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TILGHER-GENOVA

Carlo Penco

INTUITION IN MATHEMATICS ?

Wittgenstein's Remarks*

In this paper I intend to trace the development of the concept of "intuition" in Wittgenstein's philosophy of mathematics. I will focus my attention on the terminological shifts (*Anschauung, Einsicht, Intuition*) and on the connected changes of points of view. The paper is therefore mainly exegetical; but it is intended to provide material which will enable us to tackle the three following problems:

(1) *A historiographical problem: how far is the discussion on Kant and intuition, which developed in the early XX century, echoed in Wittgenstein's work?* Heir of eighteenth century discussion, *Frege* considers intuition as something irrelevant to mathematics: something subjective, arbitrary, which has to be weeded out of rigorous demonstrative procedures. Intuition (both sensible intuition, as in Kant's *Critique of Pure Reason,* and pure intuition, as in Kant's *Logic*) "cannot serve as the ground of our knowledge of the laws of arithmetic".[1] At the beginning of the nineteenth century, the crisis of logicism leads to a re-evaluation of Kant's conception of intuition as a ground for all mathematics. Such a position is accepted in different ways by Hilbert and Brouwer (and by Frege himself in his later writings[2]). *Hilbert,* against Frege's doubts, chooses as a starting point the kantian statement according to which without *sensible intuition* no object

Epistemologia IV (1981), pp. 77-94.

can be given; mathematics is grounded on the intuition of concrete objects, signs on paper, whose form is directly recognizable. *Brouwer,* following a widespread tradition from Gauss to Kronecker, founds all mathematics on the intuition of number. But he talks of *pure intuition,* and therefore he seems to be more rigorously kantian than Hilbert, as Cassirer points out.[3] In this way Brouwer stresses the autonomy of mathematics in respect of logic: at each step of the proof of a theorem we have a step in the mathematical construction. (We have to remember that, according to Brouwer, mathematics is intended as a mental activity which originates from the perception of time, i.e. from the splitting of a moment of life into two different objects. The splitting constitues the *basic intuition* of mathematics.)[4]

With Hilbert and Brouwer intuition is brought back from a peripheral to a central role in mathematics: for both, mathematics deals with entities whose existence and objectivity is somehow granted by intuition, that is by direct recognition of perceptual (Hilbert) or mental (Brouwer) constructions. Intuition is necessary at each step of the proof and guarantees its certainty. After an earlier sympathy for these views, Wittgenstein distances himself from both. His discussion of intuition can be considered as a dialogue with Frege, Hilbert and Brouwer, and their interpretation of Kant.[5]

(2) *A problem of criticism of Wittgenstein's ideas: what are the value and the limits of an "intuitionist" interpretation of Wittgenstein?* The link between Wittgenstein and intuitionism is now widely taken for granted.[6] Richardson [1976] suggests that Wittgenstein's later conception of philosophy "is dictated almost entirely by a view of language and human activity which he took over from Brouwer", and his writings are to be interpreted "under the assumption that his ideas do indeed have their origin in the Brouwer lecture". Without denying Brouwer's influence on Wittgenstein's return to philosophy and on some specific observations on mathematics, we prefer to point out Wittgenstein's progressive drawing away from Brouwer and intuitionism, as clearly stated in his notes on intuition in mathematics. The measure of Wittgen-

stein's rejection of intuitionism is important also with regard to a fundamental problem of interpretation: *is Wittgenstein's philosophy of mathematics a "revisionist" one?* (that is, such as to request a modification or a reconstruction of mathematical practice along constructivist lines?) As recently argued by Wright [1980], on this point the distance between Wittgenstein and intuitionism is very deep. Intuitionism supports the view of a proof as a mean of constructive description of mathematical entities, therefore calls for a reformulation of large parts of classical mathematics; on the contrary Wittgenstein's view of proof as a mean of conceptual change allows a more liberal attitude. The attempt of this paper is to show the path which brought Wittgenstein from an early approval of Brouwer's ideas, to a sharp rejection of intuitionism and of his theoretical claims.

(3) *A theoretical problem: can the discussion on intuition in mathematics still have a central role in the choice between empirical and transcendental point of view in mathematics?* In Russell, even more than in Frege, intuition is reduced to a psychological support; this reduction is taken up by neopositivism, namely by Ayer [1936]. He accepts the appeal to intuition as of purely psychological value, which is however a source of possible mistakes (a "risk" for the geometrician). In the fourth chapter of his book, Ayer claims to refute Kant's ideas on synthetic a priori judgements, based on intuition; and his rejection of the synthetic a priori is followed by the rigid dichotomy between empirical a posteriori and tautological a priori science. Starting with a criticism of this rigid dichotomy (quoting Ayer [1936]), *Lakatos* [1976], [1977] again takes up the idea of empiricism in mathematics. However two points give rise to doubts:

(a) If we come back to Gödel – quoted in Lakatos [1976] as a representative of empiricist revival in mathematics – we can think of some specific faculty of mathematical intuition which give us a knowledge of sets, numbers and simple axioms (just as sensory perception gives us a knowledge of physical objects and atomic facts). In the explicit attempt to provide an alternative to Kant's philosophy of mathematics, Gödel [1964] argues that sets, numbers, and other mathematical objects are not subjective forms, but part of a reality which is to be grasped through data with which we have a relation different from sensory perception. Empiricism in mathematics then goes hand in hand with the more sophisticated realism, as, more recently, in Putnam [1975].

(b) Lakatos believes it necessary to lay aside "the naïve school concepts of static rationality like a priori – a posteriori, analytic – synthetic" (p. 218). He wants in this way to stress his rejection of the neopositivist dogma of the contraposition between a priori and a posteriori sciences. However, in doing so, he remains anchored to the line laid down by neopositivism: although rejecting some aspects of it, he doesn't succeed in getting over their interpretative horizon. He rejects one dogma and at the same time implicitly accepts another one, which lies at the root of the first: the restrictive interpretation (and consequent elimination) of the kantian concept of synthetic a priori.

It is paradoxical, but we have to recognize that in a way, the Fregean criticism of Kant' synthetic a priori lies at the source of the revival of empiricism in mathematics, which is the very attitude Frege thought he had refuted once and for all. Among the alternative to this empiricistic current we have hints of a re-evaluation of Kant's synthetic a priori in authors like Sellars ([1968] ch. 9) and Hintikka [1973] (See also Miller [1975].). Wittgenstein reflexions on intuition point in a similar direction, which, starting from an original interpretation of Kant, leads to a transcendence of themes and concepts used by Kant and to a search for a new conceptual apparatus; but this transcendence is the outcome of an *Aufhebung,* not a mere rejection through incomprehension.

1. *"Anschauung": Language Provides the Intuition of Symbol and Rules for Usage (1912-1929).*

In the *Notebooks* Wittgenstein notes:

> Light on Kant's question "How is pure mathematics possible?" through the theory of tautologies. (19.10.1914)

In the conclusion of the notes dictated to Moore in the same period we read:

> logical laws are *forms* of thought and space and time are *forms* of intuition. (*NM* 117)

The Kantian flavour of this conclusion helps us to interpret the puzzling statements found in the *Notebooks*: tautologies in logic, like equations in mathematics, show the forms by which we organize our language and moreover they are the conditions upon which we organize our picture of the world of experience.

In this way we can read the discussion on intuition in mathematics sketched in *Tractatus* 6.233-6.234 as an answer to Kant. Is intuition necessary in mathematics? As against Frege and Russell, the answer is affirmative; but intuition is not considered as derived from some not clearly defined source of knowledge, but as provided by language itself, namely by calculating procedures. To say that the intuition needed for mathematics is provided by calculation implies that the process of calculation is not grounded in empirical intuition ("calculation is not an experiment" [*TLP* 6.234]), but, on the contrary, it provides the a priori *forms* of intuition and representation of the world of experience. In the conversations with Waismann, Wittgenstein clarifies this point:

> Space, time and number are *forms* of representation (*Darstellung*). They are designed to express every possible experience, and for this reason it is wrong to base them on actual experience. (*WWK* 214)

The term *Anschauung* is found again in *WWK* with a partial schift of sense, in a context at first similar to Hilbert's, but it is connected also to *TLP* 6.113, 6.127: the essence of the philosophy of logic is that logical propositions are such that we can recognize them as true from the symbol only, taking no account of sense and reference, but only following the "rules of the signs". We can conclude

> To a certain extent it is true that mathematics is based on intuition: namely the intuition of symbols; and in logic it is the same kind of intuition that is employed when we use a tautology. (*WWK* 219)

In this Wittgenstein gets near to Hilbert and recognizes "the legitimate aspect of formalism", which is the view according to which mathematical symbols have no meaning (in the sense of Frege's *Bedeutung*) (*WWK* 103-105). However, Hilbert's discourse on the intuition of concrete symbols and their process of transformation tends to consider mathematics as a description of finite objects; Wittgenstein on the other hand stresses that mathematics is not a science that describes objects (signs of ink on paper), but it is the way we describe objects:

> mathematics is always a calculus. The calculus does not describe anything. It can be applied to everything that allows of its application. (*WWK* 106)[7]

2. *Einsicht: Kant and the Synthetic a Priori (1929-1933).*

The statement given in *TLP* that the process of calculation provides the intuition needed in mathematics is to be interpreted primarily as a plea against logicism. This is clarified by the remark against the reduction of the arithmetic to logic which is developed by Wittgenstein from the end of nineteen twenties onwards. In *WWK* (105-107) he acknowledges that an equation such as 28 + + 16 = 44 can be applied to a tautology:

$$(\exists\, 28\, x)\, \varphi x\ .\ (\exists\, 16\, x)\, \psi x\ .\ \mathrm{Ind.} :\supset: (\exists\, 44\, x)\, \varphi x \vee \psi x$$

But the tautology cannot provide what is given by calculation: to find the number on the right hand which renders the equation a tautology we have to use a calculus which is independent of the tautology. Here the calculus indeed provides the intuition needed to understand the mathematical equation.

This point is developed in *Philosophische Bemerkungen* 101 ff., and in *Philosophische Grammatik* § 19, against the reduction of mathematics to conceptual analysis. A long discussion concludes:

> No investigation of concepts, only insight into the number-calculus can

> tell us that $3 + 2 = 5$. That is what makes us rebel against the idea that
>
> $$(\exists\, 3\, x)\, .\, \varphi x\, .\, (\exists\, 2\, x)\, .\, \psi x\, .\, \mathrm{Ind.}: \supset .\, (\exists\, 5\, x)\, .\, \varphi x \vee \psi x$$
>
> could be the proposition $3 + 2 = 5$. For what enables us to tell that this expression is a tautology cannot itself be the result of an examination of concepts, but must be recognizable from the calculus. (*PG* 347)

In the parallel passage in *PB* the discussion terminates with an explicit reference to Kant:

> What I said earlier about the nature of arithmetical equations and about an equation's not being replaceable by a tautology, explains – I believe – what Kant means when he insists that $7 + 5 = 12$ is not an analytic proposition, but synthetic *a priori.* (*PB* 108)

We have here the reversal of the Fregean position as stated in the *Grundlagen der Arithmetik.* Moreover, the appeal to Kant absent from the passage quoted from *PG,* is reconsidered some paragraphs further on, in the discussion on induction (*PG* § 31). The account given by Waismann ([1936], ch. 9) is misleading, mainly in the conclusion by which the principle of induction "doesn't constitute, as Poincaré thought, a synthetic *a priori* judgement; it doesn't constitute a truth, but instead establishes a *convention*". In this connection, in *PG,* we find no reference at all to "convention", and Wittgenstein's detailed discussion is focused on the contrast between extensional and intensional points of view; moreover it seems to be nearer to Poincaré's than to Waismann's conventionalistic interpretation. In fact attention is focused on the constructivist aspect: "The discovery of the periodicity is really the construction of a new symbol and a new calculus". That means it is the construction of a new concept which does not derive from a simple analysis of given concepts. The chapter concludes:

> Isn't what I am saying what Kant meant, by saying that $7 + 5 = 12$ is not analytic but synthetic *a priori?* (*PG* 404)

In these discussions between 1929 and 1933 Wittgenstein seems to close a chapter, completing his criticism of the reduction of arithmetic to analytical science, a reduction which was centred on the exclusion of intuition from the calculus. Again restoring a place to intuition as a fundamental feature for the understanding of calculus, against Frege and Russell, Wittgenstein revives the kantian view, in a way different from the one developed by the later Frege in 1924 and 1925. More coherently Kantian, Wittgenstein does not regard mathematical terms as referring to objects in a strong sense; on the contrary he considers them as referring to formal concepts, given in intension, such as rules or laws, and not as extension. The references to Kant are not coincidential or limited to this context: in a lecture given in Cambridge, while drafting these remarks, Wittgenstein comments on Broad's philosophy and asserts: the right sort of approach in philosophy is the *transcendental method,* "which can be characterized briefly as Kant's critical method without the peculiar applications Kant made of it" (*WCL* 73).

3. *Einsicht-Intuition: Intuition at each Step of the Proof? (1929-1933).*

In *PB* we find for the first time a reference to the problem of the necessity of an intuition at each step of a proof; this theme will have a central role in the ensuing years (see §§ 5 and 6):

> Is it like this: I need a new insight at each step in a proof? This is connected with the question of the individuality of each number. Something of the following sort: Supposing there to be a certain general rule (therefore one containing a variable), I must recognize each time afresh that this rule may be applied *here*. No act of foresight can absolve me from this act of *insight*. Since the form to which the rule is applied is in fact different at every step. (*PB* 149)

Wittgestein makes a margin note which is reconsidered in *PG* 301 and *PU* 186: "Act of *decision,* not *insight*". This transition from the term "insight" to the term "decision" represents an attempt to avoid the danger of falling into psychologism, and a new emphasis on the constructive and operative features of mathematics, against the propensity to found it on mental processes. After a first period of a substantial adherence to the fundamental brouwerian themes (mathematics as activity, real numbers as laws and not extensions, etc.), we can find in the writings of Wittgenstein a growing negative *attitude towards the theoretical* apparatus *of intuitionism* (already in *PB*: e.g. *PB* 174). Not only does the idea of intuition at each step begin to crack, but the very meaningfulness of the fundamental intuitionistic concepts begins to be doubted: "when intuitionists speak of the 'basic intuition' – is this a psychological process? If so, how does it come into mathematics?" (*PG* 322).

Beyond the practice of intuitionist mathematics, intuitionists are still linked to a psychologistic view of mathematics, and tend to look into the mental sphere to find the *existence* of mathematical objects, constructions perceived by intuition and recognized by introspection. In this intuitionists join formalists and other philosophers of mathematics. Everybody asserts some thing like: "This state of affaire, existence, can be proved only thus and not thus". But "they don't see that by saying that they have simply defined what *they* call existence" (*PG* 374, cf. *PG* 295).

4. *Einsicht, Intuition, Instinkt: the Psychological Problem of Intuition (1933 ff.).*

While in *PB* the terms *Einsicht* and *Intuition* seem to be sometimes interchangeable, this doesn't happen in *PG*. It seems that in *PG* the discussion of intuition explicitly takes two distinct paths:

a) *Einsicht* (heir of the discussion in which the term *Anschauung* has been used): criticism of logicism and the re-evaluation of intuition as a reconsideration of the synthetic a priori in mathematics: from here the discussion on "perspicuity" of mathematical proof (cf. § 7).
b) *Intuition*: criticism of intuitionist philosophy and of intuition considered as a mental process which would establish the calculus: from here two fundamental themes of the later Wittgenstein: the analysis of "following a rule" (§ 5) and the concept of "technique" (§ 6).

We have however to consider another aspect, connected with the everyday language, when we say that mathematicians often work with intuition, "*by instinct*" (*PG* 295). Usually, in fact, speaking of intuition we mean the phenomenon by which "one knows immediately which others only know after long experience or after calculation". It is pointless to use the term in other deeper senses, because "if we *all* knew by intuition, and by intuition alone, this isn't what we could possibly call intuition" (*LFM* 30). If we want to give the term intuition a definite meaning, we could, for instance, substitute it with "guessing right". This expression "would show the value of an intuition in a quite different light. For the phenomenon of guessing is a psychological one, but not that of guessing right" (*BGM* III 26). We could also say that the first (psychological) sense of the term is about people's behavior, the second (non-psychological) sense is about mathematics (*LFM* 29). To speak simply of intuition in mathematics without taking note of the necessary distinctions means to refer not to a *mathematical* truth, but only to a physical or psychological one (ref. *BGM* III 44), to something which is not so directly related to calculation (ref. *BGM* V, *Z* 699). Here we might speak of some "intuitively known *empirical* fact", not of some mathematical fact. In this sense intuition doesn't grasp the mathematical truth of induction (*BGM* III 43): that in the division 1/3 the remainder *will* always be 3, is an intuitively known empirical fact, not yet a mathematical truth; the latter is recognized when we acknowledge that in the division 1/3 "we *must* keep on getting 3 in the result" (the transition from "it will be like this" to "it must be like this" characterizes the concept formation in mathematics: *BGM* III 29 ff.).

In respect of intuition Wittgenstein develops an anlysis similar to the one developed in *Philosophische Untersuchungen* on the concept of "understanding": intuitions, as psychological processes, are analogous to those characteristic *accompaniments* or manifestations of understanding (*PU* 152); nowhere does Wittgenstein deny the existence of such processes or their importance from a psychological point of view, but he points out that it is not at all evident what these have to do with the nature of mathematics. Philosophical analysis of understanding, as of intuition, deals with the structure of these activities, not with the psychological processes which accompany them: "the mental process of understanding is of no interest to us (any more than the mental process of an intuition)" (*PG* 271).

5. *Intuition and "Following a Rule" (1935 ff.).*

In *Brown Book* § 5, Wittgenstein resumes the criticism of the "intuition at each step of the proof"; the new feature in respect to *PB* and *PG* (not intuition, but decision) is the importance given to the concept "following a rule"; the problem is stated in this way:

> [...] in order to follow the rule "Add 1" correctly a new insight. intuition, is needed at every step. But what does it mean to follow a rule correctly? (*BB* 141-142)

Here Wittgenstein sistematically introduces the criticism of the psychologistic reduction of "knowing" and "understanding" to mental acts, already made in *PG.* To say that an intuition is needed in order to follow the rule, is like saying that the correct step at every point is the one in accordance with the rule as it was *meant,* or intended. Under this lies the idea that "in the misterious act of meaning the rule, you made the transitions without really making them", you have performed the steps *mentally.* Against this view Wittgenstein says:

> It is no act of insight, intuition, which makes us use the rule as we do at the particular point of the series. It would be less confusing to call it an act of decision, though this too is misleading, for nothing like an act of decision must take place, but possibly just an act of writing or speaking (*BB* 143)

What is implied in the notion of intuition at each step is that "something must make us do what we do". We face the confusion between cause and reason. "*We need have no reason to follow the rule as we do.* The chain of reason has an end."

It seems that Wittgenstein here is trying to isolate the concept of "following a rule" as a primitive concept, neither based on intuition, nor on any other psychological process. The view given in *BB* is resumed and widened in the discussion of "following a rule" which Wittgenstein will develop from the '40 on. He will come back ironically on the view criticized in *BB* on the idea that "something makes us do what we do":

> What is it that compels me? – the expression of the rule? – Yes, once I have been educated in this way. But can I say it compels me to follow it? Yes: if one thinks of the rule, not as a line that I trace, but rather as a spell that holds us in thrall. (*RFM3* VII 27)

And elsewhere, referring to the *Tractatus* and then to mathematical logic generally:

> My symbolical expression was really a mythological description of the use of a rule (*PU* 221)

What then does remain in Wittgenstein's reflexions about the relationship between intuition and acting in accordance with the rule? In *PU* there appears the following passage:

> If you have an intuition in order to develop the series 1, 2, 3, 4,... you must also have one in order to develop the series 2, 2, 2, 2,... (*PU* 214. ref. *BGM* I 3 and *LFM* 28-30)

Bearing in mind the distinction given at § 4 we can say: intuition as a mental process thought by people to underlie acting according to a rule, gives only a *natural expression of a uniformity* (as a magic sign that produces the series and would not be the expression of a rule [*RFM3* 27]). On the other hand acting according to a rule presupposes the *recognition of a uniformity*; in this sense then, we can consider intuition as the structure of acting according to a rule (see § 7). However, considered as an "inner voice" which doesn't mislead me, intuition will always be "an unnecessary shuffle" (*PU* 213).

6. *Intuition and Technique.*

In the *Cambridge lecture 1939* we have another detailed discussion of the topic. Even from the first lecture Wittgenstein focuses his attention on the concept of technique, and particularly on the technique of counting: no particular intuition justifies the series of natural numbers as they are given in the technique of counting:

> [...] there is no discovery that 13 follows 12. That's our technique – we *fix*, we teach our technique that way. (*LFM* 83)

And, as with any other technique, we could change it, to adapt it either to different circumstances (we could give examples) or to our personal taste. This doesn't mean, however, that the technique of counting is a convention, based on the agreement of opinion; on the contrary:

> [...] truths of logic are determined by a consensus of opinions. Is this what I am saying? No. There is no *opinion* at all; it is not question of *opinion*. They are determined by a consensus of *action*. [...] There is a consensus, but it is not a consensus of opinion. We all act the same way, walk the same way, count the same way.
> In counting we do not express opinions at all. There is no opinion that 25 follows 24 – nor intuition. We express opinions by means of counting. (*LFM* 183-184, ref. *PU* 242 and *RFM3* VI 39)

Technique is a set of rules and of certain ways of using them; it is adopted (put in the archives) not because it corresponds to an intuition or to a truth, but, on the contrary, because it answers to a necessary function: with it people agree in what they do *and then* they can speak of truth and falsity. At this point the break with intuitionist philosophy is definitive:

> Intuitionism comes to saying that you can make a new rule at each point. It requires that we have an intuition at each step in calculation, at each application of a rule; ... and they go on to say that the series of cardinal numbers is known by a ground-intuition... We might as well say that we need, not an intuition at each step, but a decision. Actually there is neither. You don't make a decision: you simply do a certain thing. It is a question of a certain practice.
> Intuitionism is all bosh – entirely. Unless it means an inspiration. (*LFM* 237)

This quotation condenses the results of the analysis carried on from the first moment at which Wittgenstein came under the influence of intuitionism (and here he gives value to it, as an inspiration) to this final, sharp break. Intuitionism wants to reconstruct a mathematics providing a certainty once and for all, a super-mathematics. But Wittgenstein sees the matter differently: on one hand we have the *expression* of the rule; on the other hand a *technique* for applying the rule. The correct application has no justification external to the technique. There is therefore no justification which could provide us with certainty in making the step in the proof that we learn when we learn the technique. Intuitionists want to pass from a proof which simply "convince" us to a proof with "indubitable steps". But "all you've got for this step is a rule. And the use you make of the rule I suppose is the convincing one. There isn't a super-use" (*LFM* 237-238).

7. *Intuition and Perspicuity.*

In the years 1939-1940 Wittgenstein returns to the criticism of the logicist reduction. He scarcely speaks again of "intuition", but he again uses a concept already found in *PB* 107 to explain the meaning of intuition (insight), that is "immediately visible".[8]

The difference between a proof in Russell calculus (or in the stroke calculus) and a proof made in the numerical calculus is the following: in the latter the figures of the proof have a "characteristic visual shape" so as to be recognizable at a glance; in the former, at least in the case of large numbers, this doesn't occur (ref. *BGM* II 10-14). As the invention of periodicity represents a discovery, an invention of a new symbol and of a new calculus (*PG* § 31), the transition from the stroke calculus system to the decimal system means the invention of a new "system of abbreviation" which is at the same time a system of new signs and a "system for applying them for the purpose of abbreviation". In such a case then – Wittgenstein points out – "it *is* a new way of looking at the old system of signs" (*BGM* II 12).

This theme is a constant in the writings of Wittgenstein: *the way of looking at* (here *Anschauungsart*). Mathematics constructs the forms of intuition, that is the forms of our way of representing the world; in the discussion of the recognition of a numerical series in *PU* 143 ff. Wittgenstein speaks of a way of looking at things (indian mathematicians: look at this!). This is probably the context of the genesis of the fundamental concept of "perspicuous representation", the "form of account we give" (*Darstellungform*), the "way we look at things" (*PU* 122). From this point of view it is useful to read the discussion of the proof developed in *BGM* II: the picture which convince us by its perspicuity is assumed as a proof, it is taken as a model and introduces a new way of looking at and recognizing signs: it introduces a new "sign technique" (*BGM* II 39.41.54).

We can summarize Wittgenstein's account[9]: mathematical propositions are *synthetic* because they are grounded on a technique which is constituted in our way of looking at and of using signs. Mathematical propositions are, nevertheless, *a priori*, and their sense is not to express physical and psychological facts which are the conditions which make that technique possible (*BGM* V 1). Such a technique doesn't derive from experience: on the contrary it determines and constitutes the forms of the representation by which you can *see* the facts:

> It is interesting to know *how many* vibrations this note has! But it took arithmetic to teach you this question. It taught you to see this kind of fact. Mathematics – I want to say – teaches you, not just the answer to a question, but a whole language-game with questions and answers. (*BGM* V 15)

8. *Concluding Remarks.*

Many fundamental concepts of Wittgenstein's philosophy derive from his remarks on mathematics. From one point of view such remarks are deeply Kantian and antagonist towards the three schools: mathematics is concept formation, construction of forms by which we look at the world of facts, and not a recognition of objects somehow endowed with existence. In that foundational schools fall down in the same misunderstanding: they look for the *objectivity* of mathematics in the *existence* of mathematical entities: now ideal existence (Frege: numbers as logical objects), now concrete (Hilbert: signs on paper), now mental (Brouwer: constructions given in introspection). The way of considering intuition, particularly the psychological reduction of the concept, has contributed to the elaboration of this misunderstanding. The ambiguities of Wittgenstein's discussion originate from that: on one hand he tends to re-evaluate intuition against logicism, reviving Kant's theories; on the order hand he is brought to reject the concept, just because it is now used almost exclusively in the psychological sense of mental process. In this way his discussion of intuition leads him to give a role to intuition in mathematics, being careful to separate the intuition conceived as *Anschauung,* from the intuition conceived as a mental process. In this discussion we can find the genesis of new concepts central to Wittgenstein's philosophy: following a rule, technique, perspicuous representation. Such concepts, like others in Wittgenstein's philosophy, are then to be considered not as neutral epistemological tools, but as concepts which have a specific polemical value, and which are alien to other trends of thought.

Department of Philosophy
University of Genoa

NOTES

* I thank Paolo Leonardi, Paolo Agosto and Bernard A. Worthington for their criticism of a first draft of this paper.

1. Frege [1884], p. 12.
2. In his later writings Frege criticizes the reductionist program and his previous ideas, namely in the paper *Neuer Versuch der Grundlegung der Arithmetik* (1924-1925). It is of some interest to us to notice: (1) Frege excludes sensory perception as a possible source of knowledge in arithmetic and geometry. (2) He considers an a priori knowledge, not based on logic, but on intuition (*Anschauung*) (*Nachgelassene Schrifte,* Hamburg, 1969, pp. 292-293). As in Kant, it concerns space and time. (3) This source of knowledge is distinct from the sensory and the logical ones; it is called, at first, "geometrical", and all mathematics originates from it.
3. Ref. Frege [1884], § 89, Hilbert-Bernays [1934], Hilbert [1925], Cassirer [1929], pp. 133 ff., Brouwer [1912].
4. Ref. Brouwer [1952], [1940]. On the idea of intuition at each step of the proof, and on the priority of mathematics over logic ref. e.g. Heyting [1958].

5. I have discussed this problem in Penco [1979].
6. See Von Wright [1958], Dumett [1959], Richter [1965], Hacker [1972].
7. I have discussed the theme of application in mathematics in Penco [1981] and Penco [1981 a].
8. *Unmittelbar sichtbar. Uebersichtlichkeit* (perspicuity) and *Uebersehrbarkeit* (surveyability) will become in this period two central themes in the discussion against logicism. We could envisage an influence from Hilbert; however Wittgenstein's discussion takes a direction opposite to Hilbert's. He here joins more easily the criticism given in the same period by Bernays [1941] on the uncertainty of long finitistic proofs, which have a large probability of mistakes.
9. I have to remember the already classical interpretation of Specht [1963], expecially in the VI part (the constitution of objects in language), ch. 11.

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Wittgenstein's works and their abbreviations:

BGM	*Bemerkungen über die Grundlagen der Mathematik*, Oxford, 1956.
LFM	*Lectures on the Foundation of Mathematics, Cambridge, 1939*; New York, 1976.
NM	*Notes Dictated to E. Moore*, (in *TB*).
PB	*Philosophische Bemerkungen*, Oxford, 1964.
PG	*Philosophische Grammatik*, Oxford, 1969.
PU	*Philosophische Untersuchungen*, Oxford, 1953.
RFM3	*Remarks on the Foundations of Mathematics*, 3rd ed., Oxford, 1978.
TB	*Tagebücher 1914-1916*, Oxford, 1961.
TLP	*Tractatus logico-philosophicus*, London 1961.
WWK	*Wittgenstein und der Wiener Kreis*, Oxford, 1967.
Z	*Zettel*, Oxford, 1967.
WCL	*Wittgenstein's Lectures. Cambridge 1930-1932*, J. King and D. Lee (eds.), Oxford, 1980.

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