The idea of quantity at the origin of the legitimacy of mathematization in physics

Michel Paty

in Gould, C. (ed), *Constructivism and Practice: Towards a Social and Historical Epistemology*, Rowman & Littlefield, Lanham (Md.,USA), 2003, p. 109-135.

The idea of quantity at the origin of the legitimacy of mathematization in physics

Michel PATY*

SUMMARY

Newton's use of mathematics in mechanics was justified by him from his neo-platonician conception of the physical world that was going along with his «absolute, true and mathematical concepts» such as space, time, motion, force, etc. But physics, afterwards, although it was based on newtonian dynamics, meant differently the legitimacy of being mathematized, and this difference can be seen already in the works of eighteenth century «Geometers» such as Euler, Clairaut and d'Alembert (and later on Lagrange, Laplace and others). Despite their inheritance of Newton's achievements, they understood differently the meaning and use of mathematical quantities for physics, in a way that was more neutral to metaphysics.

The continental reception and assimilation of Newton's *Principia* had indeed occured as its budding onto Leibniz' calculus and a cartesian conception of rationality (spread in particular by the malebranchist disciples of Leibniz). This new thought of the legitimacy of mathematization is clearly at variance with Descartes' identification of physics with geometry, but it nevertheless can be traced back to Descartes' conception of magnitudes, as it was developed and analyzed from the notion of dimension in his *Regulæ ad directionem ingenii* (in particular, rule 14). This idea can be followed afterwards with further philosophical or mathematical specifications through authors such as Kant, Riemann and others.

This inquiry into the original thought of magnitudes, and of physical magnitudes conceived through mathematization, leads us to suggest an extension of meaning for the concept of physical magnitude that puts emphasis on its relational and structural aspects rather than restraining it to a simple «numerically valued» acception. Such a broadening would have immediate implications on our comprehension of «non classical» aspects of contemporary physics in the quantum area and in dynamical systems.

* Equipe REHSEIS (UMR 7596), CNRS et Université Paris 7-Denis Diderot, 37, rue Jacob, F-75006 Paris.
Courrier électronique : paty@paris7.jussieu.fr

CONTENTS

To the memory of Marx Wartofsky who used to consider «the philosophy of science in the broad context of [the] historical, analytic, and synthetic components of the philosophical enterprise»[1].

1

INTRODUCTION
PHYSICAL THEORY, QUANTITIES AND PROBLEMS OF ONTOLOGY.

My aim in this reflection on the concept of quantity or magnitude is twofold : first, to inquire the *legitimacy of mathematization of physics*, and second, as a consequence of it, to consider the possibility in that science to *extend the meaning of the concept of magnitude* as it is commonly taken, i. e. quantities endowed with numerical values. Such an extension would be particularly appropriate to simplify problems met in the «interpretation» of quantum physics.

I would like to introduce my approach to this question with an evocation of something that stands in its background, namely the problem of *realism*, most often identified with that of *ontology*, provided that reality be implicitly assimilated to substance. It often seems to me that there is some misunderstanding on ontological questions when we speak about contemporary science, and in particular about physics. Already in XVIIIth century the notion of substance has been systematically criticized and rejected as a residue of scholastic thought. We shall come back later on to the claims of the physico-mathematicians

[1] Wartofsky [1968], p. v.

of that time (the Geometers, as they use to call themselves) as to which physics deals essentially with *relations* of physical quantities, expressed mathematically. There were no claim for any «absolute» ontology whatsoever : this lesson has largely been retained for nowadays science and philosophy of science, and I wonder whether XXth century's fights against ontology are aimed at an effective and real target.

«Ontology» being referred to «things», and these to «reality», the target is therefore actually *reality*, as it is clearly the case in the interpretation debate on quantum physics. But «physical reality», when its existence is asserted as constituting the proper object of physics, whatever be the idea one forms about it, is definitely no more thought as «substance»[2], and can coexist with a «relativity of ontology», as we shall discuss soon.

THEORY, SIGNIFICATION AND PROBLEMS OF ONTOLOGY

In his book *Conceptual Foundations of Physical Thought*, Marx Wartofsky, penetrantly and not so commonly, if we consider today empiricist claims in philosophy of science, insisted on the *theoretical dimension of science*, particularly in physics, that permits to overcome the limitations inherent to conceptions of mere deductive or covering-law model, with respect to *semantic meaningfulness*. It is so because *theory is not closed* in propositional language, observational observation or measurement statements refered to measurement, or theoretical terms reducible to empirical ones, in a pure phenomenalist way, *but is aimed at objects*, or things, to which we refer the properties under study[3].

If we were not ready to accept this, we would have to change our notion of a physical universe to a universe made up of sensible data (in the line of George Berkeley, Stuart Mill, Ernst Mach and, I would say, Niels Bohr), and to change knowledge into a pure pragmatic enterprise. It happens that *theory* explains laws in an upper understanding level, above simple models and laws, and it has to be considered at its proper relevant level, where it «[carries as well] its interpretation with it», a trait that mere law is unable to exhibit[4]. Jean Largeault wrote in his own way, about physics, in a converging direction, that «the task of theories is (…) to determine what facts let undeterminate»[5].

But then we get into problems, because the ontology that is (or was) usually associated with theories about things appears to be not so simple. First,

---

2 On this, see, for instance, Cassirer [1910].

3 Wartofsky [1968], p. 276-287.

4 *Ibid*., p. 282. Marx Wartofsky, in this book, takes as equivalent the expressions «model, conceptual framework and theory», in the sense that «all observation and measurempent is theoretical or within the framework of some theoretical model» (*ibid*., p. 283). He considers as possible, compatible to this view, two conceptions, the realist and the constructivit ones, each of them raising serious philosophical questions, the second being merely instrumental and the first having to deal with the criteria of truth (p. 286-287).

5 Largeault [1984], p. 155.

and generally speaking, we have to face the philosophical question on ontology raised from our systems of language and leading to what Willard V. Quine calls the «relativity of ontology» (circularity makes ontological questions meaningless and the choice of an ontology can only be pragmatic)[6]. «Relativity of ontology» appears also, and is akin to some degree of conventionalism, when we inquire, with respect to a scientific theory, into the foundations of our systems of concepts and the roots of our basic notions of reference taken as given data or as provisional evidences. Note that this is not so new, three centuries after Blaise Pascal's considerations, in his essay «De l'esprit géométrique» and in his «Pensée» on the disproportion of man in the universe, trying endlessly to get into the reasons of the reasons, in a regressive analysis of our basic notions[7].

As for him, Marx Wartofsky used to speak of the «historicity of epistemologies and of ontologies» and sketched, in a seminar given in Paris in 1994[8], the «three stages of the historical constitution of the scientific object ». He pointed, for the classical age of XVIIth century (a part of the second stage, after Greek culture), a conception of the «scientific object defined in space and time and totally accessible to measure», in which «the formalism is congruent with magnitudes»[9]. The realist epistemology, inspired from the astronomy and the physics of the time and their objects adequate to it, was not a philosophical choice, but was implicated by scientific practice. In opposition to this conception, comes that of the third stage, with the Einstein-Bohr debate on quantum mechanics, which exhibits two possible ontologies : the «realist» and the «constructivist» ones.

The constructivist (I would say the *operational constructivist*, Bohr) admits classical realism for daily life and chooses an ontological discontinuity for scientific knowledge, developing his complementarity conception. The realist (I would say the *critical realist*, Einstein), conceives a change for the construction of the classical as well as the scientific object with a continuity for the ontologic criteria. Marx Wartofsky considered contemporary physics as being in a constructivist stage, but this might change with time, for there was, according to him, a conceptual continuity in the debate between realism and constructivism, and one could not say that one of these epistemologies is wrong or right. He noticed, however, that the conventions are tested by experiment, which makes some difference with strict constructivism. As for him, he considered himself as

---

6 Quine [1969]. Jean Largeault (who translated Quine's book into french), understood Quine's «relativity of ontology» as a «relativity of the points of view». He made pertinent observations on the «epistemic relativism» and the conventionalism of Poincaré and Quine as opposed to an «ontologic relativism» that ignores the difference between hypothetical convention and fact of nature (Largeault [1984], 151-156).

7 Pascal [1657] and *Pensées* («Disproportion of man»), *in* Pascal [1993], p. 527.

8 Seminar given (in french) to the REHSEIS research group of epistemology and history of science, of which he was a member during his sabbatical year 1993-1994, and where I had the pleasure to welcome him for the second time (the first one was in 1996-1997). He gave his lecture on may the 10th, 1994 (the following quotations are from my personal notes).

9 «The scientific object is object in space and the time, totally accessible to measure. The formalism is congruent with magnitudes».

sometimes a realist with constructivist tendency, and the reverse.

Nevertheless, is not critical realism a position of this kind ? With the reserve that we have to avoid ambiguities when using the word «constructivism». I mean by *critical realism* a position that includes *symbolic constructions* in the representations of *reality*, and that is conceived as a *programme* for the theoretical constructions of physics.

Having thus settled the general philosophical and metaphysical background of the scenery, let us come back to the problem of «ontology» considered from the point of view of a *constructed* or *elaborated* science.

What we know in mathematics, and in physics as well, said Henri Poincaré, are relations («*des rapports*»)[10]. This is due to the fact that physics deals with *concepts* (symbolic and mental entities) that are expressed by *magnitudes* in the form of *quantities*. And, as if to complete Poincaré's statement, Einstein said that realism in science is a program, in the following sense : we admit that physical theory is aimed at describing or representing objects that are supposed *to exist independently* of our possibilities of observation and measurement (otherwise necessary to compare our representation with phenomena). This predicate of existence referred to an external world is only an assumption, indeed a very general one[11]. If this is ontology, it is meant in a broad acception of the word : it is, in a sense, an ontology of *constructed relations aimed at* something real, not of a real considered as such («in itself»)[12]. Furthermore, nothing compels us to it : we are free to choose it or not. But in both cases, we have to be consistent in our representation of things or phenomena.

It is the problem of this consistency that I want to explore for physics. Instead of asking questions of ontology, and of reducing to these the problems of signification or meaning of *scientific statements*, I would like to inquire directly the nature of these statements in the case of physics, in order to know whether they correspond to *things* and *states of things* in the above sense. That is, entangled in a consistent and, so to speak, organic way inside the theoretical scheme, without introducing, *in their definition as things*, restrictions that refer to conditions external to them (asking only, with respect to observation and experiment, an *a posteriori* agreement between theory and observation).

This leads us to two major problems that are specific of physics as a science. The first is about the reason of the privilege given to *mathematization* in the process of conceptualization and theory making in physics. And the second can be formulated as : what is meant, in this perspective, by «*physical magnitudes expressed mathematically*».

These two problems are, actually, one and the same, as we shall see from an inspection and a meditation on magnitudes in physics, their properties

---

10 Poincaré [1905], chapters 10 and 11.

11 Einstein [1949], p. 674-675 ; Paty [1993], p.474-478.

12 Paty [1988], in particular chapters 1 and 10. On ontology, see also Largeault [1984], p. 142-150, commenting René Thom's assertion that true knowledge is about being, not about the subject. (The referred article by R. Thom, «Le problème des ontologies régionales en science» (1982), can be found in Thom [1990]).

and meaning.

2.

PHYSICS : CONCEPTUALIZATION AND MATHEMATIZATION.

MATHEMATIZATION AND STRUCTURATION OF PHYSICAL THEORIES

Tight relationship with mathematics (or with mathematization) and with quantitative observation and experiment makes the specificity of physics among the sciences. Already notable at the beginning of modern science, it ensured physics an enduring leadership on the other branches of knowledge, for physics was considered as a model for scientific rationality. In physics, phenomena are represented through concepts that are expressed in the form of magnitudes or quantities, endowed with exact definition in a mathematical way. The relations of physical concepts (for instance, distance and space coordinates, duration and time, force, etc.) are relations between these magnitudes, that take generally the form of equations or of quantitative propositions such as principles (of inertia, of relativity, or conservation principles, etc.). Equations are the mathematical expression of laws (laws of motion, laws of nature …) and the principles, formulated as general, ascertained properties of physical phenomena, provide the condition to express mathematically magnitudes and their relations.

This picture of what, esssentially, a physical theory is made of can be traced back to the beginning of mechanics, and is still adequate to describe the physics of present days. Let us now inquire further on what these quantities are, or better, on what is usually understood when we speak of, or deal with *physical magnitudes or quantities*.

WHAT IS UNDERSTOOD BY «PHYSICAL MAGNITUDES» OR «QUANTITIES» ?

Our way to understand what is meant by *physical quantity* is tributary to the shifts entailed by the evolution of physics through our use of this concept. Since XIXth century, the importance taken by experimental means, activity, and practice, has enhanced the weight given to the possibility of measuring with exactness, this term of «exactness» being understood as synonymous of numerical precision. After all, experiments end with numbers, and so should be, according to the usual views, the meaning given to the concept of magnitude or quantity : they ought to be endowed *with numerical values*. Such a tendency has increased since XVIIIth century, when powerful methods of approximations were developed in astronomy (perturbations calculated through expansions in series in the three-

body problem[13]), and above all in XIXth century when all daily phenomena of optics, electricity, magnetism, warmth, chemistry, were either assimilated by physical theories or submitted to systematic quantitative study[14].

Let us add to this the construction of high precision scientific instruments and the general context of industrialization. And consider also that the requirement of numerical precision was reinforced and justified rationally and theoretically with the elaboration of a theory of errors, related with the mathematical theory of probability and to the so-called laplacean determinism (although the word «determinism» did not yet exist when Pierre-Simon Laplace gave an effective definition of it in the now classical statement of his *Essay on the philosophy of probability*[15]) : the «mot d'ordre» by then could be formulated as : exactness and probability, overcoming their effective duality (probability being understood in the «frequencial» or «subjective» meaning of deriving knowledge from an uncomplete set of data).

Quantity is understood as a «*measure*», continuous or discontinuous. The elaboration of physics, starting from mechanics with the scientific revolution of XVIIth century, has since gone along by dealing with quantities conceived according to «measure». «Order and measure», Descartes said, but «*measure*» had in his expression the old meaning of being subject to proportions, and not that of «*measurement*», as «measure» would be generally understood afterwards, corresponding with more or less direct experiment. This further acception would restrict the meaning with which physical quantities were to be most often considered, and this meaning was : quantities as taking definite *numerical values*, revealed by measurement. Most physical magnitudes - if not all, as many have thought it for a long time - are indeed of this type : space coordinates and distances, time and duration, velocity, acceleration, force, mass, energy, electric charge, electromagnetic or gravitational field as defined in space and time, etc. These concepts are represented by continuous quantities, with the help of diffential and integral calculus, and these quantities can be put in relation with some measurement device that determines their numerical values as a function of other quantities taken as varying parameters.

And so physics was standing, and the concept of physical magnitude, when quanta came. Physical magnitudes, endowed with numerical values, could only be, according to the usual conception, those that can be directly measured. But on this we shall come at the end. For now, we face the following situation : physical theories are mathematized, and this happens through their use of mathematically expressed quantities. But this is merely a description of what they are, not a justification. We have to go further in our inquiry about quantities and mathematization.

---

13 See the astronomical works of Clairaut, Euler, d'Alembert, Laplace, etc.

14 See, particularly, the works of Fresnel, Ampère, Faraday, Regnault, Joule, Fizeau, Mascart, etc.

15 Laplace [1814], as an Introduction to his previously composed *Analytical theory of probability* (Laplace [1812]), p. vi-vii in Laplace's *Complete work* edition.

HISTORICAL ELABORATIONS : FROM QUALITIES TO QUANTITIES

Let us go directly to the essential. For this, we must not forget from where our concepts and their meaning come, because their structuration in the present is made from the flesh of the past. (Precisely, if there is no obvious ontology of the things we deal with, the rescue might be to know how their tissue has been woven).

The idea of *magnitude*, already in Antiquity and in Middle Age, contained a conceptual meaning (it was anciently, *quality*) in association, eventually, with numerical values (intensities or degrees of a quality). For Aristotle, for instance, time was the number of motion. But the concept of motion itself was a complex one, implying power and continuous cause, and the concept of velocity stayed with an ontologic, qualitative, meaning, that was related with difference of nature between motion and rest[16].

A slow shift occurred during Middle Age, as it is known, from qualities to quantities, in XIVth centuty, with the scholastic masters of the Universities of Oxford and Paris (Robert Gosseteste, William of Ockham, Jean Buridan, Nicole Oresme…), through the study of the variation of intensity of the «quality of motion», or *velocity*, with time, and the invention of the concept of «*impetus*», a dynamical impulsion conceived as an internal action transferred to the body in motion. These were important steps towards geometrization and mathematization of motion[17].

History, here, teaches another thing : the necessity to get first the right physical principles in order to be able to perform mathematization. Such is one of Galileo Galilei's lessons : with his totally new conception of «*impeto*», that was no more the *cause*, but the *effect* of motion, he did put on the forefront two essential ideas : the conservation of motion and the law of inertia[18]. Galileo's quantification of motion corresponded to an effective vanishing of quality, motion being set on the same ontological status as rest. Motion or velocity did not affect the properties of bodies. Motions of various kinds could therefore be unified, velocities (or, actually, quantities of motion, or impulsions) could be composed and the change of motion in the free fall of bodies could be studied «quantitatively», that is to say through magnitudes or quantities. The last step of physical theory construction was the choice of the good quantity (or concept) to study the laws of motion. *Time* was this concept and entered in physics as a fundamental variable[19].

The shift from qualities to quantities was decisive in the making of physical theory, by which laws were formulated as equations between the quantities carrying the conceptual contents.

---

16 Aristotle [Phys]. Cf. Clagett [1959]. On the evolution of the concepts of time, space, velocity, relativity, vacuum, see respectively Paty [1994b, 1998c, 1997c, 1999c, 1998b]. More generally, on the changes in the relations between philosophy and physics, cf. Paty [1998a].

17 See, in particular, Duhem [1913-1959], Crombie [1952], Clagett [1959].

18 Koyré [1935-1939]. See also Clavelin [1968], Drake [1970].

19 Paty [1994b].

Of the subsequent history of the construction of physics through mathematization, we shall only mention another decisive step, the construction of «*instantaneous time*» from the notion of time conceived as duration (continuous flow and quantity), suited to formulate the «law of causality» of newtonian dynamics. This invention (to let all details aside) was correlated with the new calculus (of fluxions for Newton, differential and integral for Leibniz) although Newton meant to stay with «synthetic geometry» in his elaboration of dynamics. But his geometry of limits defined and used in the *Principia* is equivalent to his calculus of fluxions[20]. What interests us, at this stage, is the rise of a new kind of magnitude, that would be explicited later on : continuous quantities conceptualized through differential and integral calculus, i.e. analysis, the «new analysis».

The impulse was thereof given. Physics would be build afterwards, in all its branches, through analytization, with the differential conception of space, time and other required quantities.

THE JUSTIFICATIONS FOR THE USE OF MATHEMATIZATION IN PHYSICS.

To justify the mathematical character of magnitudes and laws in physics, Galileo invoked the idea that the «Book of Nature» is written in the language of figures and numbers. «Its type letters», he wrote, speaking of the Universe, «are triangles, circles, and other geometrical figures, without which it would be impossible to a human being to understand a single world of it». And he added that all properties of external bodies in nature can be attributed, in ultimate analysis, to the notions of «magnitudes, figures, numbers, and slow or fast, and those have effects on our sensorial perceptions, and are, so to speak, the true essence of the things»[21].

As to Isaac Newton, he expressed, in his *Principia*, the laws of mechanics and of gravitation in a geometrical way, giving effect to the intention claimed right from the title of his book, *The mathematical principles of natural philosophy*[22]. These «mathematical principles» were, actually, more related to a general conception of geometry («synthetic geometry» called for in the Preface[23]) rather than to the analytical one, although his «geometry of limits» (of the «first and last reasons of quantities»[24]), through which he formulated the problems of mechanics and astronomy and got his results, was conceptually equivalent to the fluxion calculus he had elaborated in mathematics. This difference might be related to his conception of the mathematization of mechanics and of the laws of physics.

---

20 Paty [1994a].

21 Galileo, in *Il Saggiatore* (Galileo [1623]).

22 Newton [1687]. See Whiteside [1970].

23 Newton [1687], Newtons's Preface to the first edition.

24 Newton [1687], Book 1, Section 1, Cajori ed., p. 29-39.

Newton's use of mathematics in mechanics was justified by him from his neo-platonician conception of the physical world that was going along with his «absolute, true and mathematical concepts» such as space, time, motion, force, etc.[25]

But physics afterwards, although it was based on newtonian dynamics, meant differently the legitimacy of being mathematized, and this divergence can be seen already in the works of XVIIIth century «Geometers» such as Leonhard Euler, Alexis Clairaut and Jean le Rond d'Alembert (and later on, Joseph-Louis Lagrange, Pierre-Simon Laplace and others). Despite their inheritance of Newton's achievements, they understood the meaning and use of mathematical quantities for physics differently from him, in a way that was more neutral to metaphysics.

Take, for instance, d'Alembert's justification of «analysis» in his works on dynamics or on hydrodynamics (or astronomy as well) : analysis was inherent to his thinking of mechanical concepts. He thought dynamics from the start through the basic concepts of motion and the corresponding magnitudes (space, time, velocity, impulsion, acceleration, …) as conceived and expressed with the use of differential calculus, giving a formulation of the three principles of dynamics (inertia, composition of motions, equilibrium) in such terms. He was able in this way, from his expression of the second principle of motion, to add directly to a given velocity the differential of another one, getting Newton's «second law» (of accelerated force) as a corollary. He gave an original formulation also of the third «principle» of motion (that of «equilibrium», equivalent to Newton's third law, of «action and reaction», but in terms of destroyed or compensated motions), and obtained as a neat result his famous general («d'Alembert's») «principle», actually a powerful «theorem» of dynamics, directly demonstrated[26]. Lagrange's systematic algebraic construction of «Analytical mechanics» would rest on the same type of direct justification, due to the conceptual mathematical definitions of physical magnitudes. On the same «conceptual mathematical» basis were mathematical physics, and henceafter theoretical physics, elaborated[27].

About the mathematization of hydrodynamics (whose theoretization he was the first to perform through the invention and use of the partial differential equations calculus[28]), d'Alembert stated very clearly the conditions that were required in the introduction to his *Essay of a new theory of the resistence of fluids* : «The sciences called physico-mathematical (…) consist of the application of calculus to the phenomena of nature. (…) The invention of differential and integral calculus has allowed us to follow in some way the motion of bodies up to their elements or ultimate particles». Then, d'Alembert wrote (I would like to put emphasis on it) : «It is *only with the help of these calculations* that we can

---

25 Newton [1687], Scholium to the Definitions, Cajori ed., p. 6.

26 D'Alembert [1743]. See Paty [in press, a].

27 Paty [1994a].

28 D'Alembert [1747, 1749-1752]. See Grimberg [1998].

penetrate inside Fluids, and discover the play of their parts, the actions that mutually exert, the ones on the others, these inumerous atoms of which a Fluid is composed, and that appear at the same time united and divided, dependent and independent the ones from the others». It is so because «the inner mechanism of Fluids is *so poorly analogous* to that of solid bodies which we touch, and follows *laws that are so much different* (…)»[29].

Interestingly enough, d'Alembert stated in the above quotation the necessity of theory, of a physical and mathematized theory, as the *only way* to get knowledge of such bodies, definitely substituting imagination and arbitrary hypothese.

In the same writing, d'Alembert explained how it is only when a physical principle about the object (or type of phenomena) under consideration has been obtained that mathematization is possible : «I thought what I needed was to look for these principles and the manner in which I had to apply the calculus, if possible». Mathematization is governed by the type of physical properties that are considered : such a requirement is post-newtonian and, in a way, un-cartesian.

It is nevertheless fundamentally to René Descartes that d'Alembert gave the credit of having made possible, as a matter of principle, the mathematization of physics. Descartes's invention of algebraic geometry, which d'Alembert used to call «application of algebra to geometry» and which he qualified as «an idea among the widest and the happiest that human spirit has ever had»[30], will, as the geometer-encyclopedist wrote in his *Preliminary Discourse to the Encyclopédie*, «always be the key of the deepest researches, not only in the *Sublime Geometry* (*Géométrie Sublime*) [i.e. *Analysis* in the sense of differential and integral calculus], but in all physico-mathematical sciences»[31]. This remark points at the most fundamental reason of the mathematization of physics, with the mathematical treatment of physical concepts expressed with the form of *continuous quantities*.

The new conception of the mathematization of physics that has been shared by most scientists since XVIIIth century up to the present times, and that has been formulated in the clearest way by d'Alembert, is somewhat at variance with Descartes' identification of physics with geometry[32], for it was admitted, with Newton, that bodies, even when considered only under their «essential properties», are not reducible to mere spacial «extension». Besides extension, bodies have properties such as impenetrability and attraction that are not reducible to it, and mechanics (and, more generally, physics) differ from geometry and mathematics in that it deals with variations in time : mechanics, said again

---

29 D'Alembert [1749-1752], Introduction to the book published in 1752, p. vii-xliii (my emphasis, M.P.).

30 D'Alembert, article «Application …» of the *Encyclopédie* (Diderot & d'Alembert [1751-1780], vol. 1, published in 1751), and *Eclaircissements à l'Essai sur les Eléments de philosophie*, in the 1987 ed of d'Alembert [1758], chapter 13.

31 D'Alembert [1751], éd. 1965, p. 94.

32 Descartes [1637, 1644].

d'Alembert (and Lagrange after him), expresses as geometry varied with time[33]. The new views on mathematization can notwithstanding be traced back to Descartes' conception of magnitudes, as he developed and analyzed them from the notion of dimension in his *Rules for the direction of mind* (*Regulæ ad directionem ingenii*) written around 1628 (in particular, rule 14)[34].

3.

INTELLIGIBILITY REFERRED TO ORDER AND MEASURE. ON MAGNITUDES OR QUANTITIES

DESCARTES' «MATHESIS UNIVERSALIS». THE TWO FUNCTIONS OF MATHEMATICS IN THE USE OF REASON

Beyond the extension he had given to mathematics in their unifying methods and in their operations, Descartes conceived «his mathesis universalis» as having an even more general dimension, that revealed the faculties of intelligence itself, able to be manifested in other areas of knowledge and even in metaphysics. In particular, it was suited to the knowledge of the real physical world, by making use of mathematics in various domains of physics. Descartes 's *Rules for the direction of the mind* and, afterwards, the *Discourse on Method*, claim a twofold function of mathematics in the exercise of reason[35]. Firstly, they serve as a model and as a guarantee for certainty in the linking of propositions. Rule 1 states that the power of *mathesis universalis* can be oriented towards the formation or the acquisition, by the mind, of the ability to form «firm and true judgements on everything that is presented to him»[36]. Secondly, they rule the expression of magnitudes by which we represent the world. As to this second function, considering the natural sciences concerned by mathematics such as astronomy, music, optics, mechanics, and eventually others, Descartes saw clearly, he said, that we have to «bring to mathematics everything in which we examine *order and measure*» («*l'ordre et la mesure*»), without specifying the particular object of this measure[37].

Through his considerations, Descartes did not so much intend to elaborate a physics, or a mechanics, from mathematics, which he, actually, never fully tried or achieved, than to think the intelligibility of the objects of these sciences. He would, indeed, perform some mathematical approach of mechanics

---

33 D'Alembert [1743] Première partie, [1759], chapitre 16, p. 367-402, and article «Dimension» of the *Encyclopédie* in Diderot & d'Alembert [1751-1780], vol. 4 (1754). See Paty [1977, 1998c].

34 Descartes [1628]. See Paty [1997, 1998a].

35 Descartes [1628, 1637].

36 Descartes [1628].

37 Descartes [1628]. See the comment to Rule 4.

and of optics, insofar as they are related to quantities ; and he would further propose to geometrize physics as a consequence of his identification of matter and spatial extension[38]. But the fondamental claim with respect to physics that is to be found in his *Rules*, and also in his further works up to the *Principles of Philosophy*, was about the need and necessity of laws[39].

ORDER AND MEASURE. QUANTITIES, PLURALITIES, RELATIONS

Actually, to Descartes, physics was a science of magnitudes («*grandeurs*») that are subject to proportions, and its mathematization in principle through laws was immediately justified, under the sign of the exigency for intelligilility, related to *mathesis universalis*. In this sense, it was primarily the very road to knowledge that led to the mathematization of magnitudes concerning the real world, at variance with the neo-platonician reasons invoked by Newton (the «true and mathematical»world as opposed to the «apparent and common» or «sensible» one).

In Rule 14, Descartes defined *magnitude in general*, relative to any object, by making use of the concept of *dimension* considered as spacial extension of geometry, taken as the archetype of any magnitude at reach of order and measure. «The dimension is the real extension of the body, when we make abstraction of anything except the figure…», he stated. The relation between magnitudes (allowing to know one from another) is at the same time the expression of their ontological identity.

Descartes performed in his text a conceptual analysis of the aspects of extension that are related to differences of proportions, and identified them as *dimension*, *unit*, and *figure*. *Dimension* is «the mode and manner by which we consider a subject as measurable», and this concerns not only the three spatial dimensions, but other magnitudes as weight, velocity, etc. *Measure* is refered to division in equal parts (it can be only an intellectual division), and is the reverse of *counting* : «If we consider the parts in relation with the whole, we say that we count ; if, on the contrary, we consider the whole as divised into parts, we measure it»[40]. And it is the task of the physicist (and not of the mathematician) to examine the well-foundedness in reality of «dimension» understood in this sense, that is, *magnitude*.

As for *unit*, it is the common nature of the things that are compared. And *figures* are of two types : pluralities and magnitudes. *Pluralities* are, for example, points or any element ordered in space, whereas *magnitudes* (or quantities) are continuous and indivisible (like the area of a triangle, or a square). All the relations than can exist between figures of the same kind «must be referred to two essentiel points, that are *order and measure*».

---

38 Descartes [1637, 1644]. See Paty [1997].

39 See Koyré [1965].

40 Descartes [1628], Rule 14.

Descartes made at that point a consideration about possible simplifications of problems involving continuous magnitudes : these «can, thanks to a borrowed unit, be sometimes brought completely to a plurality, and always at least in part. The plurality of units can afterwards be disposed in an order such that *the difficulty, relative to the knowledge of measure, depends finally of the order only*»[41].

CONSONANCES AND PREFIGURATIONS

It is tempting to see in these reflections a consonance with the analysis, that was to be performed two centuries after by Bernhard Riemann, of the mathematical concept of *multiplicity* or *manyfold*, of which the three spatial dimensions are a particular case, and even some prefiguration of the idea of *topology* («*analysis situs*», indicated by Leibniz and Euler, and first developed by Riemann). We might think of a correspondence between the following couples : order / measure, plurality / magnitude and, in further terms, topology / metrics.

One might also see, in the last quoted Descartes' sentence, an insight into a difficulty of principle to be met in the analysis of continuous quantities that would be explicit when they would be treated by differential equations, such as the impossibility to solve these equations, despite their full adequation to describe a physical situation : one would then have to leave up the exact (metrical) calculations and consider «qualitative», «structural», or topological features, as Henri Poincaré would do far later.

Let us conclude this evocation of Descartes's founding conception of magnitudes by stating that the quantitative aspect of magnitudes, that make them «measurable», must not be understood in the restricted sense of numerical determinations only, to which it has often been reduced. What was important, for Descartes, was the relation into which the magnitude is expressed, that is its form (for example in an algebraic relation). «Measure» meant, to Descartes, the *relational aspect* of magnitudes or quantities.

We can, today take profit of this lesson : the conceptual content of a magnitude, even a mathematized one, does not vanish when it is attributed a value with a number, and remains given in the relation that determines it. Physical magnitudes conceived through a mathematical form, and aimed at describing or representing objects and phenomena of the physical world, will have exactly the relations as their mathematical forms have[42]. Therefore, the system of physical concepts is thread by the mathematization of the magnitudes expressing these concepts.

This justification of the use of mathematics in physics is self-consistent and does not refer to any other philosophical claim than intelligibility

---

41 Descartes [1628], Rule 14 (my emphasis, M.P.).

42 See, for instance, the analysis of the physical meaning of Lorentz' transformation formulas as demonstrated in Einstein's 1905 work on special relativity, with its consequences on the new concepts of space and time : Paty [1993], chapter 4.

through «order and measure». It was to be considered later on, but uncessarily in my view, as being related to Descartes' further essential identification of matter with spacial extension, and of physics with geometry, as exposed in his *Principles of philosophy*[43]. Repudiating this identification, Newton believed he could found mathematization of physics (that was still to be a geometrization) on his neo-platonistic view of the world, refering to «absolute, true and mathematical» quantities of the real world as opposed to «relative, apparent and common» ones, the latter being «not the quantities themselves, whose name they bear, but those sensible measures of them (either accurate or inaccurate), which are commonly used instead of the measured quantities themselves»[44],.

However, the «continental tradition» of newtonian mechanics that was to determine the ways of mathematical and theoretical physics of XVIIIth and XIXth centuries (that is, of classical physics) would justify mathematization much more in the line of a cartesian conception of intelligibility and magnitudes (actually modified to take into account the effects, on the acquisition of knowlege, of sensorial data initially in a lockean way) than in Newton's one[45]. Indeed this «tradition», that started after Christiaan Huygens and Gottfried Wilhelm Leibniz with the cartesian disciples of the latter and the malebranchists (cartesian) circles (Jacob and Johann Bernoullis, Michel de l'Hospital, Pierre Varignon, etc.)[46], adopted and developed the leibnizian differential and integral calculus and formulated in its terms (those of the «new analysis») the problems opened in the lines of Newton's *Principa*. This «graft» (or synthesis ?) of newtonian physics by leibnizian formalism thus occurred in a cartesian philosophical ground, and would blossom with the outstanding works of Euler, Clairaut and d'Alembert, then of Lagrange, Laplace and others[47]. That is why the cartesian conception of intelligibility and correlative justification of the mathematization of physical magnitudes has henceforth, at least implicitly, underlined the developements of theoretical physics up to our times[48].

The central idea of this justification, rather neutral with respect to metaphysics, can be followed with further philosophical or mathematical specifications through authors such as d'Alembert, Kant, Ampère, Riemann, Poincaré, Hermann Weyl, Einstein….

THE KANTIAN METAPHYSICS OF MAGNITUDES

---

43 Descartes [1644].

44 Newton [1687], Scholium of Definitions, Cajori's ed., p. 6, 11.

45 Paty [1977, 1994a, in press, a].

46 See, for instance, the edition of Johann Bernoulli's works, Bernoulli [1989-1991] ; and for a more general outlook, Blay [1992]. On Leibniz' calculus, cf. Leibniz [1849-1863, 1989].

47 See, in particular, on Clairaut : Greenberg [1995], Passeron [1994] ; on d'Alembert : Paty [1977], Emery & Monzani [1989], Grimberg [1998] ; on Lagrange : Martin-Viot [1994] ; on Laplace : Merleau-Ponty [1986].

48 On the contemporaneous conceptions, see Paty [1986, 1988, 1993].

The most significant attempt, from a properly philosophical point of view, after that of Descartes, to base the intelligibility of the tangible world of physics on understanding, remains that of Immanuel Kant in his *Critique of the pure reason*. After the «Transcendental æsthetics» («science of all the principles of *a priori* sensitivity»), conditioning knowledge as formed by the understanding and the apprehension of phenomena[49]), comes the «Transcendental analytic», that is the «decomposition of all our *a priori* knowledge in elements of the pure knowledge of the understanding»[50]. The synthetic principles of pure understanding include those deal essentially with the idea of magnitude and with the possibility to apply mathematics to phenomena. To these principles are added those of the «Analogies of experiment» and of the «Postulates of the empirical thought in general». The principle of the «Axioms of intuition» defines *magnitudes*, *extensive* as well as *intensive*, by stating that «all intuitions are intensive magnitudes». That of the «Anticipation of perception» based on the idea that «in all phenomena, the real, that is an object of the sensation, has a magnitude that is a degree», allows to constitute in the transcendental subject the condition of the apprehension of continuous magnitudes, extensive as well as intensive. The variation in the degrees of continuous magnitudes was directly inspired by the thought of the derivated and differential magnitudes of analysis and of newtonian physics[51].

Extensive magnitudes are such that the representation of the parts makes possible that of the whole, and they are formed therefore on the representation of spatial distances. *Intensive magnitudes* are relative to the degree of the caused sensation. To conceive them, Kant imagined a gradual change of empirical consciousness into pure consciousness by a progressive and continuous diminution of sensation, in such a way that the real would disappear completely and «it would remain only a purely formal (*a priori*) consciousness of the *various in space and time*».

It was consequently possible to apply mathematics to natural phenomena. Actually, both kantian *principles*, of the *axioms of intuition* and of the *anticipations of perception*, in his words, «are related to phenomena according to their simple possibility, and teach us how these phenomena can be produced, following the rules of a mathematical synthesis, according as well to their intuition as to the real of their perception. One can therefore use, in one as in the other, numerical magnitudes and, with them, the determination of the phenomenon as a magnitude»[52].

THE RIEMANIAN ANALYSIS OF MULTIPLICITIES

---

49 Kant [1781-1787], fr. transl., p. 781-811.

50 Kant [1781-1787], fr. transl., p.

51 *Ibid.*, p. 902-914.

52 *Ibid.*, p. 916.

It is appropriate, this time from a more precisely conceptual point of view, to make a special mention of Riemann's 1854 *Dissertation* «On the hypothese that serve as foundations to Geometry»[53]. Riemann studied systematically in it the properties that can be formulated mathematically for a continuous *variety* - or magnitude -, of any kind, with a number *n* of dimensions, and their eventual relationship with physical magnitudes. These properties are either topological either metrical, and Riemann proposed to establish a direct connection between the metrical relationships of the three dimensional space and the properties of physical bodies. By doing this, as one knows, he prepared, although being unaware of it, the mathematical framework of a geometrized physical theory - of gravitation - that the general theory of relativity was to be.

In his study, Riemann faced first the problem to construct, with the general concept of magnitude as a starting point, the concept of a magnitude with multiple dimensions. Such magnitudes are conceived according to quantity, and the comparison of their parts is undertaken «for discrete magnitudes, by means of counting, for continuous magnitudes, by means of measuring. (…)»[54]. This expression reminds us of Descartes' statement quoted earlier on «measuring and counting»[55].

Among the diversity of possible cases, Riemann considered in particular that of an absence of measure, whose researches, he commented, «form a general branch of the theory of magnitudes, independent of metrical determinations, and in which they [magnitudes] are not considered as existing independently of the position, neither as expressible by means of a unit, but as regions in a variety».

By characterizing the difference between topology and metrics, Riemann derived from it a consequence, for space, that would be of a fundamental importance : one must distinguish, for «uncommensurably large» spaces, between the «unlimited» (what has no limits), that belongs to extension (topological) relationships, and the «infinity», that belongs to metrical relationships. The first property is general and qualitative, so to speak, while the second one depends on the metrics. (And we know that a metrics postulated as euclidian had led to identify them).

With respect to extended magnitudes, Riemann established a distinction between their properties within a general (purely mathematical) theory as the one he considered, and their physical determinations. With the first, «one supposes nothing more than what is already contained in the concept of such magnitudes», while the second corresponds to properties of the physical universe, that are not given by the first one. In other words, the metrics of space is not given *a priori* and will be provided by physics, and Euclide's postulate has, actually, an empirical origin ; it is relative to extension in conformity with men' experience, to the «empirical concepts on which the metrical determinations of extension are

53 Riemann [1854]. Cf Paty [1993], chapter 7.

54 Riemann [1854].

55 Descartes [1928], Rule 14. See above.

based», namely, «the concept of solid body and that of light ray»[56]. Now, these latter «cease to subsist in the infinitely small». «It is therefore quite legitimate», Riemann goes on, «to suppose that metrical relationships of space in the infinitely small are no more adequate to the hypotheses of geometry, and this is what it would effectively be necessary to admit, once one would obtain from there a simpler explanation of phenomena. The question of the validity of the hypotheses of geometry in the infinitely small is linked to the question of the intimate principle of metrical relationships in space[57].

The conceptual clarifications possibilited by Riemann's general theory of magnitudes would be effective, and respond to his wish of preventing thought to remain hindered «by too narrow views» and «progress in the knowledge of the mutual dependence of things to find an obstacle in traditional prejudices». These effects would be felt in mathematics as well as in physics. Besides the possibility of non euclidian geometries and the physical character of space metrics, such a reflection opened another perspective for mathematical as well as for physical study of space, alternative to the metrical approach : the approach of topology, that could be, in some conditions, more «explanatory» than the first one. As an effect, it opened also at the same time the way of qualitative study for the solutions of differential equations systems and of the corresponding physical phenomena, where the *«structural» characteristics* of the relationships, and the associated *types of physical behavior*, appear more significant than «exact» (in the sense of quantitative, numerical) determinations of the particular magnitudes. It would appear to be so with Poincaré's pioneer works on the three body problem and on the properties of dynamical systems, whose further inheritance consitutes today an important part of physics.

Let us mention, on the other hand, that contemporary researches on problems such as that of quantum gravitation let increasingly conceivable that topological properties of magnitudes of any dimensions appear also as a conceptual tool that could be indispensable for the physics to come (quantum gravitation, etc.).

4

TYPES OF MAGNITUDES FOR PHYSICS AND THE «QUALITATIVE» OF THE QUANTITATIVE (ORDER IN RELATION)

PHYSICAL MAGNITUDES, CONCEPTUAL ABSTRACTIONS AND THE DEVELOPMENT OF MATHEMATICS

56 Riemann [1854].

57 Riemann [1854].

The mathematization of physical magnitudes took mainly the form of differential and integral calculus that, in its origin, had allowed to surpass the antinomy between continuous magnitudes taken at a point and a singular instant. Partial derivative equations, mathematically developed in conjunction with their utilization in the elaboration of fluids mechanics (by the pioneer works of d'Alembert[58], followed by those of Euler), became the «language» of the physics of continuous media and of fields (first thought with the support of a material medium such as ether, caloric, etc.). These mathematical forms became the indispensable way of physical thought that could henceforth spread its domain and its objects.

To classical magnitudes such as those of spacial coordinates, time, speed, mass, force, moment of inertia, work, energy, etc., others were added, as potential, electrical charge, field defined in space and time with finite velocity propagation (replacing newtonian instantaneous «action at a distance», but that it was difficult to think independently from an ether) and, later on, others, more «abstract», that we shall not evoke at length (see, for instance, «quantum numbers», «spin», etc.). These magnitudes require, from their very definition, various mathematical forms, besides numbers, functions and the differential forms already mentionned. Depending on the needs or conveniences of their relationships between them, these magnitudes can take the mathematical form of complex numbers, vectors, tensors, matrices, spinors, functions with integrable squared defined in Hilbert spaces and linear operators acting on these functions - socket of quantum mechanics -, distributions, etc.

The thought of magnitudes has enriched with all the development of mathematics, and physics has been fed with it, integrating to the expression of its concepts the new mathematical objects and theories and the associated calculation methods, whose invention it sometimes contributed to raise. It deals, in the expression of its laws, with magnitudes of various types, increasingly abstract as to their form and far away from that, intuitive and generative, of simple spacial dimension.

The concept of *entropy*, introduced by Rudolf Clausius, was one of such abstract entities that appear at first sight more mathematical than physical in an intuitive sense. Rather than a quantity directly interpreted on a measurement scale, its proper characteristics are to be relative with time (it means time's variation for a given system) and to express an *order* more than a measure in the sense of a distance or a graduation. Not without argument, Pierre Duhem saw in the second principle of thermodynamics (the increase of entropy for closed systems) a breaking down with mechanical conceptions, as it can only be formulated in such an abstract and non intuitive, and even quasi axiomatical, manner[59]. New quantities of this kind can actually be given, after their introduction, a more «intuitive» content in the usual sense of «intuitive», not so much of a possible mechanical model but of a direct theoretical function in the

---

58 D'Alembert [1749-1752].

59 See Duhem [1906] and his various works on thermodynamics and energetics.

thought of phenomena (consider also, for instance, previous to entropy, the concept of *potential* in electrodynamics[60]).

Such magnitudes within some time have been transcribed in terms of observable quanties, as for entropy those of statistical mechanics with Boltzmann's formula (or principle)[61]. Not all magnitudes dealt with in theoretical physics are in the situation of being reducible to directly mesurable ones : more challenging are quantities met in some recent chapters of physics and that are not simply endowed with numerical values. Are they only mathematical quantities used in physical theory, but not truly physical ones ? We shall come to these when concluding. But, as we speak of physics, the following statement is shared in common by all those cases : the legitimacy and the physical meaning of these magnitudes, abstract by their form but that become «concrete» and «intuitive» for our representation, are derived on the one hand from the demands of the understanding and, on the other hand, from repeated confrontation to experiment, through reproductibility of phenomena and predictivity.

MAGNITUDES OR QUANTITIES ? REDISCOVERING RELATION AND «QUALITY» OR ORDER UNDER THE RELATION

Problems are met with in physics (even in classical physics) such that although magnitudes can be defined by referring to physical situations, and can be put in relations by exact deterministic laws, they do not provide a precise description of the considered system or phenomenon. Such situations reflect actually similar mathematical properties for equations. It is not unusual that one does not know how to integrate differential equations that nevertheless represent exactly the motion of a physical system, or that one can do it only by approximations (for example, already in the three interacting bodies problem). Or again, it may happen that physical processes are represented with the help of divergent series (as d'Alembert noticed it already in XVIIIth century, for mechanical processes).

One may wonder, about situations of this type, what does that mean as to the mathematical (or, rather, mathematized) representation of these motions or processes, although the latter is wholly justified by the relationships between the magnitudes in play. The ones will maintain that it is still the exact form of the equation that represents the phenomenon, even if we do not know how to solve it exactly. The others, on the contrary, more preoccupied with approximate results and with numerical values corresponding to the possibilities of measurements, will let aside the idea of a *theoretical representation*, considered as inoperative, and will favour mere *models* with practical solutions.

We meet here, it seems to me, a limitation of the «quantitative» in the usual meaning, that is to say that of numerical determination, and some kind of

---

60 And Paul Langevin's comments about it (Langevin [1933]).

61 $S = kLogW$, where S is the entropy, W the probability, k the Boltzmann's constant.

«qualitative» is needed. In other words, the «qualitative» comes to the help of the «quantitative». But this «qualitative» is to be understood as assuming the quantitative, and is not going back to ontology or substance. It is actually nothing else than the idea of *relationship*, extended in that of *structure*, given in the *form* of the mathematized magnitudes themselves. Remember that this idea was at the heart of Descartes' conception.

The elaboration of physics, starting with mechanics, has gone along by dealing with *magnitudes* or *quantities* conceived according to «*measure*» but, as we stated at the beginning, the word «measure» underwent insensibly in the meantime a shift of its meaning, being more commonly understood as «*measurement*» than as mathematical *relation* (as it was in Descartes' sense). The increasing importance of *experiment* together with the concern for *precision* of experimental data that had become efficient with the theory of errors, not to forget a general positivist attitude crystallized in XIXth century science, led the concept of measure to refer more to *observation* than to *intelligibility*. Measured or measurable quantities have meant from that time uniquely, and perhaps restrictively, quantities *taking numerical values*, as it was most usually the case in classical physics.

But one could ask whether *physical magnitudes of another kind* than those being purely «with numerical value» could not be defined as well, assuming a larger meaning for «measure» and a wider spectrum of possibilities for the (mathematical) forms of relations a physical magnitude could be made of. One could also think of other modes of «relation» than «measure» understood in the sense of «metrics», reminding what Descartes referred to «*order*» and Riemann to «*topology*». Such questions are far from being illegitimate, and might be fruitful, considering some pecular aspects of contemporary physics, either in the *theory of dynamical systems* or in *quantum physics,* and possibly in other areas as well. Could we not imagine such «magnitudes of another kind» as having, for example, a definite mathematical expression but not being themselves directly put in correspondence with numerical values, this being let to their elements only ?

It might well be that such «unusual» mathematical representations of magnitudes would more and more fit cases met with in present physics, and indeed simplify our understanding of them. Take, for instance, a «physical» magnitude, whatever it might be, let us say the state of a «system», that would have the mathematical form of a linear superposition of elementary or referential magnitudes in the previous «numerically valued» sense ; or, indeed, another one, conceived to determine the first, that would be a matrix operator, whose elements only would be numerically valued. Or again a type of magnitude that would express not trajectories in space and time of the parts of a dynamical system but some characteristics of its equilibrium states and behaviour patterns.

And also, considering some kinds of physical systems having *properties* that cannot be reduced to properties in the sense of *directly measurable quantities*, univoquely defined or determined (for example, a position in space), would this trait be more problematic than, say, to consider a *topological property* independently of a metrical one ?

Such entities of the kind just evoked have generally been considered as non physical ones. When physicists have nevertheless to deal with such «formal expressions», they afford them a physical meaning only indirectly : they elaborate rules of transcription and interpretation that relate such «complex» quantities to quantities directly endowed with numerical values. Such is, for example, in quantum physics, the «reduction» rule for the «measurement» process, associated with the observational philosophy known as Bohr's «complementarity».

Other chapters of today physics involve types of «magnitudes» that escape the usual standards for physical quantities and are nevertheless quite powerful, and from which one gets as much intelligibility as in the usual classical cases. But these magnitudes are regarded generally as purely «formal» or «mathematical» ones, without direct physical counterparts. On the contrary, magnitudes in the usual, classical and traditional sense, such as coordinates on a trajectory, may have no definite physical content. What, then, is *physical* in such cases ? and *significantly physical*, considering the theoretical understanding of the phenomenon ? Would it not be more appropriate to consider as *physical* what is *theoretically meaningful* (and corroborated by phenomena, experiments, etc.), even if it was originally introduced in a pure formal way ?

Many examples of such transformations of content and signification through extension of meaning can be found in the history of physics and of mathematics (think only the extension of the concept of number, from integers to fractionals, to real and to complex numbers), and we can consider as well with the above point of view some genuine problems posed by various areas of present physics. In *quantum physics*, one describes *states* as linear superpositions of «eigenstates», functions or vectors (defined in Hilbert spaces), and *magnitudes* characterizing them (called «observable quantities») as linear operators acting on such states. These are the theoretical tools to deal with physical quantum processes, when, on the other hand, the data relative to measured quantities are only used to fill in and determine the components of these «functions» or «forms».

In the *study of dynamical systems*, although the theory is fully deterministic in the classical sense, it cannot give any exact prevision for trajectories, due to the amplification of small variations of the given initial state. We are inclined to ask, for such a case, whether the physical meaning of trajectory coordinates is actually a most significant one, considering what theory can provide. Another domain for concerns of this kind would possibly be *quantum cosmology*, in need of a conciliation between the continuous field of general relativity, defined in the usual four dimension space-time, and a quantum behavior that ignores spacial specifications : could we legitimately (scientifically) imagine magnitudes having a structure adequate to this property ? What kind of physical magnitudes would it be ? How would it be physically legitimated ?

Considering that physical magnitudes expressed mathematically are legitimated by the intelligibility they provide, in conformity with physical reality as it is given from the experience of phenomena, we suggest the possibility to

extend the meaning of the concept of «physical magnitude», in such a way that it can include simply «quantum physical state» for the *quantum domain*, «attractors», orders of stability, or other structural property for the *physics of dynamical systems*, and further dimensions or topological properties for *quantum gravity*.

PHILOSOPHICAL SIGNIFICATION OF A PROPOSED EXTENSION OF MEANING OF THE CONCEPT OF «PHYSICAL MAGNITUDES»

The extension of meaning proposed above, if it is to be confirmed from an overall consistency[62], would considerably simplify our understanding of the corresponding fields of knowledge. Such simplification would eventually be radical for the «interpretation problems». It would, as an immediate consequence, re-establish the use and meaning of the *objects* of a theory as description and representation. Consequently, it would allow to speak again about physics in terms of realism without being suspected to go back to the «old ways of thinking». But this, I think, we could have already known before, if we admitted that mental and symbolic description of the real (external) world is always indirect, as a «representation» of it. Indeed, the kind of realism we consider here is *critical realism*, that of *symbolic constructions* for the *representations of «reality»*, and conceived as a *programme* for scientific elaborations[63].

Let us first consider *quantum states* and *magnitudes*. As *quantum theory*, for instance, permits to explain so many sets of phenomena and to perform powerful models of them, it would be tempting to think of it as a fundamental theory about a given *world of objects*. Such is indeed the spontaneous way for physicists to practize it, although they get into problems when (but only then) they come to think about the *transition* from this *quantum domain to the classical one*, that of measuring apparatuses. The usual «standard» (copenhaguian) interpretation claims that there is no such thing as a scientific *object* (i. e. an entity endowed with *properties*) that would be a *conceptually signified* (*signifié conceptuel*) existing in the theory and that the conceptually signified (the so-called «object» of the theory, or state of the system) exists only in relation with given (and optional) conditions of preparation for measurement.

But the practice of scientists working with quantum physics objects actually (and factually, even if it dares not explicitly do so concerning «philosophical» matters) against this standard interpretation, by elaborating a *new objectivity* conceived in a sense similar to the usual one, but making use of concepts and magnitudes that have a *broader meaning* than the classical ones. The essential epistemological modification has been, actually, an (implicit) *extension of meaning* of the concept of *physical magnitude* or *quantity*, to entities that are not endowed with simple numerical values.

---

62 See, for a more detailed discussion in this respect about the concept of «quantum physical state», Paty [1999b, in press, b].

63 Paty [1988], in particular chapters 1 and 10 ; [1993], chapter 9.

The notion of *physical quantum state* differs from the ordinary notion of *physical state*, that is generally brought to magnitudes directly observable by instruments governed by the laws of classical physics. It is true that a *quantum state* is only indirectly accessible to experiment, but this does not deny the possibility *to get knowledge of it*. *Magnitudes* characterizing the state are as well not directly accessible, since they are not simply *numerically valued*. It is therefore necessary to conceive an *extension of meaning* of the concepts of *physical magnitude* and of *physical state* beyond their classical acceptions.

This extension is legitimated by the *phenomena*, in an acception of this term that does not reduce them to their apprehension by *perception*, but that conceives them according to the *understanding*, that is to say to their capacity to be brought to our *knowledge*, and this is essentially realized by the very formalism of quantum theory. This explicit extension has been, actually, prepared by an implicit one that works in practice already. But it also has been guess quasi explicitly by quantum theoretical physicists that were turned towards the formal properties of the theory, such as Max Born, Werner Heisenberg, Paul Adrian Dirac, John von Neumann and others. In their works, classical physical magnitudes were substituted by «quantum magnitudes» that differed from the former by their formal expression. For example, non commutative *q-numbers*, as proposed by Dirac in order to replace ordinary *c-numbers* would have immediately suggested an extension of meaning as the type we are suggesting[64]. But these pioneers did not however feel themselves authorized to propose from the start these formal constructions as directly conceivable as physical magnitudes, through a simple extension of meaning, because of the interpretation questions raised by then. Quantities of this kind remained merely mathematical, their relationship with physical phenomena being ruled by the «interprétation».The stumbling block was essentially the transition from the classical to the quantum, with the problem of measurement in the quantum sense.

To consider as physical magnitudes with the full meaning of the word quantities such as a *quantum state vectors* in the form of coherent linear superpositions and «observable operators» with probabilist eigenvalues, that means to leave up the tight connection, or even the identification, of *properties* with *what is or can be measured*, and to adopt a different notion of *property*, that refers to the system or state *as it is intellectually built* through a process of abstraction and theory elaboration that integrates factual data. Properties thought in that way are *no more contextual* and can be said *intrinseque* : such are, in this perspective, the properties of elementary quantum «particles» (photon, quark, etc…), and of quantum fields.

Such *properties* do not depend in any way on the circumstances of their observation, but they are *reconstituted* from experimental observations that provide values of quantities corresponding to contextual properties, with assigned probabilities measured by frequencies of events. In this respect, probabilities, far from being a limitation of knowledge, *allow the determination*, from the spectral

64 Dirac [1926]. Cf. Mehra & Rechenberg [1982], vol. 4, p. 162 sq., Darrigol [1992].

distribution of their components, of these «intrinsic» or global magnitudes - which are actually the true worries of the theory.

It is possible, generally speaking, to refer the determination of properties to two distinct features of the relationship between theoretical magnitudes : those that correspond respectively to *previsions* (*contextual* properties) and to *predictions* (*intrinsic* properties). Previsions, in the case of theories we speak of, are *contingent*, simply probable or unassignable, and are limited to the characterization of numerically valued magnitudes ; while predictions correspond to *intrinsic and structural* theoretical features, carried by magnitudes of a more complex form, that integrate, with the help of functions (or amplitudes) of probability in the case of quantum physics, magnitudes of the first kind. Such a distinction could be met as well in phenomena of a very different type than those of quantum physics, such as those pertaining to the dynamics of non linear systems[65].

REFERENCES

D'ALEMBERT, Jean le Rond [1743]. *Traité de dynamique*, David, Paris, 1743. 2nd ed., modif. and augm., David, Paris, 1758.

d'ALEMBERT, Jean le Rond [1746-1747]. *Réflexions sur la cause générale des vents* (Original version : manuscript in latin, «grand prix» for 1746 of Berlin Academy), David, Paris, 1747.

D'ALEMBERT, Jean Le Rond [1749-1752]. *Essai d'une nouvelle théorie de la résistance des fluides* (Original version : manuscript in latin, sent to Berlin Academy, 1749), Paris, David, 1752 ; repr. Culture et civilisation, Bruxelles, 1966.

D'ALEMBERT, Jean Le Rond [1751]. *Discours préliminaire de l'Encyclopédie*, in d'Alembert & Diderot (éds.) [1751-1780], vol. 1, 1751 ; re-ed., Gonthier, Paris, 1965.

D'ALEMBERT, Jean le Rond [1758]. *Essai sur les éléments de philosophie ou sur les principes des connaissances humaines*, Paris, 1758. Re-ed., followed by the *Êclaircissements* to this Essay (1765), Fayard, Paris, 1987.

D'ALEMBERT, Jean & DIDEROT, Denis (éds.) [1751-1780]. *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, 17 vols + 11 vol. de planches, Briasson, David, Le Breton et Durant, Paris, 1751-1780.

ARISTOTE [Phys.]. *Physics*. Trad. and notes by ?, ?, ?, 19??.

BERNOULLI, Johann [1989-1991]. *Der Briefwechsel von Johann Bernoulli, 2: Der Briefwechsel mit Pierre Varignon*, *Erste Teil (1692-1702)*; *Zweiter Teil (1702-)*, Bearbeit und Kommentiert von Pierre Costabel und Jeanne Peiffer, Birkhauser, Basel, 1988 (vol. 1), 1991 (vol. 2).

---

65 On this distinction, see Paty [1997].

BLAY, Michel [1992]. *La naissance de la mécanique analytique. La science du mouvement au tournant des XVIIè et XVIIIè siècles*, Presses Universitaires de France, Paris, 1992.

CASSIRER, Ernst [1910]. *Substanzbegriff und Funktionsbegriff*, Bruno Cassirer, Berlin, 1910. Engl. transl., *Substance and function*, *in* Cassirer [1923] (éd.1953), p.1-346.

- [1923]. *Substance and function and Einstein's theory of relativity*. Engl. transl. by William Curtis Swabey and Mary Collins Swabey, Open Court, Chicago, 1923; Dover, New York, 1953.

CLAGETT, Marshall [1959]. *The science of Mechanics in the Middle Ages*, The University of Wisconsin Press, Madison, 1959.

CLAVELIN, Maurice [1968]. *La philosophie naturelle de Galilée*, A. Colin, Paris, 1968.

CROMBIE, A.C. [1952]. *Augustine to Galileo. The history of science. A.D. 400-1650*, Falcon Press, London, 1952 ; new augm. ed., Heinemann, London, 1957.

DESCARTES, René [1728]. *Regulæ ad directionem ingenii* (vers 1728), in AT, vol. 10, p. 349-48 ; french transl., *Règles pour la direction de l'esprit*, Paris, Vrin, 1970.

- [1637]. *Discours de la méthode*, *suivi d'Essais de cette méthode* : *La Dioptrique*, *Les Météores*, *La Géométrie*, Leyde, 1637; in Descartes [1964-1974] (AT), vol. 6.

- [1644]. *Principia philosophiæ*, 1rst ed. princeps, Louis Elzevier, Amsterdam, 1644 ; in Descartes [1964-1974] (AT), vol. 8, p. 1-353. French transl. (1647), *Principes de la philosophie*, in Descartes [1964-1974] (AT), vol. 9, p. 1-362.

- [1964-1974]. *Oeuvres de Descartes*, published by Charles Adam & Paul Tannery, 11 volumes (1rst ed., 1896-1913) ; new rev. ed., 1964-1974 ; repr., 1996. [Edition noted AT].

DRAKE, Stillman [1970]. *Galileo Studies : Personality, tradition and revolution, University of Michigan Press*, Ann Harbour, 1970.

DUHEM, Pierre [1906]. *La théorie physique. Son objet, sa structure*, Paris, 1906 ; 2nd ed. rev. augm., 1914 ; reprinted., Vrin, Paris, 1981.

- [1913-1959]. *Le système du monde. Histoire des doctrines cosmologiques de Platon à Copernic*, 10 vols., Paris, 1913-1959.

EINSTEIN, Albert [1949]. Reply to criticism. Remarks concerning the essays brought together in this cooperative volume, *in* Schilpp [1949], p. 663-693. (Engl. transl. by P. A. Schilpp, of the original in german : Bemerkungen zu den in diesen bande Vereinigten Arbeiten, published in the german edition of Schilpp [1949], p. 493-511.)

EMERY, Monique & MONZANI, Pierre (éds.) [1983]. *Jean d'Alembert, savant et philosophe: portrait à plusieurs voix. Actes du Colloque organisé par le Centre international de synthèse - Fondation Pour la science, Paris, 15-18 juin 1983*, Archives contemporaines, Paris, 1989.

GALILEO, Galilei [1623]. *Il Saggiatore*, Roe, 1623 ; french transl., *L'Essayeur*, éd. C. Chauviré, Paris, Belles Lettres, 1979.

- [1632]. *Dialogo sopra i due massime sistemi del mondo : tolemaico e copernicano*, in Galileo [1890-1909]. French transl., *Dialogues sur les deux grands systèmes du monde*, Seuil, Paris, 1992.

- [1638]. *Discorsi e dimostrazioni mathematiche in torno di due nuove scienze*, Leyde, 1638 ; in Galileo [1890-1909] ; new ed., with introd. and notes by A. Carugo & L. Geymonat, Boringhieri, 1958. French transl. by Maurice Clavelin, *Dialogues sur deux sciences nouvelles*, Armand Colin, Paris, 1970.

- [1890-1909]. *Le Opere*, éd. Naz, 20 vols. in 21 tomes, Firenze, 1890-1909.

GREENBERG, John [1995]. *The Problem of the Earth from Newton to Clairaut*, New York, 1995.

GRIMBERG, Gérard [1998]. *D'Alembert et les équations aux dérivées partielles en hydrodynamique*, Doctorate thesis in Epistemology and history of sciences, Université Paris 7-Denis Diderot, Paris, dec. 1998.

KANT, Immanuel [1781-1787]. *Critik der reinen Vernunft*, Riga, J.F. Hartknoch, 1781 (2nd, augm., ed., 1787) ; french transl., *in* Kant, E., *Oeuvres philosophiques*, vol. 1, Paris, Gallimard, 1980, p. 705-1470.

KOYRE, Alexandre [1935-1939]. *Etudes galiléennes*, Paris, 3 vols., 1935-1939 ; 2nd ed., 1 vol., 1966. (1935-1939), Hermann, Paris, 1966.

- [1965]. *Newtonian Studies*, Harvard University Press, Cambridge (Ma), 1965. *Etudes newtoniennes* (french edition), Gallimard, Paris, 1968.

LAGRANGE, Joseph Louis [1788]. *Mécanique analytique*, in Lagrange, *Oeuvres*, vol. 11 & 12, 1888 et 1889.

LANGEVIN, Paul [1933]. *La notion de corpuscules et d'atomes*, Réunion internationale de Chimie-Physique (Paris, 1933), Hermann, Paris, 1934.

LAPLACE, Pierre Simon [1812]. *Théorie analytique des probabilités*, Paris, 1812 (reprinted, Culture et Civilisation, Bruxelles, 1967) ; 2nd ed., 1814 ; 3rd ed. rev. augm., 1820 ; in Laplace [1878-1912], vol. 7, p. 1-645.

- [1812]. *Essai philosophique sur les probabilités*, published as introduction to the 2nd ed. of *Théorie analytique*…, 1814 ; new eds. modified up to 1825 ; reprinted in Laplace [1878-1912], vol. 7, p. i-clii. Reprinted, Culture et Civilisation, Bruxelles, 1967 ; new ed. augm. with Mémoires, preface by René Thom, postface by Bernard Bru, Bourgois, Paris, 1986.

-- [1878-1912]. *Oeuvres complètes*, Gauthier-Villars, 14 vols, Paris, 1878-1912.

LARGEAULT, Jean [1984]. *Philosophie de la nature 1984*, Universalité Paris 12-Val-de-Marne, Créteil, 1984.

LEIBNIZ, Gottfried Wilhelm [1849-1863]. *Mathematische Schriften. Oeuvres*, édité par C. J. Gerhardt, 1849-1863, Halle, 7 vols, 1849-1863. Reprinted, G. Olms, Hildesheim, 1962.

LEIBNIZ, Wilhelm Gottfried [1989]. *Naissance du calcul différentiel, 26 articles*

des Acta Eruditorum. Introduction, traduction and notes by Marc Parmentier, Vrin, Paris, 1989.

MARTIN-VIOT, Florence [1994]. *L'élaboration des principes variationnels en dynamique, de Lagrange à Hamilton et Jacobi*, Doctorate thesis in Epistemology and history of sciences, Université Paris 7-Denis Diderot, nov. 1994.

MERLEAU-PONTY, Jacques [1983]. *La science de l'Univers à l'âge du positivisme. Etude sur les origines de la cosmologie contemporaine*, Vrin, Paris, 1983.

NEWTON, Isaac [1687]. *Philosophiae Naturalis principia mathematica*, London, 1687 ; 2nd ed., 1713 ; 3rd ed., 1726, edited with variations by Alexandre Koyré & I.B. Cohen, Cambridge University Press, Cambridge, 1972. Engl. transl by Andrew Motte, *The mathematical principles of natural philosophy*, 1729 ; 3rd ed., 1726 ; rev. transl. by F. Cajori, Berkeley, Univ. California Press, 1934.

PASCAL, Blaise [1657]. De l'esprit géométrique, in Pascal [1993], p. 348-355.

- [1670]. *Pensées*, in Pascal [1963], p. 493-649.

- [1963]. *Oeuvres complètes*, Preface by Henri Gouhier. Presentation and notes by Louis Lafuma, Seuil, Paris, 1963.

PASSERON, Irène [1994]. *Clairaut et la figure de la Terre au dix-huitième siècle. Cristallisation d'un nouveau style autour d'une pratique physico-mathématique* Doctorate thesis in Epistemology and history of sciences, Université Paris 7-Denis Diderot, dec. 1994.

PATY, Michel [1977]. *Théorie et pratique de la connaissance chez Jean d'Alembert*, Doctorate thesis in Philosophy, Université des Sciences Humaines, Strasbourg-2, 1977.

- [1984a]. Rapport des mathématiques et de la physique dans la pensée de d'Alembert, *Dix-huitième siècle*, n° 16, 1984, 69-79.

- [1984b] Mathématisation et accord avec l'expérience, *Fundamenta scientiae* 5, 1984, 31-50.

- [1988]. *La matière dérobée. L'appropriation critique de l'objet de la physique contemporaine* , Archives contemporaines, Paris, 1988.

- [1989]. Interprétation et construction dans le rapport des mathématiques à la physique, *Fundamenta scientiae* 10, 1989 (n° 1, jan.-march), 35-55.

- [1993]. *Einstein philosophe. La physique comme pratique philosophique*, Presses Universitaires de France, Paris, 1993

- [1994a]. Le caractère historique de l'adéquation des mathématiques à la physique, *in* Garma, Santiago; Flament, Dominique; Navarro, Victor (eds.), *Contra los titanes de la rutina.- Contre les titans de la routine*, Comunidad de Madrid/C.S.I.C., Madrid, 1994, p. 401-428.

- [1994b]. Sur l'histoire du problème du temps : le temps physique et les phénomènes, *in* Klein, Etienne et Spiro, Michel (éds.), *Le temps et sa flèche*, Editions Frontières, Gif-sur-Yvette, 1994, p. 21-58 ; also, Flammarion, Paris, 1996, p. 21-58.

- [1997]. «Mathesis universalis» e inteligibilidad en Descartes, spanish transl. by. Martha Cecilia Bustamente, *in* Albis, Victor R. ; Charum, Jorge ; Sanchez, Clara Helena ; Serrano, Gonzalo (eds.), *Memorias del Seminario en conmemoración de los 400 años del nacimiento de René Descartes*, Academia Colombiana de Ciencias Exactas, Físicas y Naturales, Coleccion Memorias, n°9, 1997, p. 135-170. [French original : «Mathesis universalis» et intelligibilité chez Descartes, *in* Chemla, Karine ; Probst, Siegmund ; Erdély, Agnès & Moretto, Antonio (eds.), *Ceci n'est pas un festschrift pour Imre Toth* (29.12.1996), to be published]

- [1997c]. Histoire rapide de la vitesse (le concept physique), in *La vitesse. Actes des 8ès Entretiens de la Villette*, Centre National de Documentation Pédagogique, Paris, 1997, p. 15-31.

- [1998a]. La philosophie et la physique, in Mattéi, Jean-François (éd.), *Le Discours philosophique*, volume 4 of the *Encyclopédie philosophique universelle*, Presses Universitaires de France, Paris, 1998, chapt. 123, p. 2104-2122.

- [1998b]. Le vide matériel, ou : La matière crée l'espace, *in* Diner, Simon & Gunzig, Edgard (éds.), *Univers du tout et du rien*, Editions de l'Université de Bruxelles, Bruxelles, 1998, p. 22-44.

- [1998c]. Les trois dimensions de l'espace et les quatre dimensions de l'espace-temps*in* Flament, Dominique (éd.), *Dimension, dimensions I*, Série Documents de travail, Fondation Maison des Sciences de l'Homme, Paris, 1998, p. 87-112.

- [1999b]. Interprétations et significations en physique quantique, *Revue Internationale de philosophie*, 1999.

- [1999c]. Les trois stades du principe de relativité, *Revue des questions scientifiques*, 170 (2), 1999, 101-148.

- [in press, a]. D'Alembert, la science newtonienne et l'héritage cartésien, *Journées d'Alembert*, *Départements de philosophie, Université de Paris X-Nanterre* (29.1.1999) and *Université de Bordeaux-2 Michel Montaigne* (16.1.1999), in press.

- [in press, b]. The concept of quantum state : new views on old phenomena, *in* Cohen, Robert S.; Howard, Don ; Renn, Jürgen ; Sarkar, Sahotra & Shimony, Abner (eds.), *John Stachel Festschrift*, Boston Studies in the Philosophy and History of science, Kluwer, Dordrecht, in press.

POINCARE, Henri [1897]. Les rapports de l'analyse et de la physique mathématique, *Revue générale des sciences pures et appliquées* 8, 1897, 857-861. Also in Poincaré [1991], p. 17-30.

- [1905]. *La valeur de la science*, Flammarion, Paris, 1905 ; 1970.

- [1991]. *L'analyse et la recherche*, Choice of texts & Introduction by Girolamo Ramunni, Hermann, Paris, 1991.

QUINE, Willard V. [1969]. *Ontological Relativity and Other Essays*, Columbia University Press, New York, 1969.

RIEMANN, Bernhard [1854]. Ueber die Hypothesen, welche der Geometrie zugrunde liege (1854), in Riemann, *Gesammelte mathematische Werke.*

Nachträge, éd. par M. Noether et W. Wirtinger, Leipzig, 1902, p. 272-287 ; french transl., Sur les hypothèses qui servent de fondement à la géométrie, in Riemann, *Oeuvres mathématiques*, french transl., Paris, 1898 (1968), p. 280-299.

SCHILPP, Paul Arthur (ed.) [1949]. *Albert Einstein, philosopher-scientist*, The library of living philosophers, Open Court, La Salle (Ill.), 1949. Repr., 1970. German transl., *Albert Einstein als Philosoph und Naturforscher*, Kohlhammer Verlag, Stuttgart, 1955.

THOM, René [1990]. *Apologie du Logos*, Hachette, Paris, 1990.

VUILLEMIN, Jules [1960]. *Physique et métaphysique chez Descartes*, Paris, 1960; new ed., 1987.

WARTOFSKY, Marx [1968]. *Conceptual Foundations of Scientific Thought*, MacMillan, London, 1968.

WHITESIDE, D.T. [1970]. The mathematical principles underlying Newton's *Principia mathematica*, *Journal for the History of Astronomy*, 1970, 116-138.