On Reichenbach's common cause principle and Reichenbach's notion of common cause*

Gábor Hofer-Szabó
Department of Philosophy
Technical University of Budapest
e-mail: gszabo@hps.elte.hu

Miklós Rédei
Department of History and Philosophy of Science
Eötvös University, Budapest
e-mail: redei@ludens.elte.hu

László E. Szabó
Theoretical Physics Research Group of HAS
Department of History and Philosophy of Science
Eötvös University, Budapest
e-mail: szabol@caesar.elte.hu

Abstract

It is shown that, given any finite set of pairs of random events in a Boolean algebra which are correlated with respect to a fixed probability measure on the algebra, the algebra can be extended in such a way that the extension contains events that can be regarded as common causes of the correlations in the sense of Reichenbach's definition of common cause. It is shown, further, that, given any quantum probability space and any set of commuting events in it which are correlated with respect to a fixed quantum state, the quantum probability space can be extended in such a way that the extension contains common causes of all the selected correlations, where common cause is again taken in the sense of Reichenbach's definition. It is argued that these results very strongly restrict the possible ways of disproving Reichenbach's Common Cause Principle.

1 Informal statement of the problem

Reichenbach's Common Cause Principle and the mathematically explicit notion of common cause formulated in terms of random events and their probabilities goes back to Reichenbach's ([1956], Section 19.) book. Both the Common Cause Principle and the related concept of common cause have been subjects of investigations in a number of works, especially in the papers by Salmon ([1978], [1980], [1984]), by Van Fraassen ([1977], [1982], [1989]), by Suppes and Zanotti ([1970], [1981]), by Cartwright ([1987]) and by Spohn ([1991]). It seems that there is no general consensus as regards the status of the Common Cause Principle and its relation to the notion of common cause. We do

*Published in *The British Journal for the Philosophy of Science*, **50** (1999), 377-399.

not wish to evaluate here the relative merits of the different analyses of the Principle and its relation to the common cause notion; nor do we aim at a critical comparison of the different suggestions as to how best to specify a technical notion of common cause. Rather, we take the notion of common cause as it was formulated by Reichenbach himself, and we investigate the following problem, raised first in Rédei ([1998]). Let $(\mathcal{S}, \mu)$ be a classical probability space, i.e. $\mathcal{S}$ be a Boolean algebra of random events and μ be a probability measure on $\mathcal{S}$. Assume that $\{(A_i, B_i) \mid i \in I\}$ is a set of pairs of events in $\mathcal{S}$ that are statistically correlated with respect to μ: $\mu(A_iB_i) > \mu(A_i)\mu(B_i)$ $(i \in I)$, and assume, further, that $\mathcal{S}$ does not contain events C_i that can be considered the common causes of the correlation between A_i and B_i, where common cause is taken in the sense of Reichenbach's definition [10], Section 19 (recalled in Section 2 below). The problem is whether the probability space $(\mathcal{S}, \mu)$ can in principle be enlarged in such a way that for each correlated pair (A_i, B_i) there exists a common cause C_i of the correlation in the larger probability space $(\mathcal{S}', \mu')$. If the probability space $(\mathcal{S}, \mu)$ is such that for every pair (A_i, B_i) in the set of correlated pairs $\{(A_i, B_i) \mid i \in I\}$ the space $(\mathcal{S}, \mu)$ can be enlarged in the said manner, then we say that $(\mathcal{S}, \mu)$ is *common cause completeable with respect to the given set of correlations* (see the Definition 5 in Section 3 for a precise formulation). We shall prove that *every* classical probability space $(\mathcal{S}, \mu)$ is common cause completeable with respect to *any finite* set of correlations. (Proposition 2 in Section 3). That is to say, we show that given *any finite* set of correlations in a classical event structure, one can *always* say that the correlations are due to some common causes, possibly "hidden" ones, i.e. ones that are not part of the inital set $\mathcal{S}$ of events.

Reichenbach's Common Cause Principle is the claim that if there is a correlation between two events A and B and a direct causal connection between the correlated events is excluded then there exists a common cause of the correlation in Reichenbach's sense. We interpret Proposition 2 as saying that Reichenbach's Common Cause Principle cannot be disproved by displaying classical probability spaces that contain a fnite number of correlated events without containing a Reichenbachian common cause of the correlations – the only justifiable conclusion one can draw is that the event structures in question are common cause incomplete.

Statistical correlations also make sense in non-classical probability spaces $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is a non-distributive, orthomodular lattice in the place of the Boolean algebra $\mathcal{S}$, and where ϕ is a generalized probability measure ("state") defined on $\mathcal{L}$, taking the place of μ. Such non-classical probability spaces emerge in non-relativistic quantum mechanics and in relativistic quantum field theory. In these theories $\mathcal{L}$ is the non-Boolean, orthomodular von Neumann lattice $\mathcal{P}(\mathcal{M})$ of projections of a non-commutative von Neumann algebra $\mathcal{M}$ determined by the observables of the quantum system. One of the difficulties in connection with interpreting quantum theory is the alleged impossibility of existence of common causes of the correlations in those quantum event structures. Note that it is not quite obvious what one means by a common cause in a quantum event structure because Reichenbach's original definition of common cause was formulated in terms of events in a classical, Kolmogorovian probability space, and the definition also makes essential use of classical conditional probabilities, which do not make sense in general in non-commutative (quantum) probability spaces. In fact, the definition of common cause in the literature analyzing the problem of common cause of quantum correlations is quite different from the Reichenbachian one: typically the definition is formulated in terms of hidden variables rather than in terms of events, and it also makes (more or less tacitly) the extra assumption that the hidden variables are *common* common causes (see below). In this paper we wish to retain all features of Reichenbach's original definition while applying it to quantum correlations, and we do this by requiring explicitly the common cause to commute with the events in the correlation. Having thus obtained a definition of common cause in quantum event structures (see Definition 6 in Section 4) we define $(\mathcal{P}(\mathcal{M}), \phi)$ to be common cause completeable with respect to a given set $\{(A_i, B_i) \mid i \in I\}$ of pairs of (commuting) correlated events in $\mathcal{P}(\mathcal{M})$ if the probability space $(\mathcal{P}(\mathcal{M}), \phi)$ can be enlarged in such a way that each of the correlations has a common cause in the enlarged non-classical probability space $(\mathcal{P}(\mathcal{M}'), \phi')$ (see the Definition 9 in Section 4 for a precise formulation). We shall prove that *every* $(\mathcal{P}(\mathcal{M}), \phi)$ is also common cause completeable with respect to the *complete* set of elements that are correlated in a given quantum state ϕ (Proposition 3 in Section 4). Proposition 3 allows us to conclude that Reichenbach's Common Cause Principle cannot be disproved even by finding quantum probability spaces that contain correlated events without containing a Reichenbachian common cause of the correlations; the only justifiable conclusion one can draw is that the quantum event structures are common cause incomplete – just like in the classical case. This conclusion seems to be in contradiction with the standard interpretation, according to which quantum correlations cannot have a hidden common cause. In Section 5 we localize the reason of this "contradiction". The essential point we make is that the standard arguments showing the impossibility of existence of a common cause of quantum correlations assume that a common cause is a *common* common cause, an assumption, we claim, is not part of Reichenbach's notion of common cause. In Section 6 we formulate a couple of open questions concerning the notion of common cause completeablity.

2 Reichenbach's notion of common cause

Let $(\mathcal{S},\mu)$ be a classical probability space, where $\mathcal{S}$ is the Boolean algebra of events and μ is the probability measure. If the joint probability $\mu(A\wedge B)$ of A and B is greater than the product of the single probabilities, i.e. if

$$\mu(A\wedge B)>\mu(A)\mu(B) \tag{1}$$

then the events A and B are said to be (positively) *correlated.*

According to Reichenbach ([1956], Section 19), a probabilistic common cause type explanation of a correlation like (1) means finding an event C (common cause) that satisfies the conditions specified in the next definition.

Definition 1 *C is a* common cause *of the correlation (1) if the following (independent) conditions hold:*

$$\begin{aligned}
\mu(A\wedge B|C) &= \mu(A|C)\mu(B|C) && (2)\\
\mu(A\wedge B|C^{\perp}) &= \mu(A|C^{\perp})\mu(B|C^{\perp}) && (3)\\
\mu(A|C) &> \mu(A|C^{\perp}) && (4)\\
\mu(B|C) &> \mu(B|C^{\perp}) && (5)
\end{aligned}$$

where $\mu(X|Y)=\mu(X\wedge Y)/\mu(Y)$ denotes the conditional probability of X on condition Y, $C^{\perp}$ denotes the complement of C and it is assumed that none of the probabilities $\mu(X)$, $(X=A,B,C,C^{\perp})$ is equal to zero.

We shall occasionally refer to conditions (2)-(5) as "Reichenbach(ian) conditions".

Reichenbach proves the following

Theorem *Conditions (2)-(5) imply (1); that is to say, if A,B and C are such that they satisfy conditions (2)-(5), then there is a a positive correlation between A and B in the sense of (1).*

Some remarks and terminology:

(i) We emphasize that, from the point of view of the explanatory role of C as the common cause of the correlation, each of the independent conditions (2)-(5) is equally important. For instance, the mere fact that an event C satisfies (4) and (5) only, i.e., the fact that C is positively statistically relevant for both A and B, is not sufficient for C to be accepted as an explanation of the correlation, since statistical relevance in and by itself is not sufficient to derive the correlation (1). This remains true even if, in addition to statistical relevance, we assume either $\mu(A\mid C)=\mu(B\mid C)=1$, or $C\subseteq A\wedge B$, since (3) can still fail, and again (1) cannot be derived.

(ii) Taking either A or B as C, the four conditions (2)-(5) are satisfied. This means that Reichenbach's definition accomodates the case when there is a direct causal link between the correlated events. To put this negatively: as it stands, Reichenbach's definition does not distinguish between direct causal influence between the correlated events and the correlation caused by a common cause. Reichenbach's definition also does not exclude the following sort of common cause: $C\neq A$ (respectively $C\neq B$) but

$$(C<A \text{ or } C>A) \text{ and } \mu(C)=\mu(A) \tag{6}$$

respectively

$$(C<B \text{ or } C>B) \text{ and } \mu(C)=\mu(B) \tag{7}$$

Such a C is a common cause in the sense of the Definition 1 but such a C should not be regarded as a meaningful common cause because C is identical with A (respectively B) up to an event of probability 0. If C is a common cause such that none of (6) and (7) holds then we shall say that C is a *proper* common cause, and in what follows, common cause will always mean a proper common cause unless stated otherwise explicitly.

(iii) It can happen that, in addition to being a probabilistic common cause, the event C logically implies both A and B, i.e. $C\subseteq A\wedge B$. If this is the case then we call C a *strong* common cause. If C is a common cause such that $C\not\subseteq A$ and $C\not\subseteq B$ then C is called a *genuinely probabilistic* common cause.

(iv) A common cause C will be called *deterministic* if

$$\begin{aligned}
\mu(A|C) &= 1=\mu(B|C)\\
\mu(A|C^{\perp}) &= 0=\mu(B|C^{\perp})
\end{aligned}$$

Note that the notions of deterministic and genuinely probabilistic common cause are not negations of each other. There does not seem to exist any straightforward relation between the notions of deterministic, genuinely probabilistic and proper common causes, as we have defined them.

Next we wish to determine the restrictions imposed on the values of the probabilities $\mu(C)$, $\mu(A|C)$, $\mu(A|C^{\perp})$, $\mu(B|C)$ and $\mu(B|C^{\perp})$ by the assumption that the correlation between A and B has C as a common cause. If we assume that there exists a common cause C in $(\mathcal{S},\mu)$ of the given correlation $\mu(A\wedge B)>\mu(A)\mu(B)$ then, using the theorem of total probability

$$\mu(X)=\mu(X|Y)\mu(Y)+\mu(X|Y^{\perp})(1-\mu(Y)) \qquad X,Y\in\mathcal{S}$$

we can write

$$\begin{aligned}
\mu(A) &= \mu(A|C)\mu(C)+\mu(A|C^{\perp})(1-\mu(C)) && (8)\\
\mu(B) &= \mu(B|C)\mu(C)+\mu(B|C^{\perp})(1-\mu(C)) && (9)\\
\mu(A\wedge B) &= \mu(A\wedge B|C)\mu(C)+\mu(A\wedge B|C^{\perp})(1-\mu(C)) && (10)\\
&= \mu(A|C)\mu(B|C)\mu(C)+\mu(A|C^{\perp})\mu(B|C^{\perp})(1-\mu(C)) && (11)
\end{aligned}$$

((11) follows from (10) because of the screening off equations (2)-(3)). So the assumption of a common cause of the correlation between A and B implies that there exist real numbers

$$r_C, r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}}$$

satisfying the following relations

$$\begin{aligned}
0 &\le r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}} \le 1 && (12)\\
\mu(A) &= r_{A|C}r_C + r_{A|C^{\perp}}(1-r_C) && (13)\\
\mu(B) &= r_{B|C}r_C + r_{B|C^{\perp}}(1-r_C) && (14)\\
\mu(A\wedge B) &= r_{A|C}r_{B|C}r_C + r_{A|C^{\perp}}r_{B|C^{\perp}}(1-r_C) && (15)\\
0 &< r_C<1 && (16)\\
r_{A|C} &> r_{A|C^{\perp}} && (17)\\
r_{B|C} &> r_{B|C^{\perp}} && (18)
\end{aligned}$$

Conversely, given a correlation $\mu(A\wedge B)>\mu(A)\mu(B)$ in a probability space $(\mathcal{S},\mu)$, if there exists an element C in $\mathcal{S}$ such that

$$\begin{aligned}
\mu(C) &= r_C && (19)\\
\mu(A|C) &= r_{A|C} && (20)\\
\mu(A|C^{\perp}) &= r_{A|C^{\perp}} && (21)\\
\mu(B|C) &= r_{B|C} && (22)\\
\mu(B|C^{\perp}) &= r_{B|C^{\perp}} && (23)
\end{aligned}$$

and the numbers $r_C, r_{A|C}, r_{B|C}, r_{A|C^{\perp}}, r_{B|C^{\perp}}$ satisfy the relations (12)-(18), then the element C is a common cause of the correlation in Reichenbach's sense.

Proposition 1 *Given any correlation $\mu(A\wedge B)>\mu(A)\mu(B)$ in $(\mathcal{S},\mu)$ there exists a non-empty two parameter family of numbers*

$$r_C(t,s), r_{A|C}(t,s), r_{B|C}(t,s), r_{A|C^{\perp}}(t,s), r_{B|C^{\perp}}(t,s)$$

that satisfy the relations (12)-(18).

Proof: Consider the system of 3 equations (13)-(15) with $t=r_{A|C}$ and $s=r_{B|C}$ as parameters. One can then express $r_C, r_{A|C^{\perp}}$ and $r_{B|C^{\perp}}$ from equations (13)-(15) as follows.

$$\begin{aligned}
r_C &= \frac{\mu(A\wedge B)-\mu(A)\mu(B)}{(\mu(A)-t)(\mu(B)-s)+\mu(A\wedge B)-\mu(A)\mu(B)} && (24)\\
r_{A|C^{\perp}} &= \frac{\mu(A)-t}{1-r_C}+t=\frac{\mu(A\wedge B)-\mu(A)s}{\mu(B)-s} && (25)\\
r_{B|C^{\perp}} &= \frac{\mu(B)-s}{1-r_C}+s=\frac{\mu(A\wedge B)-\mu(B)t}{\mu(A)-t} && (26)
\end{aligned}$$

Using the equations (24)- (26) it is easy to verify that choosing the two parameters t, s within the bounds

$$1 \geq \quad t = r_{A|C} \quad \geq \frac{\mu(A \wedge B)}{\mu(B)} \tag{27}$$

$$1 \geq \quad s = r_{B|C} \quad \geq \frac{\mu(A \wedge B)}{\mu(A)} \tag{28}$$

the conditions (12)-(18) are satisfied.

As the above proposition shows, the Reichenbach conditions allow, in principle, for a number of different common causes, each characterized probabilistically by the five real numbers r_C, $r_{A|C}$, $r_{B|C}$, $r_{A|C^\perp}$, $r_{B|C^\perp}$ satisfying the relations (12)-(18). Given a correlation $\mu(A \wedge B) > \mu(A)\mu(B)$, we call a set of five real numbers r_C, $r_{A|C}$, $r_{B|C}$, $r_{A|C^\perp}$, $r_{B|C^\perp}$ *admissible* if they satisfy conditions (12)-(18).

Definition 2 *A common cause C of a correlation $\mu(A \wedge B) > \mu(A)\mu(B)$ is said to have (be of) the* type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ *if these numbers are equal to the probabilities indicated by the indices, i.e. if the equations (19)-(23) hold.*

3 Common cause completeability – the classical case

Given a statistically correlated pair of events A, B in a probability space $(\mathcal{S}, \mu)$, a proper common cause C in the sense of Reichenbach's definition does not necessarily exist in $\mathcal{S}$. (For instance the set of events might contain only I, A, B and their orthogonal complements and hence be too small to contain a proper common cause.) If this is the case, then we call $(\mathcal{S}, \mu)$ *common cause incomplete.* The existence of common cause incomplete probability spaces leads to the question of whether such probability spaces can be enlarged so that the larger probability space contains a proper common cause of the given correlation. What is meant by "enlargement" here is contained in the Definition 3 below. Before we give this definition recall that the map $h: \mathcal{S}_1 \to \mathcal{S}_2$ between two Boolean algebras $\mathcal{S}_1$ and $\mathcal{S}_2$ is a *Boolean algebra homomorphism* if it preserves all lattice operations (including orthocomplementation). A Boolean algebra homomorphism h is an *embedding* if $X \not= Y$ implies $h(X) \not= h(Y)$.

Definition 3 *The probability space $(\mathcal{S}', \mu')$ is called an* extension *of $(\mathcal{S}, \mu)$ if there exists a Boolean algebra embedding h of $\mathcal{S}$ into $\mathcal{S}'$ such that*

$$\mu(X) = \mu'(h(X)) \qquad \textit{for all } X \in \mathcal{S} \tag{29}$$

This definition, and in particular the condition (29), implies that if $(\mathcal{S}'\mu')$ is an extension of $(\mathcal{S}, \mu)$ (with respect to the embedding h), then every single correlation $\mu(A \wedge B) > \mu(A)\mu(B)$ in $(\mathcal{S}, \mu)$ is carried over intact by h into the correlation

$$\mu'(h(A) \wedge h(B)) \quad = \quad \mu'(h(A \wedge B)) = \mu(A \wedge B) \;>\; \mu(A)\mu(B) = \mu'(h(A))\mu'(h(B))$$

Hence, it makes sense to ask whether a correlation in $(\mathcal{S}, \mu)$ has a Reichenbachian common cause in the extension $(\mathcal{S}', \mu')$. So we stipulate

Definition 4 *We say that $(\mathcal{S}', \mu')$ is a type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$* common cause completion of $(\mathcal{S}, \mu)$ with respect to the correlated events A, B *if $(\mathcal{S}'\mu')$ is an extension of $(\mathcal{S}, \mu)$, and there exists a Reichenbachian common cause $C \in \mathcal{S}'$ of type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ of the correlation $\mu'(h(A) \wedge h(B)) > \mu'(h(A))\mu'(h(B))$.*

Definition 5 *Let $(\mathcal{S}, \mu)$ be a probability space and $\{(A_i, B_i) \mid i \in I\}$ be a set of pairs of correlated events in $\mathcal{S}$. We say that $(\mathcal{S}, \mu)$ is* common cause completeable with respect to the set $\{(A_i, B_i) \mid i \in I\}$ of correlated events *if, given any set of admissible numbers $(r^i_C, r^i_{A|C}, r^i_{B|C}, r^i_{A|C^\perp}, r^i_{B|C^\perp})$ for every $i \in I$, there exists a probability space $(\mathcal{S}', \mu')$ such that for every $i \in I$ the space $(\mathcal{S}', \mu')$ is a type $(r^i_C, r_{A|C}, r^i_{B|C}, r^i_{A|C^\perp}, r^i_{B|C^\perp})$ common cause extension of $(\mathcal{S}, \mu)$ with respect to the correlated events A_i, B_i.*

Proposition 2 *Every classical probability space $(\mathcal{S}, \mu)$ is common cause completeable with respect to any* finite *set of correlated events.*

The proof of this statement proceeds by induction on the index i. One shows first that given the single pair $(A_1, B_1) = (A, B)$ of correlated events in $(\mathcal{S}, \mu)$ and any admissible numbers $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ there exists a type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ common cause completion of $(\mathcal{S}, \mu)$ with respect to (A, B). We prove this statement in two steps. In Step 1 we construct an extension $(\mathcal{S}',\mu')$ of $(\mathcal{S},\mu)$. In Step 2 we show that given any admissible numbers, the probability measure μ' can be chosen in such a way that there exists a proper common cause in $\mathcal{S}'$ that has the type specified by the admissible numbers. Finally, we shall argue that if $(\mathcal{S}^{n-1},\mu^{n-1})$ is a common cause completion of $(\mathcal{S},\mu)$ with respect to the set of $n-1$ correlations between A_i and B_i $(i=1,\dots n-1)$, then there exists a common cause completion of $(\mathcal{S},\mu)$ with respect to the n correlations. For the details see the Appendix.

4 Common cause completeability – the quantum case

Statistical correlations also make sense in non-classical probability structures $(\mathcal{L},\phi)$, where $\mathcal{L}$ is a non-distributive lattice of events and ϕ is a generalized probability measure ("state") on $\mathcal{L}$. Such non-classical probability spaces arise in quantum theory, where $\mathcal{L}$ is the non-distributive, orthomodular lattice of projections $\mathcal{P}(\mathcal{M})$ of a non-commutative von Neumann algebra $\mathcal{M}$ determined by the set of observables of a quantum system (for the operator algebraic notions used here without definition see eg. Kadison and Ringrose ([1983], [1986]) A map $\phi: \mathcal{P}(\mathcal{M}) \to [0,1]$ on such an event structure is called a *state* if it is additive on orthogonal projections in the following sense:

$$\phi(\vee_i P_i) = \sum_i \phi(P_i) \qquad P_i \perp P_j \quad i \neq j \tag{30}$$

A positive, linear functional on a von Neumann algebra $\mathcal{M}$ is called *normal* state if its restriction to the lattice of projections is a state (i.e. it is additive) in the above sense. The restriction of a normal state to a Boolean sublattice of $\mathcal{P}(\mathcal{M})$ is a classical probability measure, so normal states are the analogues of classical probability measures. If $\mathcal{M}$ acts on the Hilbert space $\mathcal{H}$, then a normal state is always of the form $\phi(X) = Tr(WX)$ with some density matrix W. We call a pair $(\mathcal{P}(\mathcal{M}),\phi)$ with a normal state ϕ a *quantum probability space.* Two *commuting* events A, B in a quantum probability space $(\mathcal{P}(\mathcal{M}),\phi)$ are called (positively) correlated if

$$\phi(A \wedge B) > \phi(A)\phi(B) \tag{31}$$

Given a correlation in a quantum probability space, we may want to ask if there is a Reichenbachian common cause in $\mathcal{P}(\mathcal{M})$ of the correlation. By a Reichenbachian common cause we mean a $C \in \mathcal{P}(\mathcal{M})$ which commutes with both A and B and satisfies the Reichenbachian conditions (2)-(5). To be explicit we stipulate the following

Definition 6 *The event $C \in \mathcal{P}(\mathcal{M})$ is a common cause of the correlation (31) between two commuting events A, B in a quantum probability space $(\mathcal{P}(\mathcal{M}),\phi)$ if*

1. *C commutes with both A and B;*
2. *the following four conditions (analogous to (2)-(5)) are satisfied*

$$\begin{aligned}
\frac{\phi(A \wedge B \wedge C)}{\phi(C)} &= \frac{\phi(A \wedge C)}{\phi(C)}\frac{\phi(B \wedge C)}{\phi(C)} \\
\frac{\phi(A \wedge B \wedge C^\perp)}{\phi(C^\perp)} &= \frac{\phi(A \wedge C^\perp)}{\phi(C^\perp)}\frac{\phi(B \wedge C^\perp)}{\phi(C^\perp)} \\
\frac{\phi(A \wedge C)}{\phi(C)} &> \frac{\phi(A \wedge C^\perp)}{\phi(C^\perp)} \\
\frac{\phi(B \wedge C)}{\phi(C)} &> \frac{\phi(B \wedge C^\perp)}{\phi(C^\perp)}
\end{aligned}$$

Similarly to the classical case, a common cause C is called proper *if it differs from both A and B by more than an event of ϕ-measure zero.*

Having this definition, we can define the type of the common cause in a quantum probability space in exactly the same way as in the classical case, and we can also speak of admissible numbers etc. Just like a classical probability space, a quantum probability space $(\mathcal{P}(\mathcal{M}),\phi)$ may contain a correlation without containing a proper common cause of the correlation in the sense of Definition 6. If this is the case, then we call the quantum probability space *common cause incomplete*, and we may ask if the quantum probability space can be enlarged so that the enlarged space contains a proper common cause. What is meant by "enlargement" is specified in the next definition, which is completely analogous to Definition 3.

Definition 7 *The quantum probability space* $(\mathcal{P}(\mathcal{M}'), \phi')$ *is an* extension *of the quantum probability space* $(\mathcal{P}(\mathcal{M}), \phi)$ *if there exists an embedding* h *of* $\mathcal{P}(\mathcal{M})$ *into* $\mathcal{P}(\mathcal{M}')$ *such that*

$$\phi'(h(X)) = \phi(X) \qquad \textit{for all } X \in \mathcal{P}(\mathcal{M})$$

By an embedding is meant here a lattice homomorphism that preserves all lattice operations (including the orthocomplementation) and such that $X \not\models Y$ implies $h(X) \not\models h(Y)$.

Definition 8 *We say that* $(\mathcal{P}(\mathcal{M}'), \phi')$ *is a type* $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ common cause completion of $(\mathcal{P}(\mathcal{M}), \phi)$ with respect to the correlated events A, B *if* $(\mathcal{P}(\mathcal{M}'), \phi')$ *is an extension of* $(\mathcal{P}(\mathcal{M}), \phi)$*, and there exists a Reichenbachian common cause* $C \in \mathcal{P}(\mathcal{M}')$ *of type* $(r_C,\ r_{A|C},\ r_{B|C},\ r_{A|C^\perp},\ r_{B|C^\perp})$ *of the correlation* $\phi'(h(A) \wedge h(B)) > \phi'(h(A))\phi'(h(B))$.

We can now give the definition of common cause competeability in the quantum case:

Definition 9 *Let* $(\mathcal{P}(\mathcal{M}), \phi)$ *be a quantum probability space and* $\{(A_i, B_i) \mid i \in I\}$ *be a set of pairs of correlated events in* $\mathcal{P}(\mathcal{M})$. *We say that* $(\mathcal{P}(\mathcal{M}), \phi)$ *is* common cause completeable with respect to the set $\{(A_i, B_i) \mid i \in I\}$ of correlated events *if, given any set of admissible numbers* $(r^i_C, r^i_{A|C}, r^i_{B|C}, r^i_{A|C^\perp}, r^i_{B|C^\perp})$ *for every* $i \in I$*, there exists a quantum probability space* $(\mathcal{P}(\mathcal{M}'), \phi')$ *such that for every* $i \in I$ *the space* $(\mathcal{P}(\mathcal{M}'), \phi')$ *is a type* $(r^i_C, r_{A|C}, r^i_{B|C}, r^i_{A|C^\perp}, r^i_{B|C^\perp})$ *common cause extension of* $(\mathcal{P}(\mathcal{M}), \phi)$ *with respect to the correlated pair* (A_i, B_i).

Proposition 3 *Every quantum probability space* $(\mathcal{P}(\mathcal{M}), \phi)$ *is common cause completeable with respect to the set of pairs of events that are correlated in the state* ϕ.

The proof of this statement is divided into two parts. In the first part we construct an extension of the quantum probability space $(\mathcal{P}(\mathcal{M}), \phi)$; this is done in two steps. In Step 1 the quantum probability space $(\mathcal{P}(\mathcal{M}), \phi)$ is embedded into the quantum probability space $(\mathcal{P}(\mathcal{H} \oplus \mathcal{H}), \phi_2)$ with a suitable state ϕ_2 extending ϕ, where $\mathcal{H}$ is the Hilbert space on which the von Neumann algebra $\mathcal{M}$ acts. In Step 2 this latter quantum probability space is embedded into $(\mathcal{P}(\mathcal{H}'), \phi')$, where $\mathcal{H}'$ is a Hilbert space constructed explicitly. We show in the second part of the proof that for *any* correlated pair (A, B) in $(\mathcal{P}(\mathcal{M}), \phi)$ and for *any* admissible set of numbers there exists in $(\mathcal{P}(\mathcal{H}'), \phi')$ a Reichenbachian common cause of type defined by the admissible numbers. For the details see the Appendix.

5 Reichenbach's Common Cause Principle and common cause completeability

Reichenbach's Common Cause Principle is a non-trivial metaphysical claim about the causal structure of the physical world: if a direct causal influence between the probabilistically correlated events A and B does not exist, then there exists a (proper) common cause of the correlation (in Reichenbach's sense). There exist both classical and quantum probability spaces that contain correlations without containing proper common causes of the correlations; hence, if one wants to maintain Reichenbach's Common Cause Principle, one must be able to claim that there exist "hidden" events ("hidden" in the sense of not being accounted for in the given event structure which is thus common cause incomplete) that can be interpreted as the common causes of the correlations. If such "hidden" common cause events exist, then there must exist an extension of the original probability space, an extension that accommodates the common causes. Propositions 2 and 3 tell us that such extensions are always possible. In other words, Propositions 2 and 3 show that a necessary condition for a common cause explanation of correlations in both classical and quantum event structures can always be satisfied. To put this negatively: one cannot disprove Reichenbach's Common Cause Principle by proving that the necessary condition (common cause completeability) for its validity cannot be satisfied.

It is generally accepted that the Reichenbachian conditions (2)-5) are just necessary conditions. If an event C must satisfy also some Supplementary Conditions (in addition to the Reichenbachian conditions) to qualify as a common cause, then a disproof of Reichenbach's Common Cause Principle requires establishing that there exists no event whatsoever that satisfies *both* the Reichenbachian conditions *and* the Supplementary Conditions. It goes without saying that such a disproof requires first the specification of the Supplementary Conditions. Propositions 2 and 3 impose strong restrictions on the possible mathematical specifications of the Supplementary Conditions: these conditions cannot be formulated in terms of the probabilities $p(C)$, $p(A|C)$, $p(B|C)$, $p(A|C^\perp)$ and $p(B|C^\perp)$. This is because the assumptions in Propositions 2 and 3 contain no restrictions whatsoever on these probabilities – beyond the Reichenbach conditions. Therefore, the hypothetical Supplementary Conditions would have to be specified entirely in terms of the algebraic/logical structure of events.

The conclusion that probability spaces are typically common cause completeable seems to contradict the received view concerning correlations predicted by quantum mechanics and quantum field theory. According to the standard interpretation an explanation of the quantum correlations by assuming a direct causal influence between the correlated quantum events is excluded by the theory of relativity, and furthermore there cannot exist Reichenbachian common causes of the correlations because, as the argument goes, the assumption of a common cause of correlations in Reichenbach's sense implies Bell's inequality (this view is present in Van Fraassen ([1989]), Skyrms ([1984]), Butterfield ([1989]) and Spohn ([1991]) and it has made its way into textbooks already (see Salmon et all ([1992])).

But there is no contradiction here at all; because the present paper's analysis differs from the standard interpretation. To see where the differences are, let us recall using the present paper's terminology and notation the standard argument in favour of the claim "the existence of Reichenbachian common causes of correlations implies Bell's inequality". Consider four pairs

$$(A_1, B_2); (A_1, B_2); (A_2, B_1); (A_2, B_2)$$

of commuting events in $\mathcal{P}(\mathcal{M})$ that are correlated in the state ϕ. Assume that there exists a *common* common cause C of the four correlations; i.e. assume that there exists a single $C \in \mathcal{P}(\mathcal{M})$ that is a common cause (in the sense of the Definition 9) of *all* four correlated pairs.

Using the notation $\phi(X|Y) = \frac{\phi(X \wedge Y)}{\phi(Y)}$ for commuting $X, Y \in \mathcal{P}(\mathcal{M})$ we can write then

$$\begin{aligned} \phi(A_i \wedge B_j|C) &= \phi(A_i|C)\phi(B_j|C) \\ \phi(A_i \wedge B_j|C^\perp) &= \phi(A_i|C^\perp)\phi(B_j|C^\perp) \\ \phi(A_i \wedge B_j) &= \phi(A_i \wedge B_j|C)\phi(C) + \phi(A_i \wedge B_j|C^\perp)\phi(C^\perp) \\ &= \phi(A_i|C)\phi(B_j|C)\phi(C) + \phi(A_i|C^\perp)\phi(B_j|C^\perp)\phi(C^\perp) \end{aligned} \qquad i, j = 1, 2 \qquad (32)$$

The elementary inequality for numbers $a_i, b_j \in [0, 1]$ $(i, j = 1, 2)$

$$-1 \leq a_i b_i - a_i b_j + a_j b_i + a_j b_j \leq 0 \qquad (33)$$

implies

$$\begin{aligned} -1 \leq \phi(A_1|C)\phi(B_1|C) - \phi(A_1|C)\phi(B_2|C) + \phi(A_2|C)\phi(B_1|C) + \phi(A_2|C)\phi(B_2|C) \\ -\phi(A_2|C) - \phi(B_1|C) \leq 0 \\ -1 \leq \phi(A_1|C^\perp)\phi(B_1|C^\perp) - \phi(A_1|C^\perp)\phi(B_2|C^\perp) + \phi(A_2|C^\perp)\phi(B_1|C^\perp) + \phi(A_2|C^\perp)\phi(B_2|C^\perp) \\ -\phi(A_2|C^\perp) - \phi(B_1|C^\perp) \leq 0 \end{aligned} \qquad (34)$$

Multiplying (34) by $\phi(C)$ and by $\phi(C^\perp)$, respectively, adding the two resulting inequalities and using (32) we obtain

$$-1 \leq \phi(A_1 \wedge B_1) + \phi)(A_1 \wedge B_2) + \phi(A_2 \wedge B_2) - \phi(A_2 \wedge B_2) - \phi(A_2) - \phi(B_1) \leq 0 \qquad (35)$$

The inequality (35) is known as the Clauser-Horne inequality, and it is known not to hold for every quantum state ϕ that predicts correlation between four projections (see eg. Summers ([1990a,b])).

The crucial assumption in the above derivation of the inequality (35) is that C is a common cause *for all four* correlated pairs; i.e. that C is a *common* common cause, shared by the different correlations. Without this assumption Bell's inequality *cannot* be derived. But there does not seem to be any obvious reason why common causes should also be common common causes, whether of quantum or of any other sort of correlations. In our interpretation of Reichenbach's notion of common cause there is nothing that would justify such an assumption; hence if such an assumption is made, it needs extra support. It should be mentioned that while the impossibility of (non-probabilistic) *common* common causes of the (non-probabilistic) GHZ correlations has been proved in the paper Belnap and Szabó ([1996]), it remains open in that paper whether non-common common causes of the GHZ correlations exist.

6 Open questions

To decide whether a particular event structure is common cause incomplete does not seem to be a trivial matter. In a previous paper the problem was raised whether the event structure defined by (algebraic relativistic) quantum field theory is common cause incomplete, and this problem is still open (see Rédei [1997, 1998]) for a precise formulation of the question). It is even conceivable that the explicitly formulated axioms that define algebraic quantum field theory – and thereby the set of all events – are not strong enough to decide the issue. Such undecidability would be especially interesting.

It also is an open mathematical question whether one can have *common cause closed probability spaces*, where a probability space is said to be "common cause closed" iff for every pair of correlated events there exists in that probability space a proper common cause of the correlation in Reichenbach's sense. It is important here that common cause means a *proper* common cause. This qualification on non-triviality is necessary; for it is not difficult to show (using standard tensor product procedures) that every quantum probability space can be enlarged in such a way that the enlarged quantum probability space is common cause closed in the improper, formal sense that for every correlated pair (A, B) there exists at least one $C \leq A, C \neq A$ such that $\phi(C) = \phi(A)$ and such a C satisfies the Reichenbach conditions.

Whether or not common cause closed probability spaces exist, it is not reasonable to expect a probabilistic physical theory to be common cause closed. This is because one does not expect to have a proper common cause explanation of probabilistic correlations that arise as a consequence of a direct physical influence between the correlated events, or which are due to some logical relations between the correlated events. One would want to have a common cause explanation of correlations only between events that are neither directly causally related, nor do they stand in a straightforward "logical consequence relation" to each other. Thus a precise notion of causal (in)dependence, different from the notion of the standard probabilistic independence (correlation) is needed. Perhaps the notion of "logical independence" (see the refs. Rédei ([1995a,b], [1998])) can be useful here. Two orthocomplemented sub-lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ of an orthomodular lattice $\mathcal{L}$ are called *logically independent* if $A \wedge B \neq 0$ for any $A \in \mathcal{L}_1$ and $B \in \mathcal{L}_2$. This is an independence condition that obtains between spacelike separated local systems in the sense of (algebraic) quantum field theory; so this logical independence condition can be viewed as a lattice theoretic formulation of "separatedness" of certain events. It seems reasonable then to expect a probabilistic physical theory $(\mathcal{L}, \mu)$ to be common cause closed with respect to the correlated elements in every two, logically independent, commuting sublattices $\mathcal{L}_1, \mathcal{L}_2$. It is not known if this is possible.

As we have argued at the end of Section 5, quantum correlations cannot have a *common* common cause in general. This raises the question of whether *classical* correlations exist that cannot have a *common* common cause. Note that Bell's inequality is never violated in classical probability theory, so one cannot obtain an answer to this question by referring to the violation of Bell's inequality. In this sense the behavior of correlations with respect to Bell's inequality is not a good indicator of (non)existence of *common* common causes. (That Bell's inequality is not a good indicator of (non)existence of non-common common causes, is clear from Propositions 2 and 3, since these propositions show that common causes can be constructed both in the classical and in the quantum case, wehereas Bell's inequality is always satisfied in the classical case and it does not hold in the quantum case.) To return to the question of this paragraph, the answer is *yes*. That is, there exist *classical* probability spaces containing different pairs of events that are correlated with respect to a fixed probability measure and which cannot have a common common cause. (The proof of this assertion will be published elswhere.)

Acknowledgement

Work supported by AKP, by OTKA (contract numbers: T 025841 and F 023447) and by the Dibner Institute MIT, where M. Rédei was staying during the 1997/98 academic year as a Resident Fellow. We also wish to thank the audiences – especially N. Belnap, R. Clifton and W. Salmon – of two seminars held in October 1997 at the Center for Philosophy of Science, Pittsburgh University.

Proof of Proposition 2

Step 1 By Stone's theorem we may assume without loss of generality that $\mathcal{S}$ is a field of subsets of a set Ω. Let Ω_1 and Ω_2 be two identical copies of Ω, distinguishable by the indices 1 and 2, and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the corresponding two copies of $\mathcal{S}$:

$$\begin{aligned} \Omega_i &\equiv \{(x,i) \mid x \in \Omega\} && i = 1,2 \\ \mathcal{S}_i &\equiv \{\{(x,i) \mid x \in X\} \mid X \in \mathcal{S}\} && i = 1,2 \end{aligned}$$

Let h_i $(i = 1,2)$ denote the Boolean algebra isomorphisms between $\mathcal{S}$ and $\mathcal{S}_i$ $(i = 1,2)$:

$$\mathcal{S} \ni X \mapsto h_i(X) = \{(x,i) \mid x \in X\} \qquad i = 1,2$$

Furthermore, let $\mathcal{S}'$ be the set of subsets of $\Omega_1 \cup \Omega_2$ having the form $h_1(X) \cup h_2(Y)$, i.e.

$$\mathcal{S}' \equiv \{h_1(X) \cup h_2(Y) \mid X, Y \in \mathcal{S}\}$$

We claim that $\mathcal{S}'$ is a Boolean algebra of subsets of $\Omega_1 \cup \Omega_2$ with respect to the usual set theoretical operations $\cup, \cap, \perp$ and that the map h defined by

$$h(X) \equiv h_1(X) \cup h_2(X) \qquad X \in \mathcal{S} \tag{36}$$

is an embedding of $\mathcal{S}$ into $\mathcal{S}'$. To see that $\mathcal{S}'$ is a Boolean algebra one only has to show that $\mathcal{S}'$ is closed with respect to the set theoretical operations of join, meet and complement, and this is a straightforward consequence of the fact that $\mathcal{S}$, itself being a Boolean algebra with respect to the set theoretical operations, is closed with respect to these operations. Checking the homomorphism properties of h is a routine task.

We now define a measure μ' on $\mathcal{S}'$ that has the property (29). Let r_i $(i = 1,2,3,4)$ be arbitrary four real numbers in the interval $[0,1]$. One can define a μ' measure on $\mathcal{S}'$ by

$$\begin{aligned} \mu'(h_1(X) \cup h_2(Y)) \equiv\; & r_1\mu(X \cap (A \cap B)) + r_2\mu(X \cap (A \cap B^\perp)) \\ & + r_3\mu(X \cap (A^\perp \cap B)) + r_4\mu(X \cap (A^\perp \cap B^\perp)) \\ & + (1-r_1)\mu(Y \cap (A \cap B)) + (1-r_2)\mu(Y \cap (A \cap B^\perp)) \\ & + (1-r_3)\mu(Y \cap (A^\perp \cap B)) + (1-r_4)\mu(Y \cap (A^\perp \cap B^\perp)) \end{aligned}$$

Since $A \cap B, A \cap B^\perp, A^\perp \cap B$ and $A^\perp \cap B^\perp$ are disjoint and their union is Ω it follows that

$$\mu'(h_1(X) \cup h_2(X)) = \mu'(h(X)) = \mu(X) \qquad X \in \mathcal{S}$$

Hence $(\mathcal{S}'\mu')$ is indeed an extension of the original probability space $(\mathcal{S}, \mu)$.

Step 2 Choose any value of the parameters t, s within the bounds specified by (27)-(28), and consider the corresponding numbers $r_{A|C} = t$, $r_{B|C} = s$ and $r_C, r_{A|C^\perp}, r_{B|C^\perp}$, the latter ones defined by (24)-(26). We claim that the probability space $(\mathcal{S}', \mu')$ constructed in Step 1 is a common cause extension of $(\mathcal{S}, \mu)$ of type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ with respect to the correlation between A and B, if the numbers r_i $(i = 1,2,3,4)$ defining μ' by the formula (37) are given by

$$\begin{aligned} r_1 &= \frac{r_C r_{A|C} r_{B|C}}{\mu(A \wedge B)} \\ r_2 &= \frac{r_C r_{A|C}(1 - r_{B|C})}{\mu(A) - \mu(A \wedge B)} \\ r_3 &= \frac{r_C r_{B|C}(1 - r_{A|C})}{\mu(B) - \mu(A \wedge B)} \\ r_4 &= \frac{r_C(1 - r_{A|C} - r_{B|C} + r_{A|C} r_{B|C})}{\mu(A^\perp \wedge B^\perp)} \end{aligned}$$

To show that $(\mathcal{S}', \mu')$ is a common cause completion of $(\mathcal{S}, \mu)$ one only has to display a proper common cause C in $\mathcal{S}'$ of the correlation. We claim that $C = h_1(\Omega) \cup h_2(\emptyset)$ is a proper common cause. Clearly,

C is a *proper* common cause if it is a common cause. To see that C is a common cause indeed one can check by explicit calculation that the following hold

$$\begin{aligned}
\mu'(h_1(\Omega)\cup h_2(\emptyset)) &= r_C && (37)\\
\mu'((h_1(A)\cup h_2(A))|(h_1(\Omega)\cup h_2(\emptyset)) &= r_{A|C} && (38)\\
\mu'((h_1(B)\cup h_2(B))|(h_1(\Omega)\cup h_2(\emptyset)) &= r_{B|C} && (39)\\
\mu'((h_1(A)\cup h_2(A))|[h_1(\Omega)\cup h_2(\emptyset)]^{\perp}) &= r_{A|C^{\perp}} && (40)\\
\mu'((h_1(B)\cup h_2(B))|[h_1(\Omega)\cup h_2(\emptyset)]^{\perp}) &= r_{B|C^{\perp}} && (41)
\end{aligned}$$

Since the numbers $r_{A|C}, r_{B|C}, r_C, r_{A|C^{\perp}}, r_{B|C^{\perp}}$ were chosen so that they satisfy the conditions (12)-(18), C is indeed a common cause.

Assume now that there exists a common cause completion $(\mathcal{S}^{n-1}, \mu^{n-1})$ of $(\mathcal{S}, \mu)$ that contains a common cause C_i of each correlation $\mu(A_i \wedge B_i) > \mu(A_i)\mu(B_i)$ $(i = 1, \ldots n-1)$. Consider the correlation between A_n and B_n. By repeating the two steps (Step 1-Step 2) one can construct a common cause completion $(\mathcal{S}^n, \mu^n)$ of $(\mathcal{S}^{n-1}, \mu^{n-1})$ that contains a common cause C_n of the correlation between A_n and B_n. To complete the induction one only has to see that $(\mathcal{S}^n, \mu^n)$ also contains common causes of each of the correlations between $h_n(A_i), h_n(B_i)$ $(i = 1, \ldots n-1)$, where h_n is the Boolean algebra embedding of $\mathcal{S}^{n-1}$ into $\mathcal{S}^n$. But $h_n(C_i)$ $(i = 1, 2 \ldots)$ are clearly common causes of the correlations between $h_n(A_i), h_n(B_i)$ $(i = 1, \ldots n-1)$ because h_n is a homomorphism preserving μ_{n-1}.

Proof of Proposition 3

Step 1 We may assume without loss of generality that $\mathcal{M}$ is acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}\oplus\mathcal{H}$ be the direct sum of $\mathcal{H}$ with itself and consider the map h_2 defined by

$$\mathcal{P}(\mathcal{M}) \ni X \mapsto h_2(X) \equiv X \oplus X \in \mathcal{P}(\mathcal{H}\oplus\mathcal{H})$$

Then h_2 is an embedding of $\mathcal{P}(\mathcal{M})$ as an orthomodular lattice into the orthomodular lattice $\mathcal{P}(\mathcal{H}\oplus\mathcal{H})$. Let ϕ_2 be a state defined on $\mathcal{P}(\mathcal{H}\oplus\mathcal{H})$ by the density matrix $\frac{1}{2}W \oplus \frac{1}{2}W$, where W is the density matrix belonging to ϕ. Clearly, ϕ_2 has the property

$$\phi_2(h_2(X)) = \phi(X) \qquad X \in \mathcal{P}(\mathcal{M}) \tag{42}$$

So the probability space $(\mathcal{P}(\mathcal{H}\oplus\mathcal{H}), \phi_2)$ is an extension of $(\mathcal{P}(\mathcal{M}), \phi)$. Since every density matrix is a convex combination of (possibly countably infinite number of) one dimensional projections, there exist vectors $\psi_k \in (\mathcal{H}\oplus\mathcal{H})$ and non-negative numbers λ_k $(k = 1\ldots)$ such that $\sum_k^{\infty}\lambda_k = 1$ and

$$\frac{1}{2}W \oplus \frac{1}{2}W = \sum_k^{\infty} \lambda_k P_{\psi_k} \tag{43}$$

(Here, and in what follows, P_ξ denotes the projection to the one dimensional subspace spanned by the Hilbert space vector ξ.)

Step 2 Let $\mathcal{H}'$ be the set of functions $g\colon \mathbb{N} \to \mathcal{H}\oplus\mathcal{H}$ from the naturals $\mathbb{N}$ such that $sup_n\|g(n)\|_2 < \infty$, where $\|\xi\|_2$ is the norm of $\xi \in \mathcal{H}\oplus\mathcal{H}$. Then the set $\mathcal{H}'$ is a complex linear space with the pointwise operations $((\kappa_1 g_1 + \kappa_2 g_2)(n) = \kappa_1 g_1(n) + \kappa_2 g_2(n))$. It is elementary to check that $\mathcal{H}'$ becomes a Hilbert space with the scalar product $\langle,\rangle'$ defined by

$$\langle g_1, g_2\rangle' \equiv \sum_{k=1}^{\infty} \lambda_k \langle g_1(k), g_2(k)\rangle_2$$

where $\langle,\rangle_2$ is the scalar product in $\mathcal{H}\oplus\mathcal{H}$ and the numbers λ_k are taken from (43). The map $h'\colon \mathcal{B}(\mathcal{H}\oplus\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ defined by

$$(h'(Q))g(n) = Q(g(n)) \qquad n \in \mathbb{N},\; g \in \mathcal{H}'$$

is an algebra homomorphism from the algebra $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of all bounded operators on $\mathcal{H}\oplus\mathcal{H}$ into the algebra $\mathcal{B}(\mathcal{H}')$ of all bounded operators on $\mathcal{H}'$; furthermore, $h'(Q_1) \neq h'(Q_2)$ if $Q_1 \neq Q_2$, and routine reasoning shows that if $Q_n \in \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ is a sequence of operators such that for some $Q \in \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ we have $Q_n\xi \to Q\xi$ for all $\xi \in \mathcal{H}\oplus\mathcal{H}$, then $(h'(Q_n))g \to h'(Q)g$ for all $g \in \mathcal{H}'$. This means that h'

is continuous in the respective strong operator topologies in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ and $\mathcal{B}(\mathcal{H}')$. It follows that if A, B are two projections in $\mathcal{P}(\mathcal{H}\oplus\mathcal{H})$, then

$$\begin{aligned} h'(A\wedge B) &= h'(s-\lim(AB)^n) = s-\lim(h'(AB)^n) = s-\lim(h'(A)h'(B))^n \\ &= h'(A)\wedge h'(B) \end{aligned}$$

It follows then that the restriction of h' to $\mathcal{P}(\mathcal{H}\oplus\mathcal{H})$ is an embedding of $\mathcal{P}(\mathcal{H}\oplus\mathcal{H})$ as an orthomodular lattice into the orthomodular lattice $\mathcal{P}(\mathcal{H}')$. Let ξ_l ($l=1,\dots dim(\mathcal{H}\oplus\mathcal{H})$) be an orthonormal basis in $\mathcal{H}\oplus\mathcal{H}$. Then the elements $g_{kl}\in\mathcal{H}'$ defined by

$$g_{kl}(n) = \begin{cases} \delta_{nk}\frac{1}{\sqrt{\lambda_k}}\xi_l & \text{if } \lambda_k\neq 0 \\ 0 & \text{if } \lambda_k = 0 \end{cases}$$

form an orthonormal basis in $\mathcal{H}'$. (δ_{nk} denotes the Kronecker symbol.) The linear operator W' on $\mathcal{H}'$ defined by

$$(W'g)(n) \equiv \lambda_n P_{\psi_n} g(n) \qquad n\in\mathbb{N},\ g\in\mathcal{H}'$$

is easily seen to be a density matrix, hence it defines a state ϕ' on $\mathcal{P}(\mathcal{H}')$. For $A\in\mathcal{P}(\mathcal{H}\oplus\mathcal{H})$ we have

$$\begin{aligned} \phi'(h'(A)) &= Tr(W'h'(A)) = \sum_{k=1}^{\infty}\sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})} \langle g_{kl}, W'h'(A)g_{kl}\rangle' \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})}\sum_{n=1}^{\infty}\lambda_n\langle g_{kl}(n), \left[W'h'(A)g_{kl}\right](n)\rangle_2 \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})}\sum_{n=1}^{\infty}\lambda_n\langle\delta_{nk}\frac{1}{\sqrt{\lambda_k}}\xi_l\lambda_n, P_{\psi_n}A\frac{1}{\sqrt{\lambda_k}}\xi_l\rangle_2 \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})}\lambda_k\langle\xi_l, P_{\psi_k}A\xi_l\rangle_2 \\ &= \sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})}\langle\xi_l, \left[\sum_{k=1}^{\infty}\lambda_k P_{\psi_k}\right]\xi_l\rangle_2 = Tr(WA) = \phi(A) \end{aligned}$$

So $(\mathcal{P}(\mathcal{H}'),\phi')$ is an extension of the quantum probability space $(\mathcal{P}(\mathcal{H}\oplus\mathcal{H}),\phi_2)$. It follows that $(\mathcal{P}(\mathcal{H}'),\phi')$ is an extension of the probability space $(\mathcal{P}(\mathcal{M}),\phi)$, where the embedding $h:\mathcal{P}(\mathcal{M})\to\mathcal{P}(\mathcal{H}')$ is given by $h'\circ h_2$.

We now claim that for any given pair of events (A,B) in $\mathcal{P}(\mathcal{M})$ that are correlated with respect to ϕ and for any given set of admissible numbers $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ the probability space $(\mathcal{P}(\mathcal{H}'),\phi')$ constructed above contains a proper Reichenbachian common cause C of type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$ of the correlation.

Indeed, given the numbers $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$, the event C defined below by (44) is a common cause of type $(r_C, r_{A|C}, r_{B|C}, r_{A|C^\perp}, r_{B|C^\perp})$.

$$C := P_\alpha \vee P_\beta \vee P_\gamma \vee P_\delta \tag{44}$$

where P_α, P_β, P_γ and P_δ are projections in $\mathcal{P}(\mathcal{H}')$ defined by

$$\begin{aligned} (P_\alpha g)(k) &= P_{\alpha_k}g(k) \\ (P_\beta g)(k) &= P_{\beta_k}g(k) \\ (P_\gamma g)(k) &= P_{\gamma_k}g(k) \\ (P_\delta g)(k) &= P_{\delta_k}g(k) \end{aligned} \qquad k\in\mathbb{N}, g\in\mathcal{H}'$$

where P_{α_k}, P_{β_k}, P_{γ_k} and P_{δ_k} are projections in $\mathcal{H}\oplus\mathcal{H}$ defined by

$$\alpha_k = \begin{cases} 0 & \text{if } \langle\psi_k,(A\wedge B^\perp\oplus A\wedge B^\perp)\psi_k\rangle_2 = 0 \\ \cos\omega_k^\alpha\alpha_k^1 + \sin\omega_k^\alpha\alpha_k^2 & \text{if } \langle\psi_k,(A\wedge B^\perp\oplus A\wedge B^\perp)\psi_k\rangle_2 \neq 0 \end{cases}$$

where

$$\begin{aligned} \cos^2\omega_k^\alpha &= r_{A|Z}(1-r_{B|Z})r_Z\frac{\langle\psi_k,(A\wedge B^\perp\oplus A\wedge B^\perp)\psi_k\rangle_2}{\phi(A^\perp\wedge B)} \\ \alpha_k^1 &= \frac{1}{\langle\psi_k,(A\wedge B^\perp\oplus A\wedge B^\perp)\psi_k\rangle_2}(A\wedge B^\perp\oplus A\wedge B^\perp)\psi_k \\ \alpha_k^1 &\perp \alpha_k^2\in A\wedge B^\perp\oplus A\wedge B^\perp \\ \left\langle\alpha_k^2,\alpha_k^2\right\rangle_2 &= 1 \end{aligned}$$

$$\beta_k = \begin{cases} 0 & \text{if } \langle\psi_k, (A\wedge B \oplus A\wedge B)\psi_k\rangle_2 = 0 \\ \cos\omega_k^\beta \beta_k^1 + \sin\omega_k^\beta \beta_k^2 & \text{if } \langle\psi_k, (A\wedge B \oplus A\wedge B)\psi_k\rangle_2 \neq 0 \end{cases}$$

where

$$\begin{aligned} \cos^2\omega_k^\beta &= r_{A|Z} r_{B|Z} r_C \frac{\langle\psi_k, (A\wedge B\oplus A\wedge B)\psi_k\rangle_2}{\phi(A\wedge B)} \\ \beta_k^1 &= \frac{1}{\langle\psi_k, (A\wedge B\oplus A\wedge B)\psi_k\rangle_2}(A\wedge B\oplus A\wedge B)\psi_k \\ \beta_k^1 &\perp \beta_k^2 \in A\wedge B\oplus A\wedge B \\ \left\langle\beta_k^2, \beta_k^2\right\rangle_2 &= 1 \end{aligned}$$

$$\gamma_k = \begin{cases} 0 & \text{if } \langle\psi_k, (A^\perp\wedge B \oplus A^\perp\wedge B)\psi_k\rangle_2 = 0 \\ \cos\omega_k^\gamma \gamma_k^1 + \sin\omega_k^\gamma \gamma_k^2 & \text{if } \langle\psi_k, (A^\perp\wedge B \oplus A^\perp\wedge B)\psi_k\rangle_2 \neq 0 \end{cases}$$

where

$$\begin{aligned} \cos^2\omega_k^\gamma &= r_{B|Z}(1-r_{A|Z}) r_Z \frac{\langle\psi_k, (A^\perp\wedge B\oplus A^\perp\wedge B)\psi_k\rangle_2}{\phi(A^\perp\wedge B)} \\ \gamma_k^1 &= \frac{1}{\langle\psi_k, (A^\perp\wedge B\oplus A^\perp\wedge B)\psi_k\rangle_2}(A^\perp\wedge B\oplus A^\perp\wedge B)\psi_k \\ \gamma_k^1 &\perp \gamma_k^2 \in A^\perp\wedge B\oplus A^\perp\wedge B \\ \left\langle\gamma_k^2, \gamma_k^2\right\rangle' &= 1 \end{aligned}$$

$$\delta_k = \begin{cases} 0 & \text{if } \langle\psi_k, (A^\perp\wedge B^\perp \oplus A^\perp\wedge B^\perp\psi_k\rangle_2) = 0 \\ \cos\omega_k^\delta \delta_k^1 + \sin\omega_k^\delta \delta_k^2 & \text{if } \langle\psi_k, (A^\perp\wedge B^\perp \oplus A^\perp\wedge B^\perp)\psi_k\rangle_2 \neq 0 \end{cases}$$

where

$$\begin{aligned} \cos^2\omega_k^\delta &= r_Z(1-r_{A|Z}-r_{B|Z}+r_{A|Z}r_{B|Z}) \frac{\langle\psi_k, (A^\perp\wedge B^\perp\oplus A^\perp\wedge B^\perp)\psi_k\rangle_2}{\phi(A^\perp\wedge B^\perp)} \\ \delta_k^1 &= \frac{1}{\langle\psi_k, (A^\perp\wedge B^\perp\oplus A^\perp\wedge B^\perp)\psi_k\rangle_2}(A^\perp\wedge B^\perp\oplus A^\perp\wedge B^\perp)\psi_k \\ \delta_k^1 &\perp \delta_k^2 \in A^\perp\wedge B^\perp\oplus A^\perp\wedge B^\perp \\ \left\langle\delta_k^2, \delta_k^2\right\rangle_2 &= 1 \end{aligned}$$

Since $\alpha_k^1, \beta_k^1, \gamma_k^1$ and δ_k^1 are unit vectors, α_k, β_k, γ_k, δ_k also are unit vectors in $\mathcal{H}\oplus\mathcal{H}$. The element C commutes with $h(A)$ and $h(B)$ because

$$\begin{aligned} P_{\alpha_k} &< A\wedge B^\perp\oplus A\wedge B^\perp \\ P_{\beta_k} &< A\wedge B\oplus A\wedge B \\ P_{\gamma_k} &< A^\perp\wedge B\oplus A^\perp\wedge B \\ P_{\delta_k} &< A^\perp\wedge B^\perp\oplus A^\perp\wedge B^\perp \end{aligned}$$

So to show that C is indeed a common cause of the said type we just have to show that the following hold

$$\phi'(C) = r_C \tag{45}$$

$$\frac{\phi'(h'(A)\wedge C)}{\phi'(C)} = r_{A|C} \tag{46}$$

$$\frac{\phi'(h'(B)\wedge C)}{\phi'(C)} = r_{B|C} \tag{47}$$

$$\frac{\phi'(h'(A)\wedge C^\perp)}{\phi'(C^\perp)} = r_{A|C^\perp} \tag{48}$$

$$\frac{\phi'(h'(B)\wedge C^\perp)}{\phi'(C^\perp)} = r_{B|C^\perp} \tag{49}$$

We show (46)-(49) by showing first that the following hold

$$\phi'(h'(A) \wedge C) = r_{A|C} r_C \quad (50)$$

$$\phi'(h'(B) \wedge C) = r_{B|C} r_C \quad (51)$$

$$\phi'(h'(A) \wedge C^{\perp}) = r_{A|C^{\perp}} r_{C^{\perp}} \quad (52)$$

$$\phi'(h'(B) \wedge C^{\perp}) = r_{B|C^{\perp}} r_{C^{\perp}} \quad (53)$$

which imply (46)-(49) if $\phi'(C) = r_C$.

We compute $\phi'(h'(A) \wedge C)$ first.

$$\phi'(h(A) \wedge C) = \phi'((h'(A) \wedge (P_\alpha \vee P_\beta) = \phi'(P_\alpha \vee P_\beta) = \phi'(P_\alpha) + \phi'(P_\beta)$$

where we have used the fact that P_α P_β P_γ and P_δ are pairwise orthogonal projections. We can compute $\phi'(P_\alpha)$ as follows

$$\begin{aligned}
\phi'(P_\alpha) = Tr(W'P_\alpha) &= \sum_{k=1}^{\infty} \sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})} \langle g_{kl}, W' P_\alpha g_{kl} \rangle' \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})} \sum_{n=1}^{\infty} \lambda_n \langle \delta_{nk} \frac{1}{\sqrt{\lambda_k}} \xi_l, \lambda_n P_{\psi_n} P_{\alpha_n} \delta_{nk} \frac{1}{\sqrt{\lambda_k}} \xi_l \rangle_2 \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{dim(\mathcal{H}\oplus\mathcal{H})} \lambda_k \langle \xi_l, P_{\psi_k} P_{\alpha_k} \xi_l \rangle_2 = Tr_2(P_{\psi_k} P_{\alpha_k}) \\
&= \sum_{k=1}^{\infty} \lambda_k |\langle \psi_k, \alpha_k \rangle|^2 = \sum_{k=1}^{\infty} \lambda_k |\langle \psi_k, \cos\omega_k^\alpha \alpha_k^1 + \sin\omega_k^\alpha \alpha_k^2 \rangle|^2 \\
&= \sum_{k=1}^{\infty} \lambda_k \big[\cos\omega_k^\alpha \langle \psi_k, \alpha_k^1 \rangle + \sin\omega_k^\alpha \langle \psi_k, \alpha_k^2 \rangle \big]^2
\end{aligned} \quad (54)$$

The second term in (54) is equal to zero because $\alpha_k^1 \perp \alpha_k^2$ by definition, and in view of the definition of α_k^1 and α_k^2 we can also write

$$\begin{aligned}
0 = \langle \alpha_k^1, \alpha_k^2 \rangle_2 &= \langle \big[A \wedge B^{\perp} \oplus A \wedge B^{\perp} \big] \psi_k, \alpha_k^2 \rangle_2 \\
&= \langle \psi_k, \big[A \wedge B^{\perp} \oplus A \wedge B^{\perp} \big] \alpha_k^2 \rangle_2 = \langle \psi_k, \alpha_k^2 \rangle_2
\end{aligned}$$

Since we have

$$\langle \psi_k, \alpha_k^1 \rangle_2 = 1$$

it follows that

$$\begin{aligned}
\phi'(P_\alpha)) &= \sum_{k=1}^{\infty} \lambda_k \cos^2 \omega_k^\alpha \\
&= \sum_{k=1}^{\infty} r_{A|C}(1 - r_{B|C}) r_C \frac{\langle \psi_k, (A \wedge B^{\perp} \oplus A \wedge B^{\perp}) \psi_k \rangle_2}{\phi(A \wedge B^{\perp})} \\
&= r_{A|C}(1 - r_{B|C}) r_C \frac{1}{\phi(A \wedge B^{\perp})} \sum_{k=1}^{\infty} \lambda_k \langle \psi_k, (A \wedge B^{\perp} \oplus A \wedge B^{\perp}) \psi_k \rangle_2 \\
&= r_{A|C}(1 - r_{B|C}) r_C \frac{1}{\phi(A \wedge B^{\perp})} \phi(A \wedge B^{\perp}) = r_{A|C}(1 - r_{B|C}) r_C
\end{aligned}$$

In a completely analogous way one obtains

$$\phi'(P_\beta)) = r_{A|C} r_{B|C} r_C$$

And so

$$\phi'(P_\alpha) + \phi'(P_\beta) = r_{A|C} r_C$$

which is (50). A similar derivation shows that

$$\phi'(h(B) \wedge C) = r_{B|C} r_C$$

which is (51). Since C commutes with both $h(A)$ and $h(B)$ we can write

$$\begin{aligned}\phi'(h(A)\wedge C^{\perp}) &= \phi'(h(A)) - \phi'(h(A)\wedge C) = \phi(A) - r_{A|C}r_C = r_{A|C^{\perp}}r_{C^{\perp}}\\ \phi'(h(B)\wedge C^{\perp}) &= \phi'(h(B)) - \phi'(h(B)\wedge C) = \phi(B) - r_{B|C}r_C = r_{B|C^{\perp}}r_{C^{\perp}}\end{aligned}$$

which establishes (52) and (53). One can compute $\phi'(P_\gamma)$ and $\phi'(P_\delta)$ exactly the same way as $\phi'(P_\alpha)$ and $\phi'(P_\beta)$, and one obtains

$$\begin{aligned}\phi'(C) &= \phi'(P_\alpha \vee P_\beta \vee P_\gamma \vee P_\delta) = \phi'(P_\alpha) + \phi'(P_\beta) + \phi'(P_\gamma) + \phi'(P_\delta)\\ &= r_{A|C}(1-r_{B|C})r_C + r_{A|C}r_{B|C}r_C + r_{B|C}(1-r_{A|C})r_C\\ &+ r_C(1-r_{A|C}-r_{B|C}+r_{A|C}r_{B|C}) = r_C\end{aligned}$$

which shows (45), which, together with (50)-(53) proves (45)-(49).

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