the realization that quantum systems could be described by functions obeying a linear wave equation, and he quickly moved to explore the consequences.

In a remarkable paper 'On the Theory of Quantum Mechanics' (Dirac, 1926a) he laid out the essentials of an approach which would serve as the basis for the later integration of wave and matrix mechanics. He demonstrated the power of this new formalism by deriving the Bose-Einstein and Fermi-Dirac statistics for an assembly of systems from elementary conditions on the permutation of the wavefunctions describing the individual systems. The foundation of this approach was the recognition that Schrödinger's theory allowed for the explicit representation of conjugate variables as differential operators. To write down the time-dependent wave equation, therefore, merely required him to make the substitutions

$$p_r = -i\hbar\frac{\partial}{\partial x}; \quad -W = -i\hbar\frac{\partial}{\partial t}$$

into the equation (3) above, treated as a wave equation, i.e.,

$$(H - W)\psi = 0. \tag{4}$$

Hence Dirac's derivation of the time-dependent equation depended on the extended phase space description described in (Dirac, 1926b). To explain:

the replacement of the conjugate variable $-W$ by the corresponding differential operator relies on the existence of a space of functions of t on which it acts. The implication is that t is also a q-number, an operator that acts by multiplication on this space of functions of *extended* configuration space.[4]

In order to set up a correspondence with the Heisenberg equations of motion Dirac is required to fix the value of the variable t, but in doing so he makes it quite clear that the functions (or superpositions of functions) that satisfy the general wave equation are functions of time and space.

> As an example of a constant of integration of the dynamical system take the value $x(t_0)$ that an arbitrary function x of the p's and q's, W and t has at a specified time $t = t_0$. The matrix that represents $x(t_0)$ will consist of elements each of which is a function of t_0. (Dirac, 1926a, p. 665)

Under the special condition that the Hamiltonian is time independent (i.e., a constant of integration), so that the energy W has a diagonal matrix representation, Dirac was able to derive the time dependence of the matrix elements of a q-number x (although, as he is at pains to point out, only for a Hamiltonian that does not involve time explicitly). This reverses the logical order of Born and Jordan's derivation of (1), which assumed the time dependence by means of Heisenberg's classical analogy. Dirac here instead shows how this time dependence arises *from* the dynamics of the quantum mechanics, that is, the Schrödinger equation.

However, he explicitly states that he views this separation of time and space as inessential, and describes the alternative (solving directly in terms of the extended phase space without considering variation in t) as more fundamental:

> It should be noticed that the choice of the time t as the variable that occurs in the elements of the matrices representing variable quantities is quite arbitrary, and any function of t and the q's that increases steadily would do. . . . It is probable that the representation of a constant of integration of the system by a matrix of constant elements is more fundamental than the representation by a matrix whose elements are functions of some variable such as t . . . (*ibid.* p. 666)

In summary, we can see that there was another motivation, independent from relativistic considerations, which led Dirac to regard energy and time

[4]Note again that the space of functions on which W acts is not identical with the space of wavefunctions that that satisfy the wave equation (4).

as conjugate variables: his expression for the time-dependent Schrödinger equation required that energy be represented by a differential operator d/dt, which was defined on a space of functions of time. Note also that Dirac did not yet have an dynamics that could apply to time-dependent Hamiltonians, since the treatment he had given assumed from the outset that the allowed energies were time independent.

2.2.3 Time in Transformation Theory

In Dirac's presentation of transformation theory, 'The Physical Interpretation of the Quantum Dynamics' (1927) it is apparently assumed from the outset that the theory involves the split between space and time that had been introduced in the earlier paper.

> These matrix elements [of a dynamical variable g] are functions of the time only. In the present paper we shall not take relativity mechanics into account, and shall count the time variable wherever it occurs as merely a parameter (a c-number). (Dirac, 1927, p. 625)

In the course of the development of the theory, which Dirac intends to to provide a generalization of matrix mechanics to address non-periodic systems and continuous observables, Dirac makes the (oft-quoted) crucial link with Schrödinger's wave mechanics:

> *The eigenfunctions of Schrödinger's wave equation are just the transformation functions ... that enable one to transform from the (q) scheme of matrix representation to a scheme in which the Hamiltonian is a diagonal matrix* (Dirac, 1927, p. 635; emphasis in the original).

These eigenfunctions are in fact the energy eigenstates, so what Dirac has found at this stage is the connection to Schrödinger's time-independent wave equation, which appears in the following form,

$$H(q_r', -i\hbar\frac{\partial}{\partial q_r'})(q_r'/\alpha') = H(\alpha')(q_r'/\alpha'). \tag{5}$$

It remains for him to provide a link to the time-dependent equation that he had derived previously, i.e., the Heisenberg equations of motion. Remarkably, he does not do so: in his presentation of the theory the time dependence of the quantum variables is assumed (condition (ii) of p. 327, *ibid.*). Neither does he try to derive the time-dependent Schr̈dinger equation as we would recognize it today. In fact, the time-dependent Schrödinger equation appears almost by accident; the only place Dirac explicitly considers time dependence of the solutions of the Schrödinger equation is in considering time varying Hamiltonians: first in general (p. 635) and then as a perturbation (p. 640).

Dirac had, at this first stage, only identified his transformations with energy eigenstates, and relied on the relation to the (extended) classical phase space to consider dynamics. The time variation of the quantum variables he considered—"constants of integration"—was particularly simple on this pseudo-classical picture, so long as the Hamiltonian was constant with time, and thus could be given by the Heisenberg equations of motion (2). However, if the Hamiltonian is time-dependent then the matrix scheme cannot have this simple time dependence. Dirac explains the problem as follows:

> For systems in which the Hamiltonian involves the time explicitly, there will be in general no matrix scheme with respect to which H is a diagonal matrix, since there will be no set of constants of integration that do not involve the time explicitly. (p. 635)

Yet the result of the derivation that he enters into to address this problem, is an equation for a Hamiltonian H that does not explicitly involve time—an equation that we immediately recognize today as the time-dependent Schrodinger equation for a wavefunction (q'/α'),

$$H\left(q_r, -i\hbar\frac{\partial}{\partial q_r}\right)(q'/\alpha') = H(q_r, p_r)(q'\alpha') = i\hbar\frac{\partial}{\partial t}(q'/\alpha'). \tag{6}$$

The form of this equation (numbered (12) in Dirac's paper) is inconsistent with his claim to have derived "Schrödinger's wave equation for Hamiltonians that involve time explicitly" (p. 636). In discussing this equation, (Darrigol, 1993, p. 341) presents an alternative derivation (not Dirac's) which follows Dirac's earlier paper in assuming that H is a constant of the motion. However, it is quite clear that this was not Dirac's intention. This is a puzzle. What could Dirac have meant by this claim?

It is very plausible that Dirac had just made a mistake in his derivation. Dirac's comment about the lack of energy eigenfunctions, and the dependence of H on t, indicates that he begins by considering solutions of the 'extended' Schrödinger equation that lie in the extended configuration space, for which there are no eigenfunctions. He begins the derivation (p. 635) by considering the Hamiltonian at an instant $t = \tau$, and the corresponding instantaneous variables q_τ, p_τ. Functions of q_τ and p_τ do not involve time explicitly, and so we can regard the Heisenberg equation of motion (for variation in τ) as acting on these functions. From this it follows that

$$H_\tau(q'_\tau \alpha') = i\hbar \frac{\partial(q'_\tau/\alpha')}{\partial\tau}.$$

However, Dirac immediately suggests (p. 636) that we write t for τ and q for q_τ, which removes the time dependence of H_τ and gives (6). Thus does Dirac arrive at the time-dependent equation for a time *in*dependent Hamiltonian.

This mistake is actually quite telling: Dirac does not yet have a conception of a time varying state, and so he is compelled to interpret this equation as one in which the Hamiltonian varies. But, comparing (5) with (6), we see that what he had found was that the variation of the state (q'/α') in time is given by the Hamiltonian operator in the position representation (rather than the energy representation). As Dirac was aware, the space of instantaneous solutions was the only one in which his transformations could be defined, but it is clear that these instantaneous spaces are to be reached by fixing a particular value of the variable t in the larger space and considering variation with respect to a parameterization of that value.

The alternative was to consider a full blown four-dimensional wave equation applying to functions of time and space, a much more formidable problem. The middle ground that Dirac had found (apparently by accident) by dealing with the problem in this manner was taken by him to correspond to a time varying Hamiltonian, but he had instead derived the equation for a time varying *state*, where the Hamiltonian may (or may not) vary with time. Yet at no point in the paper does he entertain the thought that the wavefunction can vary in time without variation of the Hamiltonian.

Why did he not immediately recognize this? At this time—before he became aware of Hilbert space methods—Dirac did not possess the modern notion of a quantum state as a vector state. Moreover, the notion of a (Schrödinger picture) instantaneous state was one that he was to remain resistant to: specifying a time parameterization served to break relativistic invariance, and meant leaving the extended phase space.[5] It seems clear that, for Dirac, the state of the system was to be defined in terms of the extended configuration space, and from there the time evolution of the 'constants of integration' (here, quantum variables) could be specified. In essence, this

[5]The modern notion of state only appeared in the second edition of Dirac's *Principles of Quantum Mechanics*. See (Brown, 2006, p. 402-403) for more details. Also note that, contrary to Hilgevoord's claim that "this view disappeared from his later work" (Hilgevoord, 2005, pp. 36–37), the use of the extended phase space was essential to Dirac's work on constrained Hamiltonian Mechanics, e.g. Dirac (1966).

resembles the modern Heisenberg picture, but without the restriction that observable quantities (quantum variables) are evaluated at an instant. According to Dirac's interpretation, the theory gave information at the level of time averages.

2.3 The Time of a Quantum Jump

The question of whether time and energy are conjugate variables is closely related to the existence of a time-energy uncertainty principle. Reading Heisenberg's famous "*anschaulichen Inhalt*" paper today (Heisenberg, 1927), one is struck by the centrality of the time-energy uncertainty relation rather than the position-momentum uncertainty relation in his informal discussions of the "intuitive content" of the theory. It is clear that this relation is central to Hesienberg's attempt to articulate a physical interpretation of the theory. As he was to later put it: "I wanted to start from the fact that quantum mechanics as we then knew it [i.e. matrix mechanics] already imposed a unique physical interpretation" (from Duncan & Janssen, p. 5).

Looking more closely, we see that Heisenberg's concern is not with time in the abstract (i.e., on a par with "space") but rather the relationship between the energy of the system and the *time of a particular event*—a "quantum jump" regarded as a real physical process. Take the following passage:

> "According to the intuitive interpretation of quantum theory attempted here, the points in time at which transitions—the "quantum jumps"—occur should be experimentally determinable in a concrete manner, such as energies of stationary states, for instance. The precision to which such a point in time can be determined is ... $h/\Delta E$, if ΔE is the change in energy accompanying the transition." (p. 189)

This illustrates his faith that the observable content of the theory should be fixed by theory, but also seems to indicate that his view was that these quantum jumps took place at determinate moments of time, albeit times about which we have limited knowledge. Moreover, Heisenberg discusses how the possibility of measuring the energy precisely depends on performing a measurement between the moments at which jumps occurred.

> In quantum mechanics, such a behavior [of quantized periodic motion] is to be interpreted as follows: since the energy is really changed, due to other disturbances or to quantum jumps, each

> energy measurement has to be performed in the interval between two disturbances, if it is to be unequivocal. (p. 194)

The implication is that a quantum system is to be understood as having a determinate energy at all times, but that this energy fluctuates due to exchanges with the environment — quantum jumps. This was the view he had taken in his previous paper regarding energy exchanges between two coupled systems, which had inspired Dirac's transformation theory.[6] Given this view, we come to appreciate why the time-energy uncertainty relation has such a central role for the interpretation of the theory: since Heisenberg regarded the physical content of the theory as corresponding to discontinuous processes of energy exchange occurring at definite times, the energy-time relation was naturally of central importance to his project of providing an intuitive grasp of the physical content of the new quantum mechanics.

It is also of interest that Jordan's view at this time is very similar to Heisenberg's, and one can imagine that this is something that they had discussed together.

> What predictions can our theory make on this point? The most obvious answer is that the theory only gives averages, and can tell us, on the average, how many quantum jumps will occur in any interval of time. Thus, we must conclude, the theory gives the probability that a jump will occur at a given moment; and thus, so we might be led to conclude, the exact moment is indeterminate, and all we have is a probability for the jump. But this last conclusion does not necessarily follow from the preceding one: it is an additional hypothesis (Jordan 1927, from Duncan and Janssen p. 17).

It seems plausible that Jordan is taking here a similar view to Heisenberg, the view that quantum jumps are physical events taking place at the some definite time. In the first part of the answer, he seems to approach Dirac's opinion that "[the theory] enables one to calculate the fraction of the total time during which the energy has any particular value, but it can give no information about the times of the transitions" (Dirac, 1927, p. 622). But Jordan goes further to say that there is information here about the *rate* of occurrence of quantum jumps. He goes even further in suggesting that the theory might be considered to supply information about the *probability* that

[6] See (Duncan & Janssen, 2012, pp. 3-8) for more details.

a jump might occur at a particular time. This is distinguished from the view that the moment of time at which the jump occurs is *indeterminate*, leaving open the possibility that the probabilities involved could be interpreted epistemically rather than objectively.

The major technical contribution of Heisenberg's paper concerns the derivation of the position-momentum uncertainty relation from the Dirac-Jordan transformation theory. On the other hand, the time-energy relation (stated without proof on p. 177) followed intuitively from the quantization of action-angle variables in classical theory, assumed to form a conjugate pair in the quantum theory (presumably motivated by the classical analogy detailed in my Section 2.1). However, the early form of Jordan's transformation theory was particularly encouraging for the prospects of a parallel derivation of the time-energy relation along the lines of the position-momentum derivation, since it assumed (in modern parlance) that canonically conjugate observables have purely continuous spectra spanning the real line. This is a necessary condition for observables to allow the derivation of a standard uncertainty principle in modern quantum mechanics, but it does not hold of typical Hamiltonians (as Pauli later pointed out).

2.4 The 1927 Solvay Conference: The Fate of the Quantum Jump

The period of time during which this 'quantum jump' interpretation of the theory remained plausible was short-lived. By the time of Heisenberg's presentation with Born at the Solvay conference of October 1927 he no longer held this interpretation of the physical meaning of the time of a quantum jump. Born and Heisenberg's presentation contains the following passage,

> If one asks the question *when* a quantum jump occurs, the theory provides no answer. At first it seemed that there was a gap here that might be filled with further probing. But soon it became apparent that this is not so, rather, that it is a failure of principle, which is deeply anchored in the nature of the possibility of physical knowledge. One sees that quantum mechanics yields mean values correctly, but cannot predict the occurrence of an individual event. (Bacciagaluppi & Valentini, 2006, p. 420)

This seems to represent a retreat to Dirac's position that only the time average was a physically meaningful quantity. However, the rest of their presentation reveals a more radical point of view. Bacciagaluppi and Valentini read their claim that "matrix mechanics deals only with closed periodic systems, and in these there are no changes. In order to have true processes ... one must restrict one's attention to a part of the system" (Bacciagaluppi & Valentini, 2006, pp. 205–6) to suggest that they shared the view of Campbell (endorsed by Heisenberg in a letter to Pauli) that time is a statistical phenomenon, absent in atomic systems but emerging at the macroscopic level like temperature or pressure.

Since the time-independent Schrödinger equation is solved by the stationary states corresponding to eigenfunctions of energy, if one makes the supposition that a system is always in such a state and the theory supplies probabilities for the 'jump' from one state to another, then it would be as if time did not exist except for these discontinuous transitions. Though Schrödinger introduces it only to reject it, his report contains a detailed analysis of this proposal, in which quantum systems considered as a whole involve no passage of time, and according to which time emerges from the theory as a macroscopic parameter related to the number of quantum jumps occurring between subsystems. According to this view, there is no change, and thus no passage of time in between quantum jumps (p. 207), and time rather emerges as a parameter related to the rate at which jumps occur.

> Limiting our attention to an isolated system, we would not perceive the passage of time in it any more than we can notice its possible progress in space. ... What we would notice would be merely a sequence of discontinuous transitions, so to speak a cinematic image, but without the possibility of comparing the time intervals between transitions. (p. 207)

According to Campbell's hypothesis, "one cannot regard the jump probability in the usual way as the probability of a transition calculated relative to unit time." (p. 451) On this view, the theory supplies probabilities for transitions between states and in terms of temporal information can only provide probabilities that one transition occur before or after another.

The alternative, says Schrödinger, is to regard the system not as occupying a single stationary state (along the lines of Bohr's earlier atomic theory) but rather as having a state that may be an arbitrary linear superposition of energy eigenstates. Taking this view—which was, of course, the view to win out—time appears in terms of the evolution of the relative phases of the eigenstates, decomposed relative to a particular basis. Now there is more to say here about the emergence of the modern notion of state, some of which is covered by Duncan & Janssen (2012), but we can see that already these developments were fatal to the idea of the quantum jump as a discontinuous transition between stationary states, which relied critically on the hypothesis that a system remain in an energy eigenstate at all times.

3 In Defense of the Notion of the Time of an Event

It is clear that (with respect to von Neumann's Hilbert space formalism) these expectations regarding the role of time in the theory were false: time and energy are not conjugate variables, the Schrödinger equation is defined for functions of space alone, and there is no such thing as the time of a quantum jump (or collapse).[7] Nonetheless, I have shown that the motivations of the authors were not simply the result of conceptual confusion (although later physicists may have been misled along those lines), and so Hilgevoord's (2005) rejoinder that time already has an appropriate representation in the theory as a parameter is misguided. Taking a sympathetic reading of their motivations, I will show that these expectations can be physically motivated, and can in fact be met with minimal mutilation of the existing formalism. Thus the fact that the standard textbook presentation of quantum mechanics is inhospitable to the introduction of 'time' as an observable need not be read as a prohibition on the introduction of the *time of an event* as an observable quantity.

The initial motivation of Heisenberg to regard the time of a quantum jump as an experimentally meaningful quantity was the idea that a quantum system remains in a stationary state of definite energy, except when it instantaneously transitions to another stationary state. So if the energy of the system could be shown experimentally to have changed from one value to another, then a quantum jump must have occurred in the meantime (p. 191), and the time at which it had occurred could be experimentally determined (up to an uncertainty depending on the energy difference). It was thus natural for Jordan to regard the transformation theory as providing probabilities for such transitions to occur within a particular interval of time. I claim that, while the requirement that a system be in an energy eigenstate at all times was mistaken, the idea that the theory should provide probabilities for events to occur during intervals of time as well in regions of space was not.

To take a straightforward example, consider an experiment consisting

[7] Of course, the dynamics of the theory can be modified to bring about the stochastic collapse process at particular moments of time, but I will not consider these alternative dynamics.

of a single radioactive atom and a Geiger counter that fully surrounds it. If the half-life of the atom is 1 hour, then the probability that the counter clicks in the first hour is 1/2, the probability it clicks in the second hour 1/4, the probability it decays in the third hours is 1/8, and so on. Thus the probability that the counter clicks at some point in the future is given by an arithmetic sequence that tends to one as t tends to infinity. However, once the counter has been observed to click, the probability that it clicks in the future in zero. This is evidently a experimentally meaningful situation, and we should expect that this phenomenological law may be derived from a detailed quantum mechanical description of the decay process. However, within the standard account of measurement there is provably no way to implement this simple scheme.

This description would involve a (Heisenberg picture) quantum mechanical state ψ in a Hilbert space $\mathcal{H}$, a Hamiltonian H describing the time evolution of the system, and a series of operators T_1, T_2, T_3 such that $\langle\psi|T_1\psi\rangle = 1/2$, $\langle\psi|T_2\psi\rangle = 1/4$, and $\langle\psi|T_3\psi\rangle = 1/8$. Since the theory has a time translation symmetry implemented by the unitary group e^{iHt}, we have that $T_2 = e^{-iHt}T_1e^{iHt}$ and $T_3e^{-iHt}T_2e^{iHt}$, where t is one hour. Even this bare bones sketch is enough to tell us something interesting about the operators T_i: if H is a self-adjoint operator with spectrum bounded from below[8] then it follows that $\langle\psi|T_{i+1}T_i|\psi\rangle \neq 0$ and so these operators T_i cannot be projections onto mutually orthogonal subspaces of $\mathcal{H}$.[9]

Thus there is no mixed state decomposition in terms of states ψ_i such that $T_i\psi_i = \psi_i$, in which case the ψ_i would correspond to the system decaying during distinct intervals of time, and neither can the T_i together serve to define a self-adjoint 'time of decay' operator. The former implication indicates the von Neumann's collapse postulate cannot be applied to this situation; the latter than his identification of observables of the theory with self-adjoint operators is ill-suited to include the time of an event as an observable quantity. Yet there seems every reason to suppose that the theory should be able to answer questions like, "When will the Geiger counter click?" or in a diffraction experiment, say, "When will a dot appear on the screen?" In failing to answer these questions, the theory would be fail to be empirically adequate. This failure would constitute a real 'problem of time' for the theory. But this problem can be overcome, and without modifying the dynamics: The problem is not with the way that quantum mechanics

[8]This corresponds to the assumption that the is a value of energy for the system below which it cannot drop.

[9]For a proof this result see Unruh & Wald (1989).

defines the state of the system, but the way that probabilities are derived from the state.

The first thing to note is that in the identification of observables with self-adjoint operators it is assumed that these these operators act instantaneously. In the Schrödinger picture (in which the states vary with time) the expectation value of an operator A in the state ψ_t is $\langle\psi_t|A|\psi_t\rangle$, whereas in the Heisenberg picture (in which the observables vary with time) the expectation value of an operator is $\langle\psi|e^{-iHt}Ae^{iHt}|\psi\rangle$. (These return the same values since $|\psi_t\rangle = e^{iHt}|\psi\rangle$.) In the Schrödinger picture it makes very little sense to ask when a particular event occurs (in the sense of a probability for it occurring during some interval of time) since we may only interrogate the state at a moment of time. However, in the Heisenberg picture we may define operators that involve more than one moment of time by integrating over t.

Consider an instantaneous measurement of position. The existence of a self-adjoint position observable implies the existence of an assignment of projection operators P_Δ to regions of space Δ such that disjoint regions of space correspond to mutually orthogonal subspaces. In the Heisenberg picture, an instantaneous measurement of P_Δ at time t apparently corresponds to asking the question "is the system located in Δ at time t?" The probability of finding a positive answer is $\langle\psi|e^{-iHt}P_\Delta e^{iHt}|\psi\rangle$, and if the system is found to be in Δ then the state is updated accordingly by the projection operator $P_\Delta(t) = e^{-iHt}P_\Delta e^{iHt}$.

To consider a measurement that takes place over more that one instant, we can integrate these operators over t. The most straightforwardly defined of these operators is the dwell time operator,[10] whose expectation value corresponds to something like the proportion of time that a system spends within a region Δ,

$$T_d = \int_{-\infty}^{\infty} P_\Delta(t)dt.$$

But while this is a self-adjoint operator (albeit one whose measurement in a concrete experimental situation is questionable), it is not appropriate for describing the time of an event since it does not assign probabilities to times (or time intervals).

Consider instead a cloud chamber experiment where we set up a detector that is sensitive to the presence of high energy particles, with the chamber located in Δ. The presence of the particle will be registered by an ionization event, which is recorded by a photosensitive emulsion. For a given state ψ

[10]See Jose Munoz et al. (2010) for a recent discussion.

we want to obtain the probability that the detector fires during some interval $I = [t_1, t_2]$, given that it fires at all. The obvious candidate for an operator that corresponds to the presence of a particle within Δ during a time interval $[t_1, t_2]$ is

$$T_\Delta(I) = \int_{t_1}^{t^2} P_\Delta(t)dt,$$

but this operator is not normalized. Nonetheless, if we assume that the particle will be detected at some time (and exactly once) then the normalization is provided by the dwell time operator, which, being self-adjoint, has a unique square root $T_d = (T_d^{1/2})$. Using the inverse of this operator, we define

$$E_\Delta(I) = T_d^{-1/2} E_\Delta([t_1, t_2]) T_d^{-1/2},$$

which returns the identity when $I = \mathbb{R}$. With this 'operator normalization' the quantity $\langle\psi|E_\Delta(I)|\psi\rangle$ can be interpreted as giving the probability that the event occurs during I rather than at some other time.[11]

The operators $E_\Delta(I)$ together form a Positive Operator Valued Measure (POVM) which maps temporal intervals I to positive operators on $\mathcal{H}$.[12] However, these operators are not mutually orthogonal projections and cannot form a Projection Valued Measure (PVM); in general, a POVM that covaries with time translations $E_\Delta(I) = e^{-iHt} E_\Delta(I - t) e^{iHt}$ cannot be a PVM.[13] Since the self-adjoint operators on $\mathcal{H}$ are in one-to-one correspondence with the set of PVMs, there is no self-adjoint operator corresponding to the operators $E_\Delta(I)$ (in the same sense that the position operator Q corresponds to the projections P_Δ). Thus von Neumann's association of observables with self-adjoint operators would exclude these event time observables.

But it is not surprising that this is so since in von Neumann's measurement schema the measurement of an observable takes place at an instant, i.e. under the condition that the time is t. The usual probabilities given by an observable are *conditional* probabilities in the following sense: they are probabilities which are valid *given that the time is* t. On the other hand, the event time observable is 'measured' over an interval of time by leaving

[11] This operator normalization scheme is suggested by Brunetti & Fredenhagen (2002) and has been interpreted (Busch, 2007) in terms of the amount of time that the system spends in the region Δ during I, but to do so ignores the physical meaning of the normalization. The more natural interpretation, I claim, is in terms of the probability of a detection event, given that such an event will certainly occur.

[12] See Brunetti & Fredenhagen (2002) for a proof of this in general.

[13] See Srinivas & Vijayalakshmi (1981).

the experiment set up and waiting for it to register an event. The condition under which such a experiment takes place is that such an event will occur sooner or later, i.e., the probabilities provided by $E_\Delta(I)$ are also conditional probabilities, with the condition being that the event takes place exactly once over all of time. The incompatibility of these normalization conditions indicates that they correspond to incompatible measurements. But how many experiments take place at an instant of time? If we are judging by empirical relevance, I suggest that the conditionalization required by event time observables is more apt—only a poor detection experiment ends without a detection event.

These observables have a close relation to the screen observables of Werner (1986), which generalize the quantum time of arrival.[14] Screen observables apply to a typical diffraction experiment where an electron, say, is emitted and some time later detected at a photoluminescent screen. Since the screen is sensitive to the presence of an electron at all times, and electrons in an ensemble will be detected at different times, an instantaneous position observable cannot suffice to describe even the spatial distribution of detection events. For these screen observables, one assumes again that the detector will fire at some time t, and so the sum of the probability of detection over all times is unity. Again, very few experimental arrangements (if any) correspond to anything like an instantaneous position measurement, which would provide probabilities for a detector spread out through all of space which fires exactly once when switched on for an instant.

The main puzzle that is raised by event time observables such as these is, in my view, one of providing a suitable update rule. While POVMs play the same predictive role as PVMs[15], event time POVMs are particularly ill-suited to supplying the means to update the state, which is typically defined for projectors through Lüders Rule, which projects the state into the eigenspace of the measured eigenvalue (normalizing according to the trace). However, the very normalization of the POVM $E_\Delta(\mathbb{R}) = I$ which made it suitable for its role as an event time observable makes it ill-suited to provide probabilities for events that occur subsequent to detection. For example, it is unclear that we can obtain a definite answer to the question: what is the probability that the particle is first detected in Δ during I and subsequently in Δ'? To answer this question appears to require a new normalization scheme since now the particle is detected in both Δ and Δ',

[14] This has been a topic of much research. See Muga & Leavens (2000) for a review.

[15] As with a projection, a normalized positive operator E supplies probabilities through taking the trace of the density operator ρ, $Tr[\rho E]$.

i.e., twice. But if we normalize along those lines then we have lost the conditional nature of the probabilities desired: the question was, what is the probability of finding the particle in Δ' given that it was *already* detected in Δ?

For answering such questions, we must go beyond operator normalization and instead consider the *extended* Schrödinger equation, defined for functions of time and space as Dirac originally envisioned it,

$$(H - W)\psi(x,t) = 0.$$

The problem with this equation is that the operator $(H - W)$ has a continuous spectrum, and so there is no vector $\psi(x,t) \in L^2[\mathcal{R}^4] = \mathcal{H}_+$ which is an eigenvector with eigenvalue 0. However, as Dirac had claimed, energy $(-W)$ and time are conjugate variables on this space of functions of space and time. Without going into the details, recent work by Brunetti et al. (2010) has shown how solutions to this equation may be written in terms of linear functionals rather than vectors. While these physical solutions don't form a Hilbert space, and define non-normalizable 'weights' on the algebra of observables of $\mathcal{H}_+$ rather than algebraic states, there is a construction which, given an operator on $\mathcal{H}_+$ representing the occurrence of an event, leads to a GNS Hilbert space representation giving the expectation values for the algebra of observables on the condition that the event in question did occur.

Since time is a self-adjoint operator on $\mathcal{H}_+$ (which, remember, is not the space of solutions of the extended Schrödinger equation), the event time operators such as $T_\Delta(I)$ are projections in this space. By conditionalizing on these events, one can calculate probabilities for subsequent events, such as another detector firing elsewhere. This theory has a good claim to be regarded as a straightforward generalization of the usual Schrödinger dynamics. First, by setting up an appropriate map from $\mathcal{H}_+$ to $\mathcal{H}$, the projections $T_\Delta(I)$ become the operator normalized time POVM $E_\Delta(I)$. Second, and most importantly, the predictions of the usual Schrödinger picture description are returned in the instantaneous limit. But note that the differences between the instantaneous form and the extended form are significant: the solutions of the extended Schrödinger equation do not form a Hilbert space, and there is no meaning to the phrase 'the state of the system' without first specifying an event, the occurrence of which can be used to give a probability assignment to further events. Thus such probability assignments correspond to a *conditional* state.

4 Coda: The Philosophy of Time

What does this have to do with the metaphysics of time? The great debate over the nature of time in fundamental physics began with the correspondence of Gottfried Wilhelm Leibniz and Samuel Clarke. Against the Newtonian view of time as a substantive physical entity (substantivalism) — the "container view" of space-time — Leibniz argued that time is nothing more than the temporal relations between independently existing events (relationism). Recent discussions of the role of time in cosmological theories (e.g., Barbour (1999)) have tended to take the line that philosophical considerations demand the elimination of time from physical theories, or at least reduction of time to some other quantity with respect to which change may be defined. However, it is implausible that Leibniz intended to deny the existence of time. For example, he wrote that "time is the order that makes it possible for events to have a chronology among themselves when they occur at different times," which is inconsistent with a reductive or eliminativist account of temporal relations.[16]

Instead, Leibniz believed that time is nothing above and beyond the temporal ordering relations that exist between events: "times, considered without the things or events, are nothing at all, and . . . consist only in the successive order of things and events." This view was in stark contrast to Clarke, who maintained that "time is not merely the order of things succeeding each other, because the quantity of time may be greater or less while the order of events remains the same." Thus what Clarke affirmed and Leibniz denied was the metrical structure of time; in a world consisting of successive events with a single linear temporal order, Clarke believed that one could (in principle) measure the amount of time between two successive events, whereas according to Leibniz this is a meaningless notion—once given the temporal orderings of events we have all there is to say about the temporal facts.

In ordinary quantum mechanics (the Dirac-von Neumann formalism) time features as a parameter that indexes the states (Schrödinger picture) or the observables (Heisenberg picture). In the Schrödinger picture, the natural way to view the time indexed states is as describing the same physical system at distinct moments of time. This seems to describe a world in which time is continuously valued, having the structure of the real line. It is hard to see how such a world could be made compatible with Leibniz's view of time as nothing but the temporal relations between events. But if we view

[16] All quotations are from translations at http://www.earlymoderntexts.com/.

the formalism as a means for providing probabilities for the occurrence of events in time, as I have suggested, we obtain a view of the world much more hospitable to Leibniz's contention that time is just the temporal ordering relations between (possible and actual) events.

References

Bacciagaluppi, G. & Valentini, A. (2006). Quantum theory at the crossroads: reconsidering the 1927 solvay conference. *arXiv preprint quant-ph/0609184*.

Barbour, J. (1999). *The End of Time*. New York: Oxford University Press.

Born, M. & Jordan, P. (1925). Über die streuung von strahlung durch atome. In van der Waerden (Ed.), *Sources of quantum mechanics* (pp. 223–252). Dover.

Brown, L. (2006). Paul A. M. Diracs the principles of quantum mechanics. *Physics in Perspective (PIP)*, *8*(4), 381–407.

Brunetti, R. & Fredenhagen, K. (2002). Time of occurrence observable in quantum mechanics. *Physical Review A*, *66*(4), 044101.

Brunetti, R., Fredenhagen, K., & Hoge, M. (2010). Time in quantum physics: from an external parameter to an intrinsic observable. *Foundations of Physics*, *40*(9), 1368–1378.

Busch, P. (2007). The time–energy uncertainty relation. In G. Muga, A. Ruschhaupt, & A. del Campo (Eds.), *Time In Quantum Mechanics – Vol. 1 (2nd. Edition)* (pp. 73–105). Springer.

Darrigol, O. (1993). *From c-numbers to q-numbers: the classical analogy in the history of quantum theory*, volume 10. University of California Press.

Dirac, P. A. M. (1925). The fundamental equations of quantum mechanics. *Proceedings of the Royal Society of London. Series A*, *109*(752), 642–653.

Dirac, P. A. M. (1926a). On the theory of quantum mechanics. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 661–677.

Dirac, P. A. M. (1926b). Relativity quantum mechanics with an application to compton scattering. *Proceedings of the Royal Society of London. Series A*, *111*(758), 405–423.

Dirac, P. A. M. (1927). The physical interpretation of the quantum dynamics. *Proceedings of the Royal Society of London. Series A*, *113*(765), 621–641.

Dirac, P. A. M. (1966). *Lectures on quantum mechanics*. Belfer Graduate School of Science, Yeshiva University.

Duncan, A. & Janssen, M. (2012). (never) mind your p's and q's: Von neumann versus jordan on the foundations of quantum theory. *arXiv preprint arXiv:1204.6511*.

Heisenberg, W. (1927). The actual content of quantum theoretical kinematics and mechanics. *Zhurnal Physik*, *43*, 172–198.

Hilgevoord, J. (2005). Time in quantum mechanics: a story of confusion. *Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics*, *36*(1), 29–60.

Hilgevoord, J. & Atkinson, D. (2011). Time in quantum mechanics. In *The Oxford Handbook of Philosophy of Time*. Oxford University Press.

Jose Munoz, I. L. E., del Campo, A., Seidel, D., & Muga, J. G. (2010). Dwell-time distributions in quantum mechanics. *Time in Quantum Mechanics*, *2*, 97.

Muga, J. G. & Leavens, C. R. (2000). Arrival time in quantum mechanics. *Physics Reports*, *338*(4), 353–438.

Olkhovsky, V. S. (2011). On time as a quantum observable canonically conjugate to energy. *Physics-Uspekhi*, *54*, 839.

Srinivas, M. & Vijayalakshmi, R. (1981). The 'time of occurrence' in quantum mechanics. *Pramana*, *16*(3), 173–199.

Unruh, W. G. & Wald, R. M. (1989). Time and the interpretation of canonical quantum gravity. *Physical Review D*, *40*(8), 2598.

Werner, R. (1986). Screen observables in relativistic and nonrelativistic quantum mechanics. *Journal of mathematical physics*, *27*, 793.