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Mathias forcing and ultrafilters

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Abstract We prove that if the Mathias forcing is followed by a forcing with the Laver Property, then any V-q-point is isomorphic via a ground model bijection to the canonical V-Ramsey ultrafilter added by the Mathias real. This improves a result of Shelah and Spinas (Trans AMS 325:2023–2047, 1999).

Keywords Mathias forcing · Ramsey ultrafilter · Laver property

Let us fix and recall the following notation. In a generic extension $\mathsf{V}^\dagger$ of V:

- for $x \in [\omega]^\omega$,

$$\lfloor x \rfloor = \{y \in [\omega]^\omega \colon x \subseteq^* y\},$$

  where $x \subseteq^* y$ means that $x \setminus y$ is finite;
- if $\mathcal{U} \subseteq [\omega]^\omega$ and $f \in \omega^\omega$, then

$$f_*(\mathcal{U}) = \{y \in [\omega]^\omega \colon \ f^{-1}[y] \in \mathcal{U}\};$$

- a V-ultrafilter is a maximal filter of subsets of $[\omega]^\omega \cap \mathsf{V}$;
- a V-ultrafilter $\mathcal{U}$ is V-Ramsey (resp. V-q-point) if each (resp. each finite-to-one) $f \in \omega^\omega \cap \mathsf{V}$ is injective or constant (resp. injective) on a set from $\mathcal{U}$.

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Shelah and Spinas [2] prove the following very interesting (see [1]) result.

Theorem ([2], Propositions 2.3 and 2.4) *Suppose that* r *is a Mathias real over* $\mathsf{V} \vDash \mathsf{CH}$ *and that* $\langle r_\xi \rangle_{\xi<\omega_2}$ *is a generic sequence of Mathias reals added to* $\mathsf{V}[r]$ *via countable support iteration. Then, for any*

$$x \in [\omega]^\omega \cap \mathsf{V}[r][\langle r_\xi \rangle_{\xi<\omega_2}],$$

if $\lfloor x \rfloor \cap \mathsf{V}$ *is a* V*-Ramsey ultrafilter, then*

$$\lfloor r \rfloor \cap \mathsf{V} = f_*(\lfloor x \rfloor \cap \mathsf{V})$$

for some bijective $f \in \omega^\omega \cap \mathsf{V}$.

In particular,

$$f^{-1}[r] \in \mathsf{V}[r] \quad \textit{and} \quad \lfloor x \rfloor \cap \mathsf{V} = \lfloor f^{-1}[r] \rfloor \cap \mathsf{V}.$$

Unfortunately the proof in [2] is somewhat demanding.[1] We prove:

Theorem *Suppose that* r *is a Mathias real over* V, $\mathcal{P} \in \mathsf{V}[r]$ *is a poset that has the Laver property in* $\mathsf{V}[r]$, $G_\mathcal{P} \subseteq \mathcal{P}$ *is a generic filter over* $\mathsf{V}[r]$, *and* $x \in [\omega]^\omega \cap \mathsf{V}[r][G_\mathcal{P}]$ *is such that* $\lfloor x \rfloor \cap \mathsf{V}$ *is a* V*-ultrafilter.*

Then

$$\lfloor r \rfloor \cap \mathsf{V} = f_*(\lfloor x \rfloor \cap \mathsf{V})$$

for some $f \in \omega^\omega \cap \mathsf{V}$.

If moreover $\lfloor x \rfloor \cap \mathsf{V}$ *is a* V*-*q*-point, then* f *can be chosen to be bijective.*

Recall that a poset $\mathcal{P}$ has the Laver property iff for any $f \in (\omega \setminus 1)^\omega$, if τ is a $\mathcal{P}$-name for an element of $\prod_{n<\omega} f(n)$, then dense in $\mathcal{P}$ is the set of p such that there is $T_p \in \prod_{n<\omega} [f(n)]^{\leqslant n+1}$ with $p \Vdash_\mathcal{P} \forall_n \ \tau(n) \in T_p(n)$.

Our theorem generalizes the result of Shelah and Spinas since V-q-points are V-Ramsey and $\mathsf{V}[r][\langle r_\xi \rangle_{\xi<\omega_2}]$ is a generic extension of $\mathsf{V}[r]$ via a poset that has the Laver property, namely via countable support iteration of the Mathias forcing.

Before starting the proof let us introduce and recall some more notation.

For $u \in [\omega]^\omega$, $U \in [\omega]^{<\omega}$, and $n < \omega$, let

$$\begin{aligned} u/U &= \{n \in u \colon U \subseteq n\}, \\ u(n) &= \text{ the } n\text{-th element of } u, \\ n^{-u} &= \sup(u \cap n), \\ n^{+u} &= \min(u/\{n\}); \end{aligned}$$

so, $u(0)^{-u} = \sup \emptyset = 0$.

[1] In the proof of the key Lemma 4.6 in [2], at the very end of Case 1, it is claimed that the desired contradiction has been reached. It seems to us that this is too optimistic.

The Mathias forcing consists of the set

$$\mathcal{Q} = \{(U, u) \in [\omega]^{<\omega} \times [\omega]^{\omega} \colon u \subseteq \omega/U\}$$

ordered by

$$(U, u) \leqslant (V, v) \iff V \subseteq U \subseteq V \cup v \ \wedge \ u \subseteq v.$$

If a filter $G \subseteq \mathcal{Q}$ is generic over V, its associated Mathias real is given by

$$r = \bigcup \{U \colon \ \exists u \ \ (U, u) \in G\}.$$

We have $\mathsf{V}[r] = \mathsf{V}[G]$ since

$$G = \{(U, u) \in \mathcal{Q} \cap \mathsf{V} \colon U \subseteq r \subseteq U \cup u\}.$$

Let $\dot{\boldsymbol{r}}$ denote the canonical name of r.

Lemma (Technical Lemma) *Suppose that*

$$\mathcal{Q} \Vdash (\dot{\mathcal{P}} \textit{ has the Laver property})$$

and

$$(U^{\circ}, u^{\circ}) * \dot{p}^{\circ} \Vdash \big(\dot{x} \in [\omega]^{\omega} \ \wedge \ \forall_{i<\omega} \ |\dot{x} \cap \dot{\boldsymbol{r}}(i)| \leqslant i\big).$$

Then, there exists

$$(U, u) * \dot{p} \leqslant (U^{\circ}, u^{\circ}) * \dot{p}^{\circ}$$

such that for any $(U^{\dagger}, u^{\dagger}) * \dot{p}^{\dagger} \leqslant (U, u) * \dot{p}$*,*

$$(U^{\dagger}, u^{\dagger}) * \dot{p}^{\dagger} \Vdash \ \dot{x} \subseteq^{*} \bigcup_{n \in u^{\dagger}} [\, n^{-u}, n^{+u}).$$

Proof of Theorem Since $\lfloor x \rfloor \cap \mathsf{V}$ is a V-ultrafilter, we have that $\lfloor x \rfloor \cap \mathsf{V} = \lfloor x' \rfloor \cap \mathsf{V}$ for any $x' \in [x]^{\omega}$ from $\mathsf{V}[r][G_{\mathcal{P}}]$. So, without loss of generality we may think that $\forall_{i<\omega} \, |x \cap r(i)| \leqslant i$. To prove the Theorem we run the following density argument.

Assuming that a condition $(U^{\circ}, u^{\circ}) * \dot{p}^{\circ}$ forces that

$$\dot{x} \in [\omega]^{\omega} \ \wedge \ \forall_{i<\omega} \, |\dot{x} \cap \dot{\boldsymbol{r}}(i)| \leqslant i \ \wedge \ \lfloor \dot{x} \rfloor \cap \mathsf{V} \text{ is a } \mathsf{V}\text{-ultrafilter},$$

get $(U, u) * \dot{p}$ by the Technical Lemma.

Let

$$u^{\text{odd}} = \{u(2i + 1)\}_{i<\omega}.$$

Then $(U, u^{\mathrm{odd}}) * \dot{p}$ forces that

$$\forall_{v \in \lfloor \dot{\boldsymbol{r}} \rfloor \cap \mathsf{V}} \quad \dot{x} \subseteq^* \bigcup \big\{ [\, u(2i), u(2i+2)) : \ u(2i+1) \in v \big\}.$$

Indeed, given

$$(U^\dagger, u^\dagger) * \dot{p}^\dagger \leqslant (U, u^{\mathrm{odd}}) * \dot{p}$$

and $v \in [\omega]^\omega$ such that

$$(U^\dagger, u^\dagger) * \dot{p}^\dagger \Vdash v \in \lfloor \dot{\boldsymbol{r}} \rfloor \cap \mathsf{V},$$

we have that $u^\dagger \subseteq^* v \cap u^{\mathrm{odd}}$; so, by the Technical Lemma,

$$(U^\dagger, u^\dagger) * \dot{p}^\dagger \Vdash \ \dot{x} \subseteq^* \bigcup \big\{ [\, n^{-u}, \ n^{+u}) : \ n \in v \cap u^{\mathrm{odd}} \big\}.$$

Define $f \in \omega^\omega$ by

$$f \restriction [\, 0, u(2)) \equiv u(1), \quad \forall_{i>0} \ f \restriction [\, u(2i), u(2i+2)) \equiv u(2i+1).$$

Clearly the condition $(U, u^{\mathrm{odd}}) * \dot{p}$ forces that

$$\lfloor \dot{\boldsymbol{r}} \rfloor \cap \mathsf{V} \subseteq f_*(\lfloor \dot{x} \rfloor \cap \mathsf{V}).$$

Since it also forces that

$$\lfloor \dot{\boldsymbol{r}} \rfloor \cap \mathsf{V} \ \text{ and } \ f_*(\lfloor \dot{x} \rfloor \cap \mathsf{V}) \ \text{ are } \ \mathsf{V}\text{-ultrafilters},$$

it must force that

$$\lfloor \dot{\boldsymbol{r}} \rfloor \cap \mathsf{V} = f_*(\lfloor \dot{x} \rfloor \cap \mathsf{V}).$$

Note that the function f is finite-to-one, so, if

$$(U^\circ, u^\circ) * \dot{p}^\circ \Vdash \ \lfloor \dot{x} \rfloor \cap \mathsf{V} \ \text{ is a } \ \mathsf{V}\text{-}\mathsf{q}\text{-point},$$

then we can find

$$(U', u') * \dot{p}' \leqslant (U, u^{\mathrm{odd}}) * \dot{p}$$

and

$$D \in [\omega]^\omega$$

such that $\omega\backslash D$ and $\omega\backslash f[D]$ are infinite, $f \upharpoonright D$ is injective, and

$$(U', u') * \dot{p}' \Vdash D \in \lfloor\dot{x}\rfloor \cap \mathsf{V}.$$

Modifying f on $\omega\backslash D$ we can get a bijective $f' \in \omega^\omega$ such that

$$(U', u') * \dot{p}' \Vdash \lfloor\dot{\boldsymbol{r}}\rfloor \cap \mathsf{V} = f'_*(\lfloor\dot{x}\rfloor \cap \mathsf{V}).$$

□

Proof of Technical Lemma For $x \subseteq \omega \times \omega$, let $(x)_i = \{j : (i, j) \in x\}$. Also, let the topology in $\mathscr{P}(\omega)$ be induced by the standard product topology of the Cantor space 2^ω by identifying $t \in 2^\omega$ with $\{n : t(n) = 1\}$; likewise for $\mathscr{P}(\omega \times \omega)$.

We will need the following well-known lemma. □

Lemma (Pure Extension Property) *The poset $\mathcal{Q}$ has Pure Extension Property, i.e., for any $(U^\circ, u^\circ) \in \mathcal{Q}$, $n \in \omega$, and a $\mathcal{Q}$-name τ such that $(U^\circ, u^\circ) \Vdash \tau \leqslant n$, there exist $i \leqslant n$ and $(U, u) \leqslant (U^\circ, u^\circ)$ with $U = U^\circ$ such that $(U, u) \Vdash \tau = i$.*

Now we can begin the proof.

Lemma 1 *If $v \in [\omega]^\omega$ and $f : [v]^{<\omega} \to \omega$, then there exists $v' \in [v]^\omega$ such that*

$$\forall_{V \in [v']^{<\omega}} \; f(V) < \min(v'/V).$$

Proof Let $v_0 = v$. For $i \in \omega$ let $n_i = \min v_i$ and $v_{i+1} = v_i / f[\mathscr{P}(i)]$. Then put $v' = \{n_i\}_{i \in \omega}$. □

Lemma 2 *Let $v \in [\omega]^\omega$ and $f : [v]^{<\omega} \to \mathscr{P}(\omega)$. Then there exist $v' \in [v]^\omega$ and $f' : [v']^{<\omega} \to \mathscr{P}(\omega)$ such that*

$$\forall_{V \in [v']^{<\omega}} \; f'(V) = \lim_{n \in v'/V} f(V \cup \{n\}).$$

We can require moreover that the convergence is so fast that

$$\forall_{V \in [v']^{<\omega}} \forall_{m \in v'/V} \forall_{n \in v'/\{m\}} \; f'(V) \cap m = f(V \cup \{n\}) \cap m.$$

Likewise, if we have $f : [v]^{<\omega} \to \mathscr{P}(\omega \times \omega)$, with the obvious modifications, e.g., the fastness condition changes to

$$\forall_{V \in [v]^{<\omega}} \forall_{m \in v'/V} \forall_{n \in v'/\{m\}} \; f'(V) \cap (m \times m) = f(V \cup \{n\}) \cap (m \times m).$$

Proof Let $\mathcal{V} = [v]^{<\omega}$. For $n \in v$, let $x_n \in (\mathscr{P}(\omega))^{\mathcal{V}}$ be given by

$$x_n(V) = \begin{cases} f(V \cup \{n\}), & V \subseteq n, \\ \emptyset, & V \nsubseteq n. \end{cases}$$

By compactness of $(\mathscr{P}(\omega))^{\mathcal{V}}$, there exist $v' \in [v]^{\omega}$ such that $\langle x_n \rangle_{n \in v'}$ converges to some $x \in (\mathscr{P}(\omega))^{\mathcal{V}}$. Put $f' = x \restriction [v']^{<\omega}$.

Further trimming of v', using that $\mathscr{P}(m)$ is finite, gives the second part. □

Lemma 3 *Suppose that*

$$\forall_{i<\omega} \ (U^{\circ}, u^{\circ}) \Vdash \dot{x}_i \subseteq \dot{\boldsymbol{r}}(i).$$

Then there exists $(U, u) \leqslant (U^{\circ}, u^{\circ})$ *such that for any nonempty* $V \in [u]^{<\omega}$ *there exists* $x^V \subseteq \max V$ *with*

$$(U \cup V, u/V) \Vdash x^V = \dot{x}_{|U \cup V|-1}.$$

Likewise, if

$$\forall_{i<\omega} \ (U^{\circ}, u^{\circ}) \Vdash \dot{\underset{\sim}{x}}_i \subseteq (i+1) \times \dot{\boldsymbol{r}}(i),$$

then there exists $(U, u) \leqslant (U^{\circ}, u^{\circ})$ *such that for any nonempty* $V \in [u]^{<\omega}$ *there exists* $\underset{\sim}{x}^V \in |U \cup V| \times \max V$ *with*

$$(U \cup V, u/V) \Vdash \underset{\sim}{x}^V = \dot{\underset{\sim}{x}}_{|U \cup V|-1}.$$

Proof We prove the first part. Let $u_0 = u^{\circ}$, $n_0 = \min u_0$, and

$$\mathcal{V}_0 = \{V : n_0 \in V \subseteq \{n_0\}\} = \{\{n_0\}\}.$$

Note that

$$(U \cup \{n_0\},\ u_0/\{n_0\}) \Vdash \dot{\boldsymbol{r}}(|U \cup \{n_0\}| - 1) = n_0.$$

Using the Pure Extension Property find $u_1 \subseteq u_0/\{n_0\}$ and $x^{\{n_0\}} \subseteq n_0$ such that

$$(U \cup \{n_0\},\ u_1) \Vdash \dot{x}_{|U \cup \{n_0\}|-1} = x^{\{n_0\}},$$

and put $n_1 = \min u_1$ and

$$\mathcal{V}_1 = \{V : n_1 \in V \subseteq \{n_0, n_1\}\}.$$

Since $|\mathcal{V}_1| = 2$, using the Pure Extension Property twice find $u_2 \subseteq u_1/\{n_1\}$ and $x^V \subseteq n_1$ for $V \in \mathcal{V}_1$ such that

$$\forall_{V \in \mathcal{V}_1} \ (U \cup V,\ u_2) \Vdash \dot{x}_{|U \cup V|-1} = x^V.$$

Put $n_2 = \min u_2$ and

$$\mathcal{V}_2 = \{V : n_2 \in V \subseteq \{n_0, n_1, n_2\}\}.$$

Since $|\mathcal{V}_2| = 4$, using the Pure Extension Property four times find $u_3 \subseteq u_2/\{n_2\}$ and $x^V \subseteq n_2$ for $V \in \mathcal{V}_2$ such that

$$\forall_{V \in \mathcal{V}_2} \ (U \cup V,\ u_3) \Vdash \dot{x}_{|U \cup V|-1} = x^V.$$

Continuing in this way we get $\langle u_i, n_i, \mathcal{V}_i \rangle_{i<\omega}$ such that for each $i < \omega$ we have

$$u_i \in [\omega]^\omega,\ n_i = \min u_i,\ u_{i+1} \subseteq u_i/\{n_i\},$$

$$\mathcal{V}_i = \{V : n_i \in V \subseteq \{n_j : j \leqslant i\}\}, \quad \text{and} \quad \forall_{V \in \mathcal{V}_i}\ x^V \subseteq n_i,$$

and

$$\forall_{V \in \mathcal{V}_i} \ (U \cup V,\ u_{i+1}) \Vdash \dot{x}_{|U \cup V|-1} = x^V.$$

Now, the condition

$$(U, u) = (U^\circ,\ \{n_i\}_{i \in \omega}),$$

and the sets x^V work. □

Lemma 4 *Suppose that $\mathcal{Q} \Vdash \dot{\mathcal{P}}$ has the Laver property, and that*

$$\forall_{i<\omega} \ (U^\circ, u^\circ) * \dot{p}^\circ \Vdash \dot{x} \subseteq \omega \ \wedge \ \forall_i \ \ |\dot{x} \cap \dot{\boldsymbol{r}}(i)| \leqslant i.$$

Then there exists a condition

$$(U, u) * \dot{p} \leqslant (U^\circ, u^\circ) * \dot{p}^\circ$$

such that for any $n \in u$ and $V \subseteq u \cap n$ the condition

$$(U \cup V \cup \{n\},\ u/\{n\}) * \dot{p}$$

forces that

$$\dot{x} \cap [\sup U, n) \ \subseteq \ [\sup U, (\sup(U \cup V))^{+u}) \ \cup \ [n^{-u}, n^{+u}).$$

In particular, any condition $(U^\dagger, u^\dagger) * \dot{p}^\dagger \leqslant (U, u) * \dot{p}$ forces that

$$\dot{x} \setminus \min u^\dagger \subseteq \bigcup_{n \in u^\dagger} [n^{-u}, n^{+u}).$$

Proof Since

$$\mathcal{Q} \Vdash \dot{\mathcal{P}} \text{ has the Laver Property,}$$

there exist $\mathcal{Q}$-names $\underset{\sim}{\dot{x}}_i$, $i < \omega$, and a condition

$$(U^\circ, u^\circ) * \dot{p} \leqslant (U^\circ, u^\circ) * \dot{p}^\circ$$

that forces that for all i

$$\underset{\sim}{\dot{x}}_i \subseteq (i+1) \times \dot{\boldsymbol{r}}(i) \ \wedge \ \forall_{j \leqslant i} \ |(\underset{\sim}{\dot{x}}_i)_j| \leqslant i \ \wedge \ \exists_{j \leqslant i} \ \dot{x}_i = (\underset{\sim}{\dot{x}}_i)_j.$$

Put $(U, u) = (U^\circ, u^\circ)$ and trim u as follows.

Use Lemma 3 to trim u so that for the trimmed u for any nonempty $V \in [u]^{<\omega}$ there exist $\underset{\sim}{x}^{V,0} \in |U \cup V| \times \max V$ such that

$$(U \cup V, u/V) \Vdash \underset{\sim}{x}^{V,0} = \underset{\sim}{\dot{x}}_{|U \cup V|-1}.$$

This implies in particular that

$$\forall_{j < |U \cup V|} \ (\underset{\sim}{x}^{V,0})_j \in [\max V]^{<|U \cup V|}.$$

Next, use Lemma 2 to trim u further so that for the trimmed u for any (possibly empty) $V \in [u]^{<\omega}$ the sequence

$$\langle \underset{\sim}{x}^{V \cup \{n\},0} \rangle_{n \in u/V}$$

converges in $\mathscr{P}(\omega \times \omega)$ to some $\underset{\sim}{x}^{V,1} \subseteq \omega \times \omega$ in such a way that

$$\forall_{m \in u/V} \ \forall_{n \in u/\{m\}} \ \ \underset{\sim}{x}^{V,1} \cap (m \times m) = \underset{\sim}{x}^{V \cup \{n\},0} \cap (m \times m);$$

in particular,

$$\forall_{m \in u/V} \ \forall_{j \leqslant |U \cup V|} \ \ |(\underset{\sim}{x}^{V,1})_j \cap m| \leqslant |U \cup V|,$$

and thus

$$\forall_{j \leqslant |U \cup V|} \ \ |(\underset{\sim}{x}^{V,1})_j| \leqslant |U \cup V|.$$

Finally, trim u again so that for the trimmed u we have

$$\forall_{m \in u} \ \forall_{V \subseteq u \cap m} \ \forall_{j \leqslant |U \cup V|} \ \ (\underset{\sim}{x}^{V,1})_j \subseteq m^{+u}.$$

It is not hard to see that the condition $(U, u) * \dot{p}$ works.

To see the last assertion of the lemma, fix

$$(U^\dagger, u^\dagger) * \dot{p}^\dagger \leqslant (U, u) * \dot{p}.$$

Suppose that $k \geqslant \min u^\dagger$ and that for some condition

$$(U^\ddagger, u^\ddagger) * \dot{p}^\ddagger \leqslant (U^\dagger, u^\dagger) * \dot{p}^\dagger$$

we have

$$(U^\ddagger, u^\ddagger) * \dot{p}^\ddagger \Vdash k \in \dot{x}.$$

Without loss of generality $|U^\ddagger| \geqslant 2$ and $k < \max U^\ddagger$. Let $m < n$ be the consecutive elements of $U^\ddagger$ such that $k \in [m, n)$. Let $V = U^\ddagger \cap [\sup U, n)$. Note that $\sup(U \cup V) = m$. Since

$$(U \cup V \cup \{n\}, u/\{n\}) * \dot{p}^\ddagger \Vdash \dot{x} \cap [\sup U, n) \subseteq [\sup U, m^{+u}) \cup [n^{-u}, n)$$

and

$$(U \cup V \cup \{n\}, u/\{n\}) * \dot{p}^\ddagger \geqslant (U^\ddagger, u^\ddagger) * \dot{p}^\ddagger \Vdash k \in \dot{x},$$

we must in fact have

$$k \in [m, m^{+u}) \cup [n^{-u}, n) \subseteq [m^{-u}, m^{+u}) \cup [n^{-u}, n^{+u}).$$

Since both m and n are in $u^\dagger$, we are done. □

This ends the proof of Technical Lemma. □

References

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