A Duality for Distributive Unimodal Logic

Adam Přenosil [1]

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod vodárenskou věží 271/2, 182 00 Praha

Abstract

We introduce distributive unimodal logic as a modal logic of binary relations over posets which naturally generalizes the classical modal logic of binary relations over sets. The relational semantics of this logic is similar to the relational semantics of intuitionistic modal logic and positive modal logic, but it generalizes both of these by placing no restrictions on the accessibility relation. We introduce a corresponding quasivariety of distributive lattices with modal operators and prove a completeness theorem which embeds each such algebra in the complex algebra of its canonical modal frame. We then extend this embedding to a duality theorem which unifies and generalizes the duality theorems for intuitionistic modal logic obtained by A. Palmigiano and for positive modal logic obtained by S. Celani and A. Jansana. As a corollary to this duality theorem, we obtain a Hennessy-Milner theorem for bi-intuitionistic unimodal logic, which is the expansion of distributive unimodal logic by bi-intuitionistic connectives.

Keywords: Modal logic, distributive modal logic, intuitionistic modal logic, positive modal logic, bi-intuitionistic modal logic, duality theory.

1 Introduction

Suppose that we are given a semantically defined expansion of classical propositional logic, such as classical modal propositional logic, and we want to add some intuitionistic flavour to it. How could we go about doing that?

Let us adopt the simple perspective that a logic is given by a category of *frames* and a contravariant functor from the category of frames to some category of algebras which assigns a *complex algebra* to each frame. The consequence relation of the logic is identified with the quasiequational logic of this class of complex algebras. A completeness theorem then consists in axiomatizing the quasivariety generated by this class.

In a set-based semantics, the frames are sets possibly with additional structure and the complex algebras are expansions of the Boolean algebra of all subsets of a frame. In poset-based semantics, the frames are posets possibly with additional structure and the complex algebras are expansions of the dis-

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tributive lattice of all *up*sets of a frame. Our task is thus to define a natural poset-based companion to a given set-based logic.

From an algebraic point of view, the natural thing to do is to drop the Boolean negation from the signature and consider the (quasi)variety generated by such reducts. This is the strategy followed by Dunn [5]. By contrast, our strategy will be to start on the semantic side.

We wish to extend an operation on the powerset of a set-based frame, say the binary operation $\circ$, to an operation on the set of all upsets of a poset-based frame with the same underlying set. There are two natural ways of doing this. One option is to define the poset-based operation as the *upper interior* of the set-based operation, that is, $u \in a \circ_+ b$ if and only if $v \in a \circ b$ for all $v \geq u$. The other option is to define the poset-based operation as the *upper closure* of $a \circ b$, that is, $u \in a \circ_- b$ if and only if $v \in a \circ b$ for some $v \leq u$.

The first of these alternatives is used in the semantics of intuitionistic logic: the intuitionistic implication $\rightarrow$ is defined as the upper interior of classical implication. The second alternative is used in bi-intuitionistic logic, introduced by Rauszer [11,12]. This logic expands intuitionistic logic by a *co-implication* connective, denoted as $a \rightarrowtail b$ here, such that $u \in a \rightarrowtail b$ if and only if there is some $v \leq u$ such that $v \notin a$ and $v \in b$. It is easy to see that for the lattice connectives $\wedge$ and $\vee$, the two extensions coincide.

The goal of the present paper is to investigate the poset-based companion of classical unimodal logic given by the connectives $\wedge$, $\vee$, $\top$, $\bot$, $\Box_+$, $\Diamond_-$ and its expansion by $\rightarrow$ and $\rightarrowtail$. We call this logic *distributive unimodal logic (DUML)* and we call the expansion *bi-intuitionistic unimodal logic (BiIUML)*. Our main result is a completeness theorem for DUML with respect to a suitable quasi-variety of modal algebras and its extension to a duality between these algebras and suitably topologized modal frames. In combination with known results, this also yields a completeness and duality theorems for BiIUML.

These results generalize and unify known completeness and duality results for intuitionistic modal logic and positive modal logic. We therefore briefly introduce these to provide some context for the present work. We also describe the relationship between the distributive unimodal logic presented here and the distributive modal logic of Gehrke et al. [8].

Intuitionistic modal logic (IML), introduced and axiomatized by Fischer Servi [6,7], expands intuitionistic propositional logic by a pair of modalities $\Box$ and $\Diamond$ which generalize the box and diamond modalities of classical modal logic. The frames for this logic are Kripke frames for intuitionistic propositional logic (that is, posets) equipped with a binary accessibility relation R required to satisfy the conditions $\geq \circ R \subseteq R \circ \geq$ and $R \circ \leq \subseteq \leq \circ R$. The box and diamond modalities are then defined asymmetrically: $u \in \Box A$ if and only if $u(\leq \circ R)v$ implies $v \in A$, whereas $u \in \Diamond A$ if and only if uRv for some $v \in A$. Observe that the box operator is persistent by definition, while the condition $\geq \circ R \subseteq R \circ \geq$ is needed to ensure the persistence of the diamond operator.

Unlike in classical modal logic, the box and diamond modalities of IML are not mutually interdefinable. Their interaction is captured by the axioms

$\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$ and $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$. The completeness theorem for IML was later extended to a duality by Palmigiano [9]. For a (slightly dated) overview of research on IML, see [14] or [13].

Positive modal logic (PML), introduced and axiomatized by Dunn [5], is the negation-free fragment of classical modal logic. A poset-based [2] relational semantics for PML analogical to the semantics of IML was provided by Celani and Jansana [3], who formulated a Priestley-style duality for PML [4]. This semantics requires that the accessibility relation R satisfy the conditions $\geq \circ R \subseteq R \circ \geq$ and $\leq \circ R \subseteq R \circ \leq$, which ensure that both of the modal operators are local with respect to partial order.

Both IML and PML thus place non-trivial requirements on the accessibility relation, and furthermore that IML does not retain the symmetry between $\Box$ and $\Diamond$ present in classical modal logic. The present paper then answers the natural question: what is the modal logic of *semantically dual* box and diamond operators defined by *arbitrary* binary relations over posets?

Finally, a modal logic which does not fit the above semantic template of posets equipped with a single relation is distributive modal logic (DML) introduced by Gehrke et al. [8], which in addition to $\Box$ and $\Diamond$ contains primitive modal operators corresponding to the classical modalities $\Box\neg$ and $\Diamond\neg$. (We chose the name distributive *uni*modal logic precisely to differentiate the present approach from DML.) This logic has a poset-based relational semantics, which differs from the semantics of IML and PML in that each of the modal operators has *its own* accessibility relation, denoted $R_\Box$ and $R_\Diamond$ in the case of $\Box$ and $\Diamond$.

As regards, the relationship between DML and DUML, one can adopt either of the following positions. Either one can consider DML to be a fragment of a generalization of classical *multimodal* logic and consider DUML to be a generalization of classical *unimodal* logic, ideally with the proviso that the *full* language of DUML should include connectives corresponding to $\Box\neg$ and $\Diamond\neg$ which are not considered here. Alternatively, one can view DUML as a special case of DML for frames where $R_\Box$ and $R_\Diamond$ are uniquely determined by their intersection, that is, $\leq \circ (R_\Box \cap R_\Diamond) \circ \leq \subseteq R_\Box$ and $\geq \circ (R_\Box \cap R_\Diamond) \circ \geq \subseteq R_\Diamond$.

The outline of the paper is as follows. In Section 2, we introduce some basic notation and briefly overview known facts concerning the representation of distributive lattice and (bi-)Heyting algebras. In Sections 3 and 4, we introduce DUML via its relational semantics, define the corresponding quasivariety of modal algebras and prove the soundness of the algebraic semantics. In Section 5 defines the canonical frame of a modal algebra and obtain a completeness theorem which embeds each modal algebra in the complex algebra of its canonical frame. In Section 6, we use the completeness theorem to show that the class of all frames for DUML is not definable by modal quasiequations relative to the class of all frames for DML. In Section 7, we consider the conditions that IML and PML place on R and prove that in our setting they correspond to canon-

[2] Strictly speaking, their semantics is formulated in terms of pre-ordered sets rather than posets, but the distinction is irrelevant for our purposes.

ical equations. The known completeness and duality theorems for these logics are therefore covered by the completeness and duality theorems for DUML. Finally, in Section 8, we formulate a duality for DUML based on the bitopological duality for distributive lattices by Bezhanishvili et al. [1] and derive a Hennessy-Milner theorem for BiIUML as a corollary.

2 Preliminaries

By relations we will mean *binary* relations, and by distributive lattices and their homomorphisms we will mean *bounded* distributive lattices and bound-preserving homomorphisms. The category of posets and monotone functions will be denoted Pos and the category of distributive lattices and their homomorphisms will be denoted DLat.

A *Heyting algebra* is a distributive lattice expanded by a binary operation $\to$ such that $a \wedge b \leq c$ if and only if $b \leq a \to c$. Dually, a *co-Heyting algebra* is a distributive lattice expanded by binary operation $\succ\!-$ (called co-implication) such that $a \leq b \vee c$ if and only if $c \succ\!- a \leq b$. A *bi-Heyting algebra* is then both a Heyting algebra and a co-Heyting algebra. It is well known that all of these classes of algebras are varieties.

Let $(W, \leq)$ be a poset and R be a relation on W. The *opposite* of $(W, \leq)$ is the poset $(W, \geq)$. We denote the diagonal relation on W by Δ_W. Given $U \subseteq W$, $R_{|U}$ denotes the restriction of R to U, $R^{-1}[U] = \{w \in W | wRu \text{ for some } u \in U\}$ and $R^{-1}[u] = R^{-1}[\{u\}]$ for $u \in W$. By a *pre-order on* $(W, \leq)$ we mean a pre-order on the W which extends $\leq$.

The upward (downward) closure of $U \subseteq W$ will be denoted $U^{\uparrow}$ ($U^{\downarrow}$). We say that U is *convex* if $U^{\uparrow} \cap U^{\downarrow} \subseteq U$. We call $U^{\uparrow} \cap U^{\downarrow}$ the *convex closure* of U.

Given any relation $R \subseteq U \times V$, we use the abbreviations $R^{\uparrow} = \leq \circ R \circ \leq$ and $R^{\downarrow} = \geq \circ R \circ \geq$. This notation will be used often in the following. We call a relation *convex* if $R = R^{\uparrow} \cap R^{\downarrow}$.

Let $(U, \leq)$ and $(V, \sqsubseteq)$ be posets. A relation $R \subseteq U \times V$ is *monotone (antitone)* if it is an upset (downset) of $U^{op} \times V$. A *monotone relation pair* $(R_{\uparrow}, R_{\downarrow})$ between U and V is a pair consisting of a monotone relation $R_{\uparrow} \subseteq U \times V$ and an antitone relation $R_{\downarrow} \subseteq U \times V$.

The *kernel* of a monotone function $f : (U, \leq) \to (V, \sqsubseteq)$ is the pre-order $\leq_f$ on $(U, \leq)$ such that $u \leq_f v$ if and only if $f(u) \sqsubseteq f(v)$. A *monotone function pair* from U and V is a pair of monotone functions from U and V with a common codomain. The *kernel pair* of a monotone function pair $f : U \to W$, $g : V \to W$ is the pair consisting of a monotone relation $\sigma_{\uparrow} : U \times V$ such that $u\sigma_{\uparrow}v$ if and only if $f(u) \leq g(v)$ holds in W and an antitone relation $\sigma_{\downarrow} : U \times V$ such that $u\sigma_{\downarrow}v$ if and only if $f(u) \geq g(v)$ holds in W. The kernels of monotone functions from a poset W are precisely the pre-orders on W and the kernel pairs of monotone function pairs from U and V are precisely the monotone relation pairs between U and V.

We now formulate known representation theorems for distributive lattices and (bi-)Heyting algebras in the notation which we will later use to formulate the completeness theorem for distributive unimodal logic. Given a poset W, let

W^+ be the distributive lattice of all upsets of W (called the *complex algebra* of W), and given a monotone function $f : U \to V$, let f^+ be the homomorphism of distributive lattices $f^{-1} : V^+ \to U^+$. This assignment yields a functor $(-)^+ : \mathsf{Pos} \to \mathsf{DLat}$.

In the opposite direction, given a distributive lattice $\mathbf{A}$, let $\mathbf{A}_\bullet$ be the poset of all prime filters on $\mathbf{A}$ ordered by inclusion, and given a homomorphism of distributive lattices $h : \mathbf{A} \to \mathbf{B}$, let $h_\bullet$ be the monotone function $h^{-1} : \mathbf{B}_\bullet \to \mathbf{A}_\bullet$. This assignment again yields a functor $(-)_\bullet : \mathsf{DLat} \to \mathsf{Pos}$.

Given a distributive lattice $\mathbf{A}$, define the function $\eta_{\mathbf{A}} : \mathbf{A} \to (\mathbf{A}_\bullet)^+$ such that $\mathcal{U} \in \eta_{\mathbf{A}}(a)$ if and only if $a \in \mathcal{U}$. Then $\eta_{\mathbf{A}}$ is an embedding of distributive lattices. Each distributive lattice is thus a sublattice of the complex algebra of some poset, hence the semantically defined quasiequational logic of complex algebras of posets distributive lattices coincides with the algebraically defined quasiequational logic of distributive lattices.

These constructions extend to Heyting (bi-Heyting) algebras and posets with suitably defined morphisms. We say that a *Heyting morphism* of posets $f : (U, \leq) \to (V, \sqsubseteq)$ is a monotone function such that $f(u) \sqsubseteq v'$ implies that $u \leq v$ and $f(v) = v'$ for some $v \in U$, and a *bi-Heyting morphism* of posets is a function $f : (U, \leq) \to (V, \sqsubseteq)$ such that both $f : (U, \leq) \to (V, \sqsubseteq)$ and $f : (U, \geq) \to (V, \sqsupseteq)$ are Heyting morphisms of posets. The functors $(-)^+$ and $(-)_\bullet$ then extend to contravariant functors between the category of posets with Heyting (bi-Heyting) morphisms and the category of Heyting (bi-Heyting) algebras and $\eta_{\mathbf{A}}$ is an embedding of Heyting (bi-Heyting) algebras if $\mathbf{A}$ is a Heyting (bi-Heyting) algebra.

3 Modal frames

Let us start by introducing the relational semantics for distributive unimodal logic which was outlined in the introduction. We will define frames for this logic, their complex algebras and their p-morphisms. We then describe subframes, simulation pairs and bisimulations.

Definition 3.1 A *(modal) frame* $\mathcal{F} = (W, \leq, R)$ is a poset $(W, \leq)$ equipped with a convex relation R.

Recall that R is convex if $R^\uparrow \cap R^\downarrow \subseteq R$. It is easily seen from the definition of p-morphisms of modal frames given below, which only cares about $R^\uparrow$ and $R^\downarrow$, that a poset equipped with a non-convex relation R is in fact p-isomorphic to the same poset equipped with the convex closure of R. No loss of generality is thus involved in assuming that R is convex. This assumption will later allow us to obtain the right correspondence results for modal axioms.

The distributive lattice $(W, \leq)^+$ of upsets of a modal frame $\mathcal{F} = (W, \leq, R)$ can now be equipped with a box operator

$$\Box_R(A) = \{u \in X \mid \text{ if } uR^\uparrow v, \text{ then } v \in A\}$$

and with a diamond operator

$$\Diamond_R(A) = \{u \in X \mid uR^\downarrow v \text{ for some } v \in A\}.$$

More compactly, $\Diamond_R(A) = (R^{\downarrow})^{-1}[A]$ and $W \setminus \Box_R(A) = (R^{\uparrow})^{-1}[W \setminus A]$. The *complex algebra* of $\mathcal{F}$ is the expansion $\mathcal{F}^+$ of the distributive lattice $(W, \leq)^+$ by these two operators . The *Heyting (bi-Heyting) complex algebra* $\mathcal{F}^+_{\rightarrow}$ $(\mathcal{F}^+_{\rightarrow,\succ\!-})$ of $\mathcal{F}$ is the unique expansion of $\mathcal{F}^+$ by Heyting (bi-Heyting) connectives.

A *(Heyting, bi-Heyting) modal quasiequation* is a quasiequation in the language of distributive lattices expanded by $\Box$ and $\Diamond$ (and $\rightarrow$, and $\rightarrow$ and $\succ\!-$). A modal frame $\mathcal{F}$ is a *model* of a set of (Heyting, bi-Heyting) modal quasiequations Σ if $\mathcal{F}^+ \vDash \Sigma$. The class of all models of Σ will be denoted $\mathrm{Mod}(\Sigma)$.

For any class of modal frames K, let $\mathsf{K}^+ = \{\mathcal{F}^+ \mid \mathcal{F} \in \mathsf{K}\}$ and likewise for $\mathsf{K}^+_{\rightarrow}$ and $\mathsf{K}^+_{\rightarrow,\succ\!-}$. The *distributive unimodal logic* of K is then the quasiequational logic of K^+, the *intuitionistic unimodal logic* of K is the quasiequational logic of $\mathsf{K}^+_{\rightarrow}$, and the *bi-intuitionistic unimodal logic* of K is the quasiequational logic of $\mathsf{K}^+_{\rightarrow,\succ\!-}$. If K is not specified, we take it to be the class of *all* modal frames. These quasiequational logics can be translated into the form of sequent calculi via the correspondence between the sequent $\Gamma \vdash \Delta$ and the inequality $\bigwedge \Gamma \leq \bigvee \Delta$.

Definition 3.2 Given modal frames $\mathcal{F} = (U, \leq, R)$ and $\mathcal{G} = (V, \sqsubseteq, S)$, a *p-morphism* $f : \mathcal{F} \to \mathcal{G}$ is a monotone function $f : (U, \leq) \to (V, \sqsubseteq)$ such that

- $uR^{\uparrow}v$ implies $f(u)S^{\uparrow}f(v)$,
- $uR^{\downarrow}v$ implies $f(u)S^{\downarrow}f(v)$,
- $f(u)S^{\uparrow}v'$ implies $uR^{\uparrow}v$ for some $v \in U$ such that $f(v) \leq v'$,
- $f(u)S^{\downarrow}v'$ implies $uR^{\downarrow}v$ for some $v \in U$ such that $f(v) \geq v'$.

A *Heyting (bi-Heyting) p-morphism* of modal frames is a p-morphism of modal frames which is a Heyting (bi-Heyting) morphism of posets.

Proposition 3.3 *If $f : \mathcal{F} \to \mathcal{G}$ is a p-morphism of modal frames, then $f^+ = f^{-1} : \mathcal{G}^+ \to \mathcal{F}^+$ is a homomorphism of their complex algebras.*

Proof. The proof is standard. $\Box$

The category of modal frames and their p-morphisms will be denoted ModPos. We now describe the images and kernels of p-morphisms.

Definition 3.4 A *subframe* of a frame $\mathcal{F} = (W, \leq, R)$ is a frame $(U, \leq_{|U}, R_{|U})$ for some $U \subseteq W$ such that

- if $u \in U$ and $uR^{\uparrow}v \in W$, then $w \leq v$ for some $w \in U$,
- if $u \in U$ and $uR^{\downarrow}v \in W$, then $w \geq v$ for some $w \in U$.

A *Heyting subframe* of $\mathcal{F}$ is a frame $(U, \leq_{|U}, R_{|U})$ for some $U \subseteq W$ such that U is closed under $\leq$ and R and

- if $u \in U$ and $uR^{\downarrow}v \in W$, then $w \geq v$ for some $w \in U$.

A *bi-Heyting subframe* of $\mathcal{F}$ is a frame $(U, \leq_{|U}, R_{|U})$ for some $U \subseteq W$ such that U is closed under $\leq$, $\geq$ and R.

Proposition 3.5 *The (Heyting, bi-Heyting) subframes of a modal frame $\mathcal{F}$ are precisely the images of (Heyting, bi-Heyting) p-morphisms into $\mathcal{F}$.*

Proof. It suffices to inspect the definition of p-morphisms. $\square$

Definition 3.6 Let $\mathcal{F} = (U, \leq, R)$ and $\mathcal{G} = (V, \sqsubseteq, S)$ be modal frames. A *simulation pair* between modal frames $\mathcal{F} = (U, \leq, R)$ and $\mathcal{G} = (V, \sqsubseteq, S)$ is a monotone relation pair $(\sigma_{\uparrow}, \sigma_{\downarrow})$ between $(U, \leq)$ and $(V, \sqsubseteq)$ such that

$$\sigma_{\uparrow} \circ S^{\uparrow} \subseteq R^{\uparrow} \circ \sigma_{\uparrow},$$
$$\sigma_{\downarrow} \circ S^{\downarrow} \subseteq R^{\downarrow} \circ \sigma_{\downarrow}.$$

A *Heyting simulation pair* is a simulation pair such that

$$\sigma_{\uparrow} \circ \sqsubseteq \, \subseteq \, \leq \circ \, \sigma_{\uparrow}.$$

A *bi-Heyting simulation pair* is a Heyting simulation pair such that

$$\sigma_{\downarrow} \circ \sqsupseteq \, \subseteq \, \geq \circ \, \sigma_{\downarrow}.$$

A *bisimulation* is a convex relation σ such that $(\sigma^{\uparrow}, \sigma^{\downarrow}) = (\leq \circ \sigma \circ \leq, \geq \circ \sigma \circ \geq)$ is a bi-Heyting simulation pair.

Observe that bi-Heyting simulation pairs are precisely pairs of the form $(\sigma^{\uparrow}, \sigma^{\downarrow})$ for some convex σ.

Proposition 3.7 *The (Heyting, bi-Heyting) simulation pairs between modal frames $\mathcal{F}$ and $\mathcal{G}$ are precisely the kernel pairs of (Heyting, bi-Heyting) p-morphism pairs from $\mathcal{F}$ and $\mathcal{G}$.*

Proof. The p-morphism pairs from $\mathcal{F}$ and $\mathcal{G}$ are in bijective correspondence with the p-morphisms from the naturally defined disjoint union of $\mathcal{F}$ and $\mathcal{G}$. The definition of (Heyting, bi-Heyting) simulation pairs is then just a repackaging of the definition of (Heyting, bi-Heyting) p-morphisms. $\square$

4 Modal algebras

Having described the relational semantics of distributive unimodal logic, we introduce the corresponding class of modal expansions of distributive lattices and show that it includes the complex algebras of modal frames.

Definition 4.1 A *box operator* on a distributive lattice $\mathbf{A}$ is a unary function $\Box : \mathbf{A} \to \mathbf{A}$ such that $\Box(a \wedge b) = \Box a \wedge \Box b$ and $\Box\top = \top$. A *diamond operator* on a distributive lattice $\mathbf{A}$ is a unary function $\Diamond : \mathbf{A} \to \mathbf{A}$ such that $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ and $\Diamond\bot = \bot$.

A *modal algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \top, \bot, \Box, \Diamond)$ such that $\mathbf{A}_{\mathsf{DLat}} = (A, \wedge, \vee, \top, \bot)$ is a distributive lattice, $\Box$ is a box operator on $\mathbf{A}_{\mathsf{DLat}}$, $\Diamond$ is a diamond operator on $\mathbf{A}_{\mathsf{DLat}}$, and $\mathbf{A}$ satisfies the *positive modal law*

$$\Diamond b \leq \Box a \vee c \Rightarrow \Diamond b \leq \Diamond(a \wedge b) \vee c$$

and the *negative modal law*

$$\Diamond a \wedge c \leq \Box b \Rightarrow \Box(a \vee b) \wedge c \leq \Box b.$$

The *opposite* of $\mathbf{A}$ is the modal algebra $\mathbf{A}^{op} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box)$.

A *Heyting (bi-Heyting) modal algebra* is an algebra which is both a modal algebra and a Heyting (bi-Heyting) algebra.

Modal algebras form a quasivariety. On bi-Heyting modal algebras, the positive and negative modal laws are equivalent to the equations

$$\Diamond(a \wedge b) \rightarrowtail \Diamond b \leq \Box a \rightarrowtail \Diamond b,$$
$$\Diamond a \to \Box b \leq \Box(a \vee b) \to \Box b,$$

hence bi-Heyting modal algebras in fact form a variety. The category of modal algebras and their homomorphisms will be denoted $\mathsf{ModDLat}$.

Note that moving from $\mathbf{A}$ to $\mathbf{A}^{op}$ transforms the positive modal law into the negative one and vice versa. We will often implicitly appeal to this symmetry between boxes and diamonds to cut our proofs down to half.

Proposition 4.2 (Soundness) *Every complex algebra $\mathcal{F}^+$ is a modal algebra.*

Proof. It suffices to verify that the positive modal law holds in $\mathcal{F}^+$. Suppose that $\Diamond_R a \subseteq \Box_R b \cup C$ and $u \in \Diamond_R a$, $u \notin c$ for some $a, b, c \in \mathcal{F}^+$. Then there are some v, w such that $u \geq vRw$ and $w \in a$. But then $v \in \Diamond_R a$ and $v \notin c$, therefore $v \in \Box_R b$, $w \in a \cap b$, and $u \in \Diamond_R(a \cap b)$. □

5 Canonical frames

We now show that each modal algebra can be embedded in the complex algebra of its suitably defined canonical frame. This means that distributive unimodal logic is precisely the quasiequational logic of modal algebras. Since by [10] quasiequational consequence can be captured by a simple calculus, this result deserves to be called a completeness theorem.

To define the *canonical frame* $\mathbf{A}_\bullet$ of a modal algebra $\mathbf{A}$, we equip the poset of prime filters on $\mathbf{A}$ with the accessibility relation $R_\mathbf{A}$ such that

$$\mathcal{U} R_\mathbf{A} \mathcal{V} \text{ if and only if } \Box a \in \mathcal{U} \text{ implies } a \in \mathcal{V} \text{ and } a \in \mathcal{V} \text{ implies } \Diamond a \in \mathcal{U}.$$

We use the notation $R^{\supseteq}_\mathbf{A} = \supseteq \circ R_\mathbf{A} \circ \supseteq$ and $R^{\subseteq}_\mathbf{A} = \subseteq \circ R_\mathbf{A} \circ \subseteq$.

The algebra $(\mathbf{A}_\bullet)^+$ is called the *canonical extension* of the algebra $\mathbf{A}$. We say that a quasiequation σ is *canonical* if $\mathbf{A} \vDash \sigma$ implies $(\mathbf{A}_\bullet)^+ \vDash \sigma$. A set of quasiequations Σ is canonical if each $\sigma \in \Sigma$ is canonical. The canonicity of Σ then implies that Σ axiomatizes the logic of $\mathrm{Mod}(\Sigma)$.

We will need some basic constructions to prove a crucial lemma about prime filters on modal algebras. Given filters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbf{A}$, we define the filter $\mathcal{U} \wedge \mathcal{V} = \{u \wedge v \in \mathbf{A} \mid u \in \mathcal{U} \text{ and } v \in \mathcal{V}\}$, the filter $\Diamond \mathcal{V} = \{a \in \mathbf{A} \mid \Diamond v \leq a \text{ for some } v \in \mathcal{V}\}$, the filter $\Box^{-1}\mathcal{V} = \{a \in \mathbf{A} \mid \Box a \in \mathcal{V}\}$, and the filter $\mathcal{U}_\downarrow(\mathcal{V}) = \{a \in \mathbf{A} \mid v \leq a \vee u_- \text{ for some } v \in \mathcal{V}, u_- \notin \mathcal{U}\}$. A $\mathcal{U}_\downarrow$*-filter* is then a filter $\mathcal{V}$ such that $\mathcal{U}_\downarrow(\mathcal{V}) = \mathcal{V}$. Observe that a directed union of $\mathcal{U}_\downarrow$-filters is again a $\mathcal{U}_\downarrow$-filter.

Lemma 5.1 *Let $\mathcal{U}$ be a prime filter and $\mathcal{V}$ be a filter on $\mathbf{A}$. If $\Diamond \mathcal{V} \subseteq \mathcal{U}$, then $\mathcal{U} R^{\supseteq}_\mathbf{A} \mathcal{V}'$ for some prime $\mathcal{V}' \supseteq \mathcal{V}$.*

Proof. We need to find suitable prime filters $\mathcal{U}'$ and $\mathcal{V}'$ such that $\mathcal{U} \supseteq \mathcal{U}' R_{\mathbf{A}} \mathcal{V}' \supseteq \mathcal{V}$. For $\mathcal{V}'$, we use Zorn's lemma to take a maximal filter extending $\mathcal{V}$ such that $\Diamond\mathcal{V}' \subseteq \mathcal{U}$. Such a filter is prime: if $a, b \notin \mathcal{V}'$, then there are some $v'_1, v'_2 \in \mathcal{V}'$ such that $\Diamond(a \wedge v'_1), \Diamond(b \wedge v'_2) \notin \mathcal{U}$, hence $\Diamond(a \wedge v'), \Diamond(b \wedge v') \notin \mathcal{U}$ for $v' = v'_1 \wedge v'_2$, $\Diamond(a \wedge v') \vee \Diamond(b \wedge v') = \Diamond((a \vee b) \wedge (v'_1 \wedge v'_2)) \notin \mathcal{U}$ and $a \vee b \notin \mathcal{V}'$.

The inclusion $\Diamond(\mathcal{V}' \wedge \Box^{-1}\mathcal{U}_{\downarrow}(\Diamond\mathcal{V}')) \subseteq \mathcal{U}$ now holds: if $v'_1 \in \mathcal{V}'$ and $\Diamond v'_2 \leq \Box a \vee u^-$ for some $v'_2 \in \mathcal{V}'$, $u^- \notin \mathcal{U}$, then $\Diamond(v'_1 \wedge v'_2) \leq \Diamond(a \wedge v'_1 \wedge v'_2) \vee u^-$, hence $\Diamond(a \wedge v'_1 \wedge v'_2) \in \mathcal{U}$. Since $\mathcal{V}'$ was chosen to be a maximal filter such that $\Diamond\mathcal{V}' \subseteq \mathcal{U}$, it follows that in fact $\Box^{-1}\mathcal{U}_{\downarrow}(\Diamond\mathcal{V}') \subseteq \mathcal{V}'$.

It now suffices to use Zorn's lemma to extend $\mathcal{U}_{\downarrow}(\Diamond\mathcal{V}')$ to a maximal $\mathcal{U}_{\downarrow}$-filter $\mathcal{U}'$ such that $\Box^{-1}\mathcal{U}' \subseteq \mathcal{V}'$. Such a filter is prime: if $a, b \notin \mathcal{U}'$, then $a \wedge u' \leq \Box v_1^-$ and $b \wedge u' \leq \Box v_2^-$ for some $v_1^-, v_2^- \notin \mathcal{V}'$, $u' \in \mathcal{U}'$, hence $(a \vee b) \wedge u' \leq \Box v_1^- \vee \Box v_2^- \leq \Box(v_1^- \vee v_2^-)$. Since $\mathcal{V}'$ is prime, $v_1^- \vee v_2^- \notin \mathcal{V}'$, thus $a \vee b \notin \mathcal{U}'$. □

Corollary 5.2 *$\mathcal{U} R^{\supseteq}_{\mathbf{A}} \mathcal{V}$ if and only if $a \in \mathcal{V}$ implies $\Diamond a \in \mathcal{U}$. $\mathcal{U} R^{\subseteq}_{\mathbf{A}} \mathcal{V}$ if and only if $\Box a \in \mathcal{U}$ implies $a \in \mathcal{V}$.*

Embedding a modal algebra into the complex algebra of its canonical frame is now straightforward. Recall that the function $\eta_{\mathbf{A}} : \mathbf{A} \to (\mathbf{A}_{\bullet})^+$ such that $\mathcal{U} \in \eta_{\mathbf{A}}(a)$ if and only if $a \in \mathcal{U}$ for each prime filter $\mathcal{U}$ on $\mathbf{A}$ is an embedding of distributive lattices.

Theorem 5.3 (Completeness) *For every modal algebra $\mathbf{A}$, $\mathbf{A}_{\bullet}$ is a modal frame and $\eta_{\mathbf{A}} : \mathbf{A} \to (\mathbf{A}_{\bullet})^+$ is an embedding of modal algebras.*

Proof. The convexity of $R_{\mathbf{A}}$ follows from Corollary 5.2. It thus suffices to prove that $\eta_{\mathbf{A}}$ preserves diamonds. It is clear that $\Diamond_{R_{\mathbf{A}}} \eta_{\mathbf{A}}(a) \subseteq \eta_{\mathbf{A}}(\Diamond a)$ by the definition of $R_{\mathbf{A}}$. The opposite inclusion is precisely Lemma 5.1. □

Lemma 5.1 also shows that the assignment $(-)_{\bullet}$ in fact extends to a functor $(-)_{\bullet} : \mathsf{ModDLat} \to \mathsf{ModPos}$. Recall that for any homomorphism of distributive lattices $h : \mathbf{A} \to \mathbf{B}$, we define the monotone function $h_{\bullet} : \mathbf{B}_{\bullet} \to \mathbf{A}_{\bullet}$ as h^{-1}.

Proposition 5.4 *If $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism of modal algebras, then $h_{\bullet} : \mathbf{B}_{\bullet} \to \mathbf{A}_{\bullet}$ is a p-morphism of modal frames.*

Proof. We only verify that $\mathcal{U} R^{\supseteq}_{\mathbf{B}} \mathcal{V}$ implies $h_{\bullet}(\mathcal{U}) R^{\supseteq}_{\mathbf{A}} h_{\bullet}(\mathcal{V})$ and that $h_{\bullet}(\mathcal{U}) R^{\supseteq}_{\mathbf{A}} \mathcal{V}'$ implies $\mathcal{U} R^{\supseteq}_{\mathbf{B}} \mathcal{V}$ for some prime filter $\mathcal{V}$ on $\mathbf{B}$ such that $h_{\bullet}(\mathcal{V}) \supseteq \mathcal{V}'$.

Suppose that $\mathcal{U}_{\mathbf{B}} R_{\mathbf{B}} \mathcal{V}_{\mathbf{B}}$. If $a \in h^{-1}[\mathcal{V}_{\mathbf{B}}]$, then $h(a) \in \mathcal{V}_{\mathbf{B}}$, $\Diamond h(a) = h(\Diamond a) \in \mathcal{U}_{\mathbf{B}}$, $\Diamond a \in h^{-1}[\mathcal{U}_{\mathbf{B}}]$ and dually for $\Box$. Therefore $h^{-1}[\mathcal{U}_{\mathbf{B}}] R_{\mathbf{A}} h^{-1}[\mathcal{V}_{\mathbf{B}}]$.

Now suppose that $h^{-1}[\mathcal{U}_{\mathbf{B}}] R^{\supseteq}_{\mathbf{A}} \mathcal{V}_{\mathbf{A}}$. Then $\Diamond\mathcal{V}_{\mathbf{A}} \subseteq h^{-1}[\mathcal{U}_{\mathbf{B}}]$, hence $\Diamond h[\mathcal{V}_{\mathbf{A}}] \subseteq h[\Diamond\mathcal{V}_{\mathbf{A}}] \subseteq \mathcal{U}_{\mathbf{B}}$. By Lemma 5.1, there is a prime filter $\mathcal{W}_{\mathbf{B}} \supseteq h[\mathcal{V}_{\mathbf{A}}]$ such that $\mathcal{U}_{\mathbf{B}} R^{\supseteq}_{\mathbf{B}} \mathcal{W}_{\mathbf{B}}$. Clearly $h^{-1}[\mathcal{W}_{\mathbf{B}}] \supseteq \mathcal{V}_{\mathbf{A}}$ for any such $\mathcal{W}_{\mathbf{B}}$. □

Theorem 5.3 and Proposition 5.4 extend to (bi-)Heyting modal algebras and (bi-)Heyting p-morphisms: if $\mathbf{A}$ and $\mathbf{B}$ are (bi-)Heyting algebras and $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism of (bi-)Heyting algebras, then $\eta_{\mathbf{A}}$ is in fact an embedding of (bi-)Heyting algebras and $h_{\bullet}$ is a (bi-)Heyting morphism of posets.

6 Relationship with distributive modal logic

As a corollary to the completeness theorem, we can show that the class of all frames for DML such that $R_\Diamond$ and $R_\Box$ are generated by the same relation is not definable by (bi-Heyting) modal quasiequations. Recall that a frame for DML was introduced by Gehrke et al. [8] as a poset $(W, \leq)$ equipped with a pair of accessibility relations $R_\Diamond$ and $R_\Box$ such that $\geq \circ R_\Diamond \circ \geq \subseteq R_\Diamond$ and $\leq \circ R_\Box \circ \leq$.[3] The box (diamond) operator of DML is defined in terms of $R_\Box$ ($R_\Diamond$) precisely as in classical modal logic. We say that $R_\Diamond$ and $R_\Box$ are generated by the same underlying relation if and only if $(R_\Diamond \cap R_\Box)^\uparrow = R_\Diamond$ and $(R_\Box \cap R_\Diamond)^\downarrow = R_\Box$.

Proposition 6.1 *The positive modal law* $\Diamond b \leq \Box a \vee c \Rightarrow \Diamond b \leq \Diamond(a \wedge b) \vee c$ *holds in a frame for DML if and only if for each* $u R_\Diamond v$ *there are* $u' \leq u$ *and* $v' \geq v$ *such that* $u R_\Diamond v'$, $u' R_\Diamond v$ *and* $u' R_\Box v'$.

Proof. We only show the harder (left-to-right) direction. Suppose that $u R_\Diamond v$ but there are no $u' \leq u$ and $v' \geq v$ such that $u' R_\Diamond v$, $u R_\Diamond v'$ and $u' R_\Box v'$. Then let $w \in b$ if and only if $w \geq v$, let $w \in a$ if and only if there is some $u' \leq u$ such that $u' R_\Diamond v$ and $u' R_\Box w$, and let $w \notin c$ if and only if $w \leq u$. It follows that $\Diamond b \leq \Box a \vee c$, but $u \in \Diamond b$, $u \notin \Diamond(a \wedge b)$ and $u \notin c$. □

Proposition 6.2 *There is no set of (bi-Heyting) modal quasiequations* Σ *such that* Σ *holds in a frame for DML if and only if* $R_\Diamond$ *and* $R_\Box$ *are generated by the same underlying relation.*

Proof. By Theorem 5.3, any such Σ is a quasiequational consequence of the axioms of modal algebras. It therefore suffices to build a frame for DML where $R_\Diamond$ and $R_\Box$ are not generated by the same underlying relation but which satisfies the relational conditions of Proposition 6.1 and its dual. This can be done in a brute-force way.

Let $\mathcal{F} = \mathcal{F}_0$ be any frame for DML which at least contains some pair of points connected by some accessibility relation. Let $\mathcal{F}_{i+1}$ be the frame obtained from $\mathcal{F}_i$ by adding for each $u R_\Diamond v$ a pair of points u', v' which satisfy exactly the condition of Proposition 6.1 and dually for each $u R_\Box v$. Finally, let $\mathcal{F}_\omega = \bigcup_{i \in \omega} \mathcal{F}_i$. The frame $\mathcal{F}_\omega$ satisfies the condition of Proposition 6.1 by construction and it is easy to see that $u(R_\Diamond \cap R_\Box)v$ in $\mathcal{F}_\omega$ if and only if $u(R_\Diamond \cap R_\Box)v$ already in $\mathcal{F}$. The relations $R_\Diamond$ and $R_\Box$ on $\mathcal{F}_\omega$ are therefore *never* generated by the same underlying relation. □

7 Modal locality conditions

We have obtained a completeness via canonicity theorem for the logic of modal frames with arbitrary (without loss of generality convex) relations. Let us now investigate what happens when we impose some conditions relating $\leq$ and R.

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[3] They in fact use the opposite order convention, that is, $\leq \circ R_\Diamond \circ \leq \subseteq R_\Diamond$.

In particular, we consider the following conditions:

$$\geq \circ R \subseteq R \circ \geq$$
$$R \circ \leq \subseteq \leq \circ R$$
$$\leq \circ R \subseteq R \circ \leq$$
$$R \circ \geq \subseteq \geq \circ R$$

The first two of these are exactly the conditions that IML imposes on R, while PML requires the first and third conditions. The following observation made already by Gehrke et al. [8] justifies calling them *locality conditions.*

Proposition 7.1 *A frame satisfies the condition* $\geq \circ R \subseteq R \circ \geq$ *if and only if* $\Diamond_R(A) = \{u \in W \mid uRv \text{ for some } v \in A\}$. *A frame satisfies the condition* $\leq \circ R \subseteq R \circ \leq$ *if and only if* $\Box_R(A) = \{u \in W \mid uRv \text{ implies } v \in A\}$.

The other two conditions can be seen as locality conditions for the backward-looking box and diamond operators which we are not considered in this paper.

In the present framework, these classes of modal frames can be axiomatized by canonical modal equations. The completeness and duality theorems proved here therefore generalize known completeness and duality theorems established for IML by Fischer-Servi [6] and Palmigiano [9] and for PML by Dunn [5] and Celani and Jansana [3].

In the propositions below, we use the notation $\mathcal{F} \vDash \geq \circ R \subseteq R \circ \geq$ to mean that the inclusion holds in $\mathcal{F}$.

Proposition 7.2 $\mathcal{F} \vDash \geq \circ R \subseteq R \circ \geq$ *if and only if* $\mathcal{F}^+ \vDash \Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

Proof. If $u \geq vRw$ but uRx implies $x \ngeq w$, let $y \in A$ if and only if $uR^{\uparrow}y$ and let $y \in B$ if and only if $y \geq w$. Then $u \in \Box A$ and $u \in \Diamond B$. If $u \in \Diamond(A \cap B)$, then there is some $z \geq w$ such that $u(\geq \circ R)z$ and $u(\leq \circ R \circ \leq)z$, hence uRz by the convexity of R, contradicting the assumption that uRz implies $z \ngeq w$.

Vice versa, if $u \in \Box A$ and $u \in \Diamond B$, then $u(\geq \circ R)v$ for some $v \in B$, hence $uRw \geq v$ for some w. But then $w \in A$ and $w \in B$, hence $u \in \Diamond(A \cap B)$. □

Proposition 7.3 $\mathcal{F} \vDash R\circ\leq \subseteq \leq\circ R$ *if and only if* $\mathcal{F}^+ \vDash \Diamond a \to \Box b \leq \Box(a \to b)$.

Proof. If $uRv \leq w$ but xRw implies $x \nleq u$, let $y \in A$ if and only if $y \geq w$ and $y \in B$ if and only if $y \nleq w$. Clearly $u \notin \Box(A \to B)$. But if $z \in \Diamond A$ and $z \notin \Box B$ for some $z \geq u$, then $z(\geq \circ R \circ \geq)w$ and $z(\leq \circ R \circ \leq)w$, hence zRw by the convexity of R, contradicting the assumption that zRw implies $z \ngeq u$.

Vice versa, if $u \notin \Box(A \to B)$, then $u(R \circ \leq)v$ for some $v \in A$, $v \notin B$, hence $u \leq wRv$ for some w. But then $w \in \Diamond A$, $w \notin \Box B$, hence $u \notin \Diamond A \to \Box B$. □

Dualizing these two propositions yields the following.

Proposition 7.4 $\mathcal{F} \vDash \leq \circ R \subseteq R \circ \leq$ *if and only if* $\mathcal{F}^+ \vDash \Box(a \vee b) \leq \Box a \vee \Diamond b$.

Proposition 7.5 $\mathcal{F} \vDash R\circ\geq \subseteq \geq\circ R$ *if and only if* $\mathcal{F}^+ \vDash \Diamond(a \succ b) \leq \Box a \succ \Diamond b$.

To prove that the above equations are canonical, it now suffices to show that if $\mathbf{A}$ satisfies the equation, $\mathbf{A}_\bullet$ satisfies the corresponding relational condition.

Proposition 7.6 *If* $\mathbf{A} \vDash \Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$*, then* $\mathbf{A}_\bullet \vDash {\geq} \circ R \subseteq R \circ {\geq}$.

Proof. If $\mathcal{U}(\geq \circ R)\mathcal{V}$, then $\Diamond\mathcal{V} \subseteq \mathcal{U}$. Extend $\mathcal{V}$ to a maximal filter $\mathcal{W}$ such that $\Diamond\mathcal{W} \subseteq \mathcal{U}$. The filter $\mathcal{W}$ is prime and if $a \notin \mathcal{W}$, then $\Diamond(a \wedge w) \notin \mathcal{U}$ for some $w \in \mathcal{W}$. But then $\Diamond w \in \mathcal{U}$, hence $\Box a \notin \mathcal{U}$ because $\Box a \wedge \Diamond w \leq \Diamond(a \wedge w)$. □

Proposition 7.7 *If* $\mathbf{A} \vDash \Diamond a \to \Box b \leq \Box(a \to b)$*, then* $\mathbf{A}_\bullet \vDash R \circ {\leq} \subseteq {\leq} \circ R$.

Proof. If $\mathcal{U}(R \circ \leq)\mathcal{V}$, then $\Box^{-1}\mathcal{U} \subseteq \mathcal{V}$. Extend $\mathcal{U}$ to a maximal filter $\mathcal{W}$ such that $\Box^{-1}\mathcal{W} \subseteq \mathcal{V}$. The filter $\mathcal{W}$ is prime and if $\Diamond a \notin \mathcal{W}$, then $\Diamond a \wedge w \leq \Box b$ for some $w \in \mathcal{W}$, $b \notin \mathcal{V}$, hence $w \leq \Diamond a \to \Box b \leq \Box(a \to b)$. But then $a \to b \in \mathcal{V}$, hence $a \notin \mathcal{V}$. □

Again, dually we obtain the following two propositions.

Proposition 7.8 *If* $\mathbf{A} \vDash \Box(a \vee b) \leq \Box a \vee \Diamond b$*, then* $\mathbf{A}_\bullet \vDash {\leq} \circ R \subseteq R \circ {\leq}$.

Proposition 7.9 *If* $\mathbf{A} \vDash \Diamond(a \succ b) \leq \Box a \succ \Diamond b$*, then* $\mathbf{A}_\bullet \vDash R \circ {\geq} \subseteq {\geq} \circ R$.

Observe that the equation $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ implies the positive modal law and the equation $\Diamond a \to \Box b \leq \Box(a \to b)$ implies the negative modal law. We conjecture that the variety of modal algebras relatively axiomatized by $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ and $\Box(a \vee b) \leq \Box a \wedge \Diamond b$ (that is, the variety of positive modal algebras introduced by Dunn [5]) is in fact the largest variety of modal algebras, and the variety relatively axiomatized by $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ and $\Diamond a \to \Box b \leq \Box(a \vee b) \to \Box b$ is the largest variety of Heyting modal algebras.

8 Duality for modal algebras

We now extend the completeness theorem for modal algebras to a duality based on the Priestley duality for distributive lattices. In order to derive a Hennessy-Milner theorem as a corollary, we in fact formulate a dual *adjunction* between modal algebras and "compactly branching" modal spaces, which restricts to a dual equivalence if we require the spaces to be compact.

The *bitopological* framework of Bezhanishvili et al. [1] will turn out to be suitable for this purpose. Bezhanishvili et al. formulate a dual equivalence between the category of distributive lattices (Heyting, bi-Heyting algebras) and suitable categories of spaces equipped with a pair of topologies. We slightly diverge from their framework in two ways. Firstly, we take the partial order $\leq$ to be part of the signature of such spaces, even though it is uniquely determined by the topologies. We therefore call them Priestley spaces rather than Stone spaces. Secondly, we generalize this dual equivalence to a dual adjunction. This involves no substantial novelty, only checking that the proof of the dual adjunction goes through without the assumption of compactness.

Definition 8.1 A *bitopological poset* $\mathcal{X} = (W, \leq, \tau_\pm)$ is a poset $(W, \leq)$ such that the *upspace* (W, τ_+) is a topological space, the *downspace* (W, τ_-) is a topological space, each $U \in \tau_+$ is an upset of $(W, \leq)$, and each $U \in \tau_-$ is a downset of $(W, \leq)$. The *opposite* of $\mathcal{X}$ is the bitopological poset $\mathcal{X}^{op} = (W, \geq, \tau_\mp)$. A *bicontinuous* function $f : (U, \leq, \tau_\pm) \to (V, \sqsubseteq, \upsilon_\pm)$ both a continuous function $f : (U, \tau_+) \to (V, \upsilon_+)$ and a continuous function $f : (U, \tau_-) \to (V, \upsilon_-)$.

Let $\mathcal{X} = (W, \leq, \tau_\pm)$ be a bitopological poset. The *join topology* of $\mathcal{X}$ is the topology $\tau = \tau_+ \vee \tau_-$. A subset of $\mathcal{X}$ is *upopen*, *upclosed* or *upcompact* if it is open, closed or compact in the *uptopology* τ_+, and it is *downopen*, *downclosed* or *downcompact* if it is open, closed or compact in the *downtopology* τ_-. A set is *upclopen* (*downclopen*) if it is upopen and downclosed (downopen and upclosed). An *upbasis* (*downbasis*) is a basis for the upspace (downspace).

A convex subset U of $\mathcal{X}$ is *closed* if $U^\uparrow$ is downclosed and $U^\downarrow$ is upclosed. The bitopological poset $\mathcal{X}$ is *compact* if it is compact in the join topology, or equivalently if and only if each cover of $\mathcal{X}$ by elements from $\tau_+ \cup \tau_-$ has a finite subcover. It is *Hausdorff* if the diagonal relation Δ_W is a closed subset of $\mathcal{X}^{op} \times \mathcal{X}$. It is a *pre-Priestley space* if it is Hausdorff, has an upbasis of upclopens, and has a downbasis of downclopens. It is a *Priestley space* if it is a compact pre-Priestley space. The category of pre-Priestley (Priestley) spaces and bicontinuous monotone functions will be denoted $\mathsf{PrePries}$ (Pries).

We now set up a dual adjunction between $\mathsf{PrePries}$ and DLat. Given a pre-Priestley space $\mathcal{X}$, let $\mathcal{X}^*$ be the distributive lattice of all upclopen subsets of $\mathcal{X}$, and given a bicontinuous monotone function $f : \mathcal{X} \to \mathcal{Y}$, let f^+ be the homomorphism of distributive lattices $f^{-1} : V^+ \to U^+$.

Let $\eta^+_{\mathbf{A}}(a) = \eta_{\mathbf{A}}(a)$ be the set of all prime filters $\mathcal{U}$ on $\mathbf{A}$ such that $a \in \mathcal{U}$ and let $\eta^-_{\mathbf{A}}(a)$ be the set of all prime filters $\mathcal{U}$ on $\mathbf{A}$ such that $a \notin \mathcal{U}$. Given a distributive lattice $\mathbf{A}$, let $\mathbf{A}_*$ be the poset of prime filters $\mathbf{A}_\bullet$ equipped with the uptopology generated by $\eta^+_{\mathbf{A}}(a)$ for $a \in \mathbf{A}$ and with the downtopology generated by $\eta^-_{\mathbf{A}}(a)$ for $a \in \mathbf{A}$. Given a homomorphism of modal algebras $h : \mathbf{A} \to \mathbf{B}$, let h_* be the function $h^{-1} : \mathbf{B}_* \to \mathbf{A}_*$.

It remains to define the co-unit of the dual adjunction. Given any pre-Priestley space $\mathcal{X}$, we define the function $\varepsilon_{\mathcal{X}} : \mathcal{X} \to (\mathcal{X}^*)_*$ such that $U \in \varepsilon_{\mathcal{X}}(u)$ if and only if $u \in U$ for each upclopen subset U of $\mathcal{X}$.

Theorem 8.2 (Bitopological Priestley dual adjunction) $(-)_* \dashv (-)^*$: $\mathsf{PrePries}^{op} \to \mathsf{DLat}$ *is an adjunction with unit* η *and co-unit* ε *which restricts to an equivalence between* Pries^{op} *and* DLat*.*

Proof. We know from [1] that restricting to Priestley spaces yields a dual equivalence. What we need to prove is that the assignment $(-)^*$ defines a functor $(-)^* : \mathsf{PrePries}^{op} \to \mathsf{DLat}$, that $\varepsilon_{\mathcal{X}} : \mathcal{X} \to (\mathcal{X}^*)_*$ is a bicontinuous monotone function for any pre-Priestley space $\mathcal{X}$, and that the triangle equality $(\varepsilon_{\mathcal{X}})^* \circ \eta_{\mathcal{X}^*} = 1_{\mathcal{X}^*}$ holds. The first claim is straightforward to prove.

To prove the second claim, we know that $(\mathcal{X}^*)_*$ has an upbasis of upclopen sets and that each upclopen subset V of $(\mathcal{X}^*)_*$ consists of all prime filters of upclopens on $\mathcal{X}$ which contain some upclopen subset U of $\mathcal{X}$. It then follows that $u \in U$ if and only if $\varepsilon_{\mathcal{X}}(u) \in V$.

To prove the third claim, $\eta_{\mathcal{X}^*}$ sends an upclopen subset U of $\mathcal{X}$ to the set V of all prime filters of upclopens on $\mathcal{X}$ which contain U. But $\varepsilon_{\mathcal{X}}$ sends a point $u \in \mathcal{X}$ to a filter which contains U if and only if $u \in U$, hence the pre-image of V under $\varepsilon_{\mathcal{X}}$ is precisely U. □

We now extend this dual adjunction to modal algebras. A convex relation

R on a bitopological poset $\mathcal{X}$ is said to be *continuous* if $(R^{\downarrow})^{-1}[U]$ is upopen for U upopen and $(R^{\uparrow})^{-1}[U]$ is downopen for U downopen. It is *compact* if $R^{\downarrow}[u]$ is downcompact and $R^{\uparrow}[u]$ is upcompact for each $u \in \mathcal{X}$.

Definition 8.3 A *pre-modal space* $\mathcal{X} = (W, \leq, R, \tau_{\pm})$ is both a modal frame $(W, \leq, R)$ and a pre-Priestley space $(W, \leq, \tau_{\pm})$ such that R is closed, $(R^{\downarrow})^{-1}[U]$ is upclopen for U upclopen, and $(R^{\uparrow})^{-1}[U]$ is downclopen for U downclopen. A *modally compact space* is a pre-modal space such that R is compact. A *modal space* is a compact pre-modal space.

Modal frames can be viewed as modal spaces with a discrete bitopology, that is, the uptopology of all upsets and the downtopology of all downsets. Modally compact frames are then frames such that for each point u, $R^{\downarrow}[u]$ is the lower closure of a finite set and $R^{\uparrow}[u]$ is the upper closure of a finite set. It is easily seen that each modal space is a modally compact space. (Since the relation R is closed, $R^{\downarrow}[u]$ is downclosed, hence also compact, for each u.)

We define the complex algebra $\mathcal{X}^*$ of a pre-modal space $\mathcal{X}$ as the expansion of the complex algebra of the underlying pre-Priestley space by the operations $\Box_R$ and $\Diamond_R$. The set of all upclopens on a pre-modal space is closed under these operations. The functions $\eta_{\mathbf{A}}$ and $\varepsilon_{\mathcal{X}}$ are defined for modal algebras and pre-modal spaces as for distributive lattices and pre-Priestley spaces.

Proposition 8.4 *If $\mathcal{X}$ is a modally compact space, then the function $\varepsilon_{\mathcal{X}} : \mathcal{X} \to (\mathcal{X}^*)_*$ is a bicontinuous p-embedding of modally compact spaces.*

Proof. Let $\mathcal{X} = (W, \leq, \tau_{\pm}, R)$. We know that $\varepsilon_{\mathcal{X}}$ is a bicontinuous monotone embedding of pre-Priestley spaces. Corollary 5.2 implies that if $uR^{\downarrow}v$, then $\varepsilon_{\mathcal{X}}(u) R^{\supseteq}_{\mathcal{X}^*} \varepsilon_{\mathcal{X}}(v)$. Vice versa, suppose that $\varepsilon_{\mathcal{X}}(u) R^{\supseteq}_{\mathcal{X}^*} \mathcal{V}$. The inclusion $\Diamond \mathcal{V} \subseteq \varepsilon_{\mathcal{X}}(u)$ holds by Corollary 5.2, hence each upclopen set in $\mathcal{V}$ intersects with $R^{\downarrow}[u]$. By the downcompactness of $R^{\downarrow}[u]$, so does their intersection. There is therefore some $v \in R^{\downarrow}[u]$ such that $\varepsilon_{\mathcal{X}}(v) \supseteq \mathcal{V}$. □

We denote the category of modally compact spaces and their bicontinuous p-morphisms by ModKSpace and the full subcategory of modal spaces by ModSpace. Using Proposition 8.4, we obtain the following duality theorem.

Theorem 8.5 $(-)_* \dashv (-)^* : \mathsf{ModKSpace}^{op} \to \mathsf{ModDLat}$ *is a dual adjunction with unit η and co-unit ε which restricts to a dual equivalence between* ModSpace *and* ModDLat.

Proof. Given Theorem 8.2, it suffices to show that η is preserves $\Box$ and $\Diamond$ and that ε is a morphism in ModKSpace. Theorem 5.3 proves the former claim and Proposition 8.4 proves the latter claim. □

We now derive a Hennessy-Milner theorem as a corollary. See [2] for a proof of the Hennessy-Milner theorem for classical modal logic.

Definition 8.6 A *modal model* over a set of atomic propositions *Prop* consists of a pre-modal space $\mathcal{X}$ and a valuation function $val_{\mathcal{X}} : Prop \to \mathcal{X}^*$. A *modally compact model* is a modal model such that $\mathcal{X}$ is a modally compact space.

A *bisimulation* between modal models $(\mathcal{X}, val_{\mathcal{X}})$ and $(\mathcal{Y}, val_{\mathcal{Y}})$ is a bisimulation σ between $\mathcal{X}$ and $\mathcal{Y}$ such that $u \sigma v$ implies that $u \in val_{\mathcal{X}}(p)$ if and only if $v \in val_{\mathcal{X}}(p)$ for all $p \in Prop$.

The valuation function extends to a unique homomorphism $h_{\mathcal{X}} : \mathbf{F}(Prop) \to \mathcal{X}^*$, where $\mathbf{F}(Prop)$ is the free bi-Heyting modal algebra generated by $Prop$. Given a pair of modal models $(\mathcal{X}, val_{\mathcal{X}})$ and $(\mathcal{Y}, val_{\mathcal{Y}})$ and a pair of points $u \in \mathcal{X}$, $v \in \mathcal{Y}$, define $u \sigma_{\mathcal{X},\mathcal{Y}} v$ to hold in case $u \in h_{\mathcal{X}}(a)$ if and only if $v \in h_{\mathcal{X}}(a)$ for all $a \in \mathbf{F}(Prop)$. In other words, $u \sigma_{\mathcal{X},\mathcal{Y}}$ if and only if $h_{\mathcal{X}}^{-1}[\varepsilon_{\mathcal{X}}(u)] = h_{\mathcal{Y}}^{-1}[\varepsilon_{\mathcal{Y}}(v)]$.

Theorem 8.7 (Hennessy-Milner theorem) *Let $(\mathcal{X}, val_{\mathcal{X}})$ and $(\mathcal{Y}, val_{\mathcal{Y}})$ be modally compact models. Then $\sigma_{\mathcal{X},\mathcal{Y}}$ is the largest bisimulation between these models.*

Proof. By Proposition 8.4, $(h_{\mathcal{X}})_* \circ \varepsilon_{\mathcal{X}}(u)$ and $(h_{\mathcal{Y}})_* \circ \varepsilon_{\mathcal{Y}}(v)$ are p-morphisms. By Proposition 3.7, σ is thus a bisimulation, and clearly the largest one. □

9 Conclusion

We have introduced distributive unimodal logic as a semantically motivated generalization of classical set-based modal logic to a poset-based setting. We defined a suitable quasivariety of modal algebras and proved a completeness theorem embedding each modal algebra in the complex algebra its canonical frame. We then discussed the relationship between distributive unimodal logic and the distributive modal logic of Gehrke et al. [8] and showed that the existing completeness and duality theorems for intuitionistic modal logic and positive modal logic are subsumed by the completeness and duality theorems for distributive unimodal logic. Finally, the completeness theorem was extended to a duality between the category of modal algebras and a cat/egory of suitably topologized modal frames and a Hennessy-Milner theorem for bi-intuitionistic unimodal logic was proved as a corollary.

The completeness and duality theorems proved here can in fact be extended to the full language of distributive modal logic (which includes modal operators corresponding to the classical modalities $\Box\neg$ and $\Diamond\neg$). Extending them to modalities of higher arity, however, seems to be substantially more difficult.

Apart from other standard areas of investigation which were left untouched (such the finite model property, decidability and correspondence theory), we can also pose a question which does not arise in any of the other modal logics considered here, namely: given some choice of connectives, is there a largest variety of modal algebras in this language? In other words, is there a most general equational condition in a given language which ensures the validity of the quasiequations defining the class of modal algebras? It seems natural to conjecture that the positive modal algebras introduced by Dunn in fact form the largest variety of modal algebras.

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