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BOOK REVIEW

Francesca Biagioli: Space, Number, and Geometry from Helmholtz to Cassirer

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Lydia Patton[1]

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There is a new energy in the study of the history and philosophy of nineteenth and twentieth century mathematics and physics. Pioneering work from Alberto Coffa, Alan Richardson, Michael Friedman, Don Howard, Gary Hatfield, Janet Folina, Michael Heidelberger, Thomas Ryckman, and kindred thinkers in the 1990s and early 2000s inspired many to take up research on the complex relations between neo-Kantianism, logical empiricism, physiology of perception, epistemology, group theory and geometry, and relativity theory.[1] Francesca Biagioli has been working in these fields for some time, and *Space, Time, and Geometry* further establishes her reputation as one of the strongest researchers in this area.

Space, Time, and Geometry avoids merely telling the history of relativity theory. Instead, the work is oriented around the careers of a scientist, Hermann von Helmholtz, and a philosopher, Ernst Cassirer. That focus allows Biagioli to delve into what mattered to scientists, mathematicians, and philosophers working at the time, including new ways of linking empirical observation with group theory, problems defining physical and mathematical quantities, and debates over the role of axioms and conventions in science (including questions about the structure and development of theories). Over the course of seven chapters, Biagioli shows that relativity theory is just one of the compounds produced from these elements over the nineteenth and twentieth century.

[1] Even as those mentioned continue to break new ground on themes including the formulation and defense of 'scientific philosophy' and the Unity of Science program, they are joined by a group of researchers who bring formal expertise and historical sensibility to an expanded set of questions. Marco Giovanelli, Flavia Padovani, Michela Massimi, Katherine Brading, Elise Crull, David Hyder, Scott Edgar, Joshua Eisenthal, Matthias Neuber, and their allies focus on the history and philosophy of physics, physiology, or mathematics. Erich Reck, Georg Schiemer, Dirk Schlimm, Jeremy Heis, Audrey Yap, and Paula Cantù delve into structuralism and formalism in nineteenth and twentieth century mathematics. Martin Kusch, Pierre Keller, Samantha Matherne, Katherina Kinzel, Paul Roth, and Frederick Beiser are among the mainstays of a circle studying the philosophy of history itself during this period.

✉ Lydia Patton
critique@vt.edu

[1] Department of Philosophy, Virginia Tech, Blacksburg, USA

The book begins (Chs. 1 and 2) with the actual encounter between Helmholtz, a leading light in nineteenth century science, and the Marburg School of neo-Kantianism: "the reception of Helmholtz in neo-Kantianism is a good example of a fruitful interaction between philosophy and the sciences" (6). Cassirer's engagement with Helmholtz's work in physics, physiology, and the theory of measurement "led Cassirer to the view that […] the core [Kantian] idea that there are increasingly higher levels of generality in the conditions of experience was confirmed by more recent developments in the history of science" (7). Cassirer was at work developing the view that he articulated in *Substance and Function* (*Substanzbegriff und Funktionsbegriff*), where he argued for a form of such generality in the realm of logic: replacing the logic of genus and species with the concept of a mathematical function (45). The traditional logic could not account for, for instance, new work on mathematical series and limits. Biagioli traces the influences of Helmholtz, Dedekind, and Klein on Cassirer's formulation of "a generalized version of the Kantian schematism of the pure concepts of the understanding by pointing out the role of the mathematical concept of function in the sciences" (47). Chapter 1 sets the stage for this by detailing the now well-known debates over Kantian epistemology, especially Kant's account of space, time, and perception.

Here the status of geometrical axioms, and thus of non-Euclidean geometry, comes to the forefront. Classic nineteenth century discussions of geometrical axioms focused on their role in measurement, perceptual experience, and physical reasoning. They serve the dual role Cassirer finds for them, on Biagioli's account: geometrical axioms are (or are the basis of, or are employed with) mathematical functions that serve as schemas, that is, as valid rules for generating evidence in experience. But how do the axioms work, and where do we acquire them? Are the axioms of geometry hypotheses, a view associated with Bernhard Riemann; empirical in origin, which Helmholtz argued in places; or conventions, as Henri Poincaré sometimes mentions?

Biagioli's Chapter 3 begins with a puzzle that she had identified in an earlier paper (Biagioli 2014, Ch. 3, §3.2): Helmholtz's 'refined' empiricism appeals to the transcendental nature of space, but Helmholtz argues that geometrical axioms are empirical in origin. Space, then, "can be transcendental without the axioms being so", as Helmholtz remarks. The idea is that space can be a mathematical structure necessary for valid measurements, but the specific axioms required for measurement (or for geometry) need not be transcendental or universally valid. Biagioli notes differences between Helmholtz and the neo-Kantians on this score. Hermann Cohen objected that the homogeneity of space "presupposes both geometry and mechanics" (74). Magnitudes do not exist in themselves independently of constructive activities and even measurement, as geometrical empiricists like Helmholtz suppose (72–73). Alois Riehl, on the other hand, argued that Helmholtz had confused 'transcendental' and 'a priori' (68).

The target of Biagioli's analysis is an explanation and defense of Ernst Cassirer's reaction to Helmholtz in Volume 4 of the magisterial *The Problem of Knowledge* (*Das Erkenntnisproblem*). The bulk of the case is made in Chapters 3 and 4 (and in §5 of Biagioli 2014).

Cassirer sees in Helmholtz a foreshadowing of the group-theoretical approaches of Sophus Lie, Henri Poincaré, and Felix Klein. A homogeneous 'space' with respect to any particular set of measurement procedures is possible only under certain group-theoretical assumptions (rigidity of transformations, for instance). In his response to J. P. N. Land in *Mind* in 1878, Helmholtz writes that the "form of outer intuition" for him is the "group of spatial transformations": "Geometry captures a fundamental feature of empirical reality insofar as such a group is required for measurements to be repeatable" (86; see Biagioli 2018 and Neuber 2018 for recent work on this subject).

Helmholtz's positions lead to complications, as Cassirer, Poincaré, and Lie note. For instance, Poincaré observes that Helmholtz defines a 'rigid' body as one that is invariant with respect to the transformations of a certain group. If those transformations in turn require 'rigid' bodies as a priori conditions, then Helmholtz's reasoning appears circular (§4.2.3; see also 168–169).

Sophus Lie addresses what he calls the "Riemann-Helmholtzsche Problem" in his 1893 *Theory of Transformation Groups* (*Theorie der Transformationsgruppen*) (117). Biagioli does not go into detail about Lie's analysis, indicating that it appears in Chapter 21 of Lie's work, edited with Friedrich Engel. There, as an earlier commentator notes,

> Lie gives two solutions of the problem. In the first he investigates […] a group possessing free mobility in the infinitesimal […] if a point and any line-element through it be fixed, continuous motion shall still be possible; but if besides any surface-element through the point and line-element be fixed then shall no continuous motion be possible. The groups in tri-dimensional space possessing […] this free mobility Lie finds to be precisely those characteristic of the Euclidean and the two non-Euclidean geometries. Strangely enough, for the seemingly analogous and simpler case of the plane or two-dimensional space these are not the only groups. There are others where the paths of the infinitesimal transformations are spirals. Without the group idea, Helmholtz had reached this reality, and […] concluded that also to characterize our tri-dimensional spaces a new condition, a new axiom, was needed, that of monodromy. It is one of the most brilliant results of Lie's second solution of the space problem, that starting from transformation-equations with three of Helmholtz's four assumptions he proves that the fourth, the famous "Monodromie des Raumes," is, in space of three dimensions, wholly superfluous. (Halsted 1899, 221–222)

Lie's work, which won the first Lobachewsky Prize in 1897, shows that Helmholtz's approach could be generalized, broadened, and deepened by analyzing the mutual dependencies of Helmholtz's geometrical axioms, and by using the formal apparatus of group theory and non-Euclidean geometry.

Biagioli focuses on the responses from Klein, Poincaré, and Cassirer, who "reformulated the argument by using Felix Klein's 1871 projective model of non-Euclidean geometry" (117). Cassirer, in particular, "used Klein's classification" of geometries "to generalize the Kantian notion of space to a system of hypotheses, including both Euclidean and non-Euclidean geometries" (118).

The concluding Chapters (5–7) of the book are rich and complex. I will try to provide a brief road map to their significance.

Biagioli argues that "the idea of a generalization of the Kantian form of spatial intuition can be traced back to Helmholtz's remarks about the use of analytic geometry in measurement and had its origins in the so-called arithmetization of mathematics. In my reconstruction, what distinguishes the [Helmholtz–Cassirer] tradition from Schlick's view of geometry is the idea that a more general understanding of spatial order in structural terms […] was necessary in order to pose the problems concerning measurement correctly" (197). On Biagioli's account, Schlick had argued that Poincaré's conventionalism was the only solution to the circularity problem Poincaré raised with Helmholtz, but Cassirer responds that Kant's form/content distinction could be rationalized and given a new structure in Helmholtzian terms. First, for Helmholtz, "whereas the notion of a three dimensional space of constant curvature can be inferred from free mobility, the choice between the specific cases of such a manifold depends on the structure of actual space," and in 1910, Cassirer's goal "was to show that such a way to reformulate Kant's form/content distinction was compatible with a Kantian architectonic of knowledge" (163). Cassirer advocates using rational criteria, including inductive conclusions from the history of science, in formulating empirical predictions using concepts and axioms (178–179).

Poincaré's circularity objection can be avoided in this way. Cassirer's account also allows for an increasingly clarified relationship between analysis and synthesis in mathematics and physics, which had become a hot topic in nineteenth century geometry and analysis. Von Staudt had argued that "the viewpoint of analytic geometry presupposes that of the geometry of position" (123), whereas "in 1874, Klein showed that both von Staudt's proof of the final theorem and his treatment of projective coordinates presupposed further assumptions about continuity, whose first analytic treatment goes back to Dedekind (1872). Since this fact seemed to call into question the feasibility of a purely synthetic foundation of geometry, Klein proposed a synthesis between analytic and synthetic methods" (123).[2] In the years that followed, Dedekind, Klein, and Cassirer developed views based on the increasing arithmetization of mathematics. While "Klein's stance led to a sharp distinction between mathematics and empirical science", paradoxically, pure mathematics was required to provide a "precise formulation, classification, and assessment of probability of the hypotheses occurring in other disciplines" (142). Dedekindian structuralism and the arithmetization of analysis became increasingly crucial to the formulation of hypotheses and predictions in the empirical sciences. Conversely, the results of the empirical sciences form an ever increasing inductive basis for pure theorizing, and so the history of science becomes crucial to philosophical and mathematical analysis: just as the Marburg School had argued consistently, from its founding moments until its apotheosis in Cassirer's work.

Francesca Biagioli's *Space, Number, and Geometry from Helmholtz to Cassirer* is a substantial and pathbreaking contribution to the energetic and growing field of researchers delving into the physics, physiology, psychology, and mathematics of the nineteenth and twentieth centuries.[3] The book provides a bracing and painstakingly researched re-appreciation of the work of Hermann von Helmholtz and Ernst Cassirer, and of their place in the tradition, and is worth study for that alone.

The contributions of the book go far beyond that, however. It is a clear, accurate, and deep account of fascinating and philosophically momentous implications of the move to relativity theory. It is not alone: Michael Friedman's *Foundations of Space–Time Theories*, for instance, is such a work, as is Roberto Toretti's *Relativity and Geometry* and Thomas Ryckman's *The Reign of Relativity*. To say that this book belongs in that company is a strong endorsement, and Biagioli's book deserves it.

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[2] A translation of Dedekind (1872) can be found in Dedekind (1963).

[3] There is no requirement of comprehensiveness in a project such as this one. I should mention that Flavia Padovani's work on Hans Reichenbach on measurement and time would be an excellent complement to much of what is achieved in *Space, Time, and Geometry*, as would Elise Crull's and Erik Banks's work on Grete Hermann.

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