Journal of Physics: Conference Series **82** (2007) 012006 doi:10.1088/1742-6596/82/1/012006

GÖDEL, TRUTH & PROOF

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Abstract: The usual way of interpreting Gödel's (1931) incompleteness result is as showing that there is a gap between truth and provability, i.e. that we can never prove everything that is true. Moreover, this result is supposed to show that there are unprovable truths which we can *know* to be true. This, so the story goes, shows that we are more than machines that are restricted to acting as proof systems. Hence our minds are 'not mechanical'.

In this paper I would like to indicate that this interpretation of Gödel goes far beyond what he really proved. I would like to show that to get from his result to a conclusion of the above kind requires a train of thought which is fuelled by much more than Gödel's result itself, and that a great deal of the excessive fuel should be utilized with an extra care.

1. Incompleteness theorem: a 'boringly technical' formulation

There are many ways in which we can present Gödel's incompleteness result. Let us, first, do it in a bluntly technical way.

Let $\boldsymbol{A}$ be an alphabet (a finite set of objects) and let $\boldsymbol{L}$ be a set of strings over $\boldsymbol{A}$. Let $\boldsymbol{M}$ be a set of subsets of $\boldsymbol{L}$. Where x and y are strings over $\boldsymbol{A}$ (especially elements of $\boldsymbol{L}$), let $x^\frown y$ denote the concatenation of x and y; and let $x =_{\boldsymbol{M}} y$ state that for every $m \in \boldsymbol{M}$ it is the case that $x \in m$ iff $y \in m$ (i.e. that either both x and y are members of m or none of them is). Let us, moreover, say that x is ***M**-persistent* iff x belongs to the intersection of $\boldsymbol{M}$ (i.e. $x \in m$ for every $m \in \boldsymbol{M}$). Let us pick up an element $a \in \boldsymbol{A}$ and let us call a set s of strings over $\boldsymbol{A}$ *a-open* iff s contains no string x together with $a^\frown x$; and let us call it *a-saturated* iff it contains $a^\frown x$ whenever it does not contain x. Let there exist a binary operation $\oplus$ such that for every $x,y \in \boldsymbol{L}$ it is the case that

$$(*)\ (a^\frown x) \oplus y =_{\boldsymbol{M}} a^\frown (x \oplus y)$$

* Work on this paper has been supported by the Research Grant No. 401/04/0117 of the Grant Agency of the Czech Republic. I am grateful to Jaroslav Zouhar for critical comments.

and for every $x \in \boldsymbol{L}$ there is a $y \in \boldsymbol{L}$ so that

(**) $x \oplus y =_{\boldsymbol{M}} y$

Moreover, let there exist a $b \in \boldsymbol{L}$ so that for every $x \in \boldsymbol{L}$

(***) $b \oplus x$ is ***M***-persistent iff *x* is.

Then it is easy to see that if the intersection of ***M*** is *a*-open, then it is not *a*-saturated. For let *c* be the element of ***L*** for which $(a \cap b) \oplus c =_{\boldsymbol{M}} c$ (its existence is guaranteed by (**)). *c* is obviously ***M***-persistent iff $(a \cap b) \oplus c$ is, and hence, in force of (*), iff $a \cap (b \oplus c)$ is. Moreover, in force of (***), *c* is ***M***-persistent iff $b \oplus c$ is; hence $a \cap (b \oplus c)$ is ***M***-persistent iff $b \oplus c$ is. This means that the intersection of ***M*** contains either both $a \cap (b \oplus c)$ and $b \oplus c$, or none of them. And this entails that if it is *a*-open, hence it does not contain both of them, then it must contain none of them, hence not be *a*-saturated.

Now one of the way of presenting Gödel's results would be saying that it is a matter of demonstration that the language of PA constitutes a special case of this pattern, in the following sense. Take ***A*** to be the vocabulary of the language of arithmetic and ***L*** the set of its well-formed formulas. Take ***M*** to be the set of all consistent theories in ***L*** containing the axioms of Peano (or, for that matter, Robinson) arithmetic (so that the relation $=_{\boldsymbol{M}}$ comes out as logical equivalence). Take *a* to be the negation-sign (so that an *a*-open set is a consistent theory and *a*-saturated set a complete theory). Let $x \oplus y$ be the result of substituting the Gödel number of *y* for the single free variable of *x* (if there is a single free variable of *x*, otherwise let $x \oplus y$ be simply *y*); then it is clear that (*) holds and also it can be proved that (**) holds[1].

Moreover, there provably exists a formula *b* with one free variable (and who is familiar with Gödel's proof knows that it is the formula **Pr**(*x*) where **Pr** is the provability predicate) so that for every formula *x*, $b \oplus x$ is ***M***-persistent (i.e. provable) iff *x* is. Thus, also (***) holds; and hence it follows Peano (or Robinson) arithmetic (and, by way of generalization, any its consistent axiomatic extension) cannot be complete. (Hence, let us notice, the problem is that there exists the *fix point* of the unprovability predicate, a statement which, in effect, says *I am unprovable*.)

This is a rather boring formulation of Gödel's result; and it seems very unlikely that spelled out in this way it would catch the attention of anybody outside the field. To become, as it did, one of the most discussed intellectual challenges of the twentieth century, it must be approached from a very different side, namely via some 'intellectually exciting interpretation'.

[1] This is the *fix point theorem* which was not originally proved by Gödel in its generality, which, however, has later found its way into the standard exposition of the incompleteness result. (See Gaifman, 2006.)

2. Incompleteness theorem: an 'intellectually exciting' interpretation

As a matter of fact, thrilling, mind-blowing or astonishing interpretations of the incompleteness phenomenon are dime a dozen. Let me mention only one of the earliest and most popular ones, due to the mathematician J. R. Lucas (1961, 112)[2]:

> Gödel's theorem seems to me to prove that Mechanism is false, that is, that minds cannot be explained as machines

Far from being of interest only for narrow specialists, this interpretation ascribes to Gödel's result the power of deciding one of the most irritating conundrums mankind has ever considered: namely *can there be a mechanical mind*?

How is this possible? How do we (and do we at all?) get from the 'boring' theorem itself to such an exciting interpretation? A usual train of thought that is supposed to get us there is approximately the following:

1. Gödel showed that any reasonable axiomatic theory of arithmetic contains an *I-am-unprovable-statement*, a statement claiming of itself that it is unprovable, and that this statement can be neither provable, nor refutable.
2. But then what the statement says is true, and hence there is a statement which is true, but unprovable.
3. Hence, as we *know* that the *I-am-unprovable-statement* is true, we can know something which we cannot prove.
4. Hence, we have methods of reaching (mathematical) truth beyond proof.
5. Hence, we, humans, can reason in a way no machine will ever be able to.

3. What exactly did Gödel prove?

Before we can evaluate the soundness of the above train of thought, let me recapitulate the bulk of Gödel's result in more familiar words. As is well known, Gödel pointed out that there exists an effective one-to-one assignment of natural numbers (and hence numerals) to formulas and sequences of formulas of PA (we will speak, as usual, about their *Gödel numbers*); and showed that given such an enumeration, we can define a predicate **Prf** so that

1. if m is the number of a proof of the formula with the number n, then **Prf**(**m**,**n**) is provable in PA (where **m** and **n** are the numerals standing for the numbers m and n, respectively); and

2. if m is not the number of a proof of the formula with the number n, then ¬**Prf**(**m**,**n**) is provable.

Provided that PA is consistent, and hence that the provability of ¬**Prf**(**m**,**n**) excludes the provability of **Prf**(**m**,**n**), it is obviously the case that **Prf**(**m**,**n**) is provable if and only if m is the number of a proof of the formula with the number n. Now defining **Pr**(x) as $\exists y$**Prf**(y,x) yields us:

[2] There are lots of others; the most popular being probably Penrose (1989).

3. If the formula with a number n is provable, then there is an m so that **Prf**(**m**,**n**), and hence **Pr**(**n**), is provable.

4. If the formula with the number n is not provable, ¬**Prf**(m,**n**) is provable for every m, and provided PA is ω-consistent (which means that for no formula $F[x]$ it can contain $\neg F[n]$ for every numeral n while containing $\exists x F[x]$), ∃x**Prf**(x,**n**), and hence **Pr**(**n**), is *not* provable.

This means that provided ω-consistency, a formula with the number n is provable iff **Pr**(n) is provable.

Moreover, Gödel showed that there is a formula ***G*,** our *I-am-unprovable-statement*, such that the equivalence ***G*** ↔ ¬**Pr**(**g**), where **g** is the Gödel number of **G**, is provable in PA. Hence ***G*** 'says of itself' that it is unprovable. This implies that ¬***G*** is provable iff **Pr**(**g**) is and as the last formula is, as we saw, provable iff ***G*** is, ¬***G*** is provable iff ***G*** is. As a result, neither ***G***, nor ¬***G*** can be provable - in pain of the inconsistency of PA.

Furthermore, as ***G*** says of itself that it is unprovable and it is indeed unprovable, it appears to be *true*, hence it is the instance of a true, but unprovable sentence.

Let us, at this point, also stress what is *not* the case:

5. It is *not* the case that whenever **f** is the number of a formula ***F***, the equivalence ***F*** ↔ **Pr**(**f**) is provable. If this were the case, PA would be inconsistent, for this would yield the provability of ***G*** ↔ **Pr**(**g**) (where **g** is the numeral for the Gödel's number of ***G***), whereas we know that **Pr**(**g**) ↔ ¬***G*** is provable.

6. It is *not* even the case that ¬**Pr**(**n**) is provable for all numbers n of unprovable formulas[3]. If this were the case, it would be the case that the unprovability of ***G*** would yield the provability of ¬**Pr**(**g**), which, as we know, yields the provability of ***G***.

We should not forget that this result was achieved with the help of the assumption of ω-consistency (we need not list consistency as a further assumption, for it follows from ω-consistency). However, this assumption is not essential; it turned out that by a modification of the proof this assumption can be discarded in favor of simple consistency[4].

However, the assumption of consistency is persistent, we cannot get rid of it. Does this compromise the result? What if PA is *not* consistent?

I think that though we do not have (and in fact cannot have[5]) a formal proof of it, it is reasonable to take it for consistent. Some theoreticians would claim that its consistency is demonstrated simply by the fact of the existence of the standard model; but this may be seen as problematic, for we cannot really *construct* the model, but only either prove its existence within set theory (the consistency of which is itself a problem), or simply claim that they know that it exists in force of the fact that we can *see* it within a Platonist heaven where it is

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[3] In fact we can prove ¬**Pr**(**n**) for not a single n; for proving that there is an unprovable formula would amount to proving that PA is consistent, which Gödel's result entails to be impossible.

[4] This was shown by Rosser (1936).

[5] One of the consequences of Gödel's result is that a theory cannot prove its own consistency, it can be proven only within another theory. But if this theory is stronger, it would be in the need of a consistency proof itself; whereas if it is weaker, it would lack the means needed to carry out the proof.

located[6]. However, I think that the more straightforward reason is that the axioms of PA are genuine *axioms* in the original sense of the word: they are formulas which capture our arithmetical intuitions that are so fundamental that it would be hard to even say what it would mean to reject them; and also the inference rules are clearly truth-preserving[7].

4. Is there a gap at all?

The existence of ***G***, of the *I-am-unprovable-statement*, seems to document the gap between truth and provability: ***G*** is true (and, moreover, is *recognizable* as true by us), but it is not provable. I will be arguing that the nature of the gap might involve more than meets the eye; but before I do so, it might be good to preempt a possible objection from the opposite side, namely the objection that the whole gap is a mere illusion. The point is that in so far as we understand PA as a first-order theory, ***G*** is in fact *not* (unconditionally) true - for it is true in some models of PA while false in others. Hence could it not be the case that our conclusion that ***G*** is true was simply mistaken?

Suppose – and disregard the apparent oddity of the idea for a moment – that beside the *genuine* natural numbers there are also certain *fake* natural numbers which are somehow very good in hiding among the genuine ones so that we have difficulties to tell the two kinds of numbers apart. Suppose that **Prf** holds between one of these numbers and the number **f** of a formula ***F*** of PA. Then we will conclude that ∃*x***Prf**(*x*,**f**), and hence **Prf**(**f**). We will conclude that ***F*** is provable, but we will be mistaken, for there will be no *genuine* natural number, i.e. a potential number of a proof, related to **f** by **Prf**.

Now return to the reasons which made us believe that ***G*** was true. The basic reason was that we believed that ***G*** were saying of itself that it is unprovable, i.e. that ¬∃*x***Prf**(*x*,**g**)↔***G***. But provided the possibility outsketched above, ¬∃*x***Prf**(*x*,**g**) might be false even if there is no proof of ***G*** – hence ***G*** does not say of itself that it is unprovable, but merely that it is 'unquasiprovable' (where not everything that is quasiprovable is provable, hence it is not the case that something is unquasiprovable if it is unprovable). In such a case, ***G*** might not be unambiguously true – it would say something that might be, provided ***G*** is unprovable, true or false depending on whether some of the fake numbers manage to sneak in among the genuine ones.

To be sure, this sounds like a fairy tale. But in fact understanding PA as a first-order theory leads to a tale very similar to this one. As was shown also by Gödel (1930), every first order theory must have a model, and so the first-order formulation of PA is bound to have more than one model, and consequently it must have models containing, besides the genuine (*standard*) numbers also the fake (*nonstandard*) ones. Hence there will be a model in which ***G***

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6 Some theoreticians argue that this was the view of Gödel himself, and indeed some claims of Gödel indicate that such 'Platonist extremism' was not alien to him. On the other hand, as Feferman (2006) points out, things are not this simple, for Gödel might be found to make also blatantly anti-Platonist claims, e.g.: 'if interpreted as meaningful statements, [these axioms] necessarily presuppose a kind of Platonism, which cannot satisfy any critical mind and which does not even produce the conviction that they are consistent'. See also Potter (2001).

7 Note that just as in case somebody were to dispute whether a prototypical example of a tree is *really* a tree, we would have to see it as undermining the very concept of tree, we would have to see disputing the consistency of axioms of *this* kind as undermining the very concept of consistency.

is true, but also a model in which ***G*** is false (and the latter one is bound to contain the non-standard numbers). In other words, we must have a model of PA+{***G***} as well as a model of PA+{¬***G***}.

Why should we believe that we cannot tell the standard numbers from the nonstandard ones? Because, someone would insist, this is a provable fact. And indeed, it is a provable fact, *if we take PA to be a first-order theory*. But why should we take PA as a first-order (rather than second-order) theory? Because, someone might want to answer, (standard[8]) second-order logic is not a real logic, for it lacks complete axiomatization – and as Gödel's proof builds on the axiomatization of arithmetic, an unaxiomatizable logic is out of the question.

But this answer misconstrues the proposal to base arithmetic on second-order logic. Gödel's system is, of course, about the *axiomatic system* of PA, and hence it is axiomatics that we must focus on even in the second-order case. The significant difference consists in *semantics* – namely in the fact that the second-order axiom of induction, in contrast to the first-order one, is taken to exclude all non-standard models; and that hence in second-order PA, there is no model of PA+{¬***G***}. As a result, ***G*** may be seen as unambiguously true, *viz.* true in the single model of PA. Hence the difference between the first-order and the second-order PA is precisely in that the latter allows for an unprovable truth[9].

Hence there appears to be no reason not to understand the intuitive PA as a *second-order* theory (save for the reason that going second-order does not mean any substantial gain in what we can prove, and thus *form the viewpoint of proving* it may be better to stay on the simpler first-order level). The question of the proper framework for explicating intuitive arithmetic is complicated and it need not have a unique, context-independent answer; but be it as it may, claiming that ***G*** is not true in force of the fact that there *must* be a model of PA+{¬***G***} is unwarranted – there *need not* be one.

5. Why is the undecidable formula true?

Does all of this mean that the gap between truth and proof – the existence of a discoverable but unprovable truth – is a brute fact? Here we must be careful. A student of Gödel's proof comes to see that ***G*** is true, and she comes to see that it is unprovable. But how does she manage to *see* that it is true? Well, because Gödel *demonstrates* it – he puts forward a transparent chain of reasons leading from the obvious to the truth of ***G***. But is "a transparent chain of reasoning", after all, not what a *proof* amounts to?

Let us recapitulate the way in which we reach the conclusion that ***G***, i.e. our *I-am-unprovable-statement*, is true. It is true because

(a) it says of itself that it is unprovable, and

(b) it is indeed unprovable.

Why is (a) the case? The reason obviously is that

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[8] For the relevant difference between first-order and second-order logic see Peregrin (1997).

[9] The nature of the difference may become clearer if we see it not as one between first-order and second-order logic (or arithmetic), but rather one between the Henkinian and the standard version of second-order logic (where the former is known to be reducible to the first-order case). See Peregrin (*ibid.*)

(aa) In PA, we can prove that ***G*** is equivalent to the formula stating that the number assigned to ***G*** does not have the property possessed by all and only numbers of provable formulas have ($\vdash_{PA} \neg$**Pr**[**g**]$\leftrightarrow$***G***)

(b), on the other hand, is the case because

(ba) the provability of ***G*** implies its refutability (provability of $\neg$***G***); whereas

(bb) PA is consistent.

Further, (ba) is the case because

(baa) the provability of ***G*** ($\vdash_{PA}$ ***G***) implies the provability of the claim that the number assigned to ***G*** belongs among the numbers of provable formulas ($\vdash_{PA}$ **Pr**[**g**]); and

(bab) the claim that the number assigned to ***G*** belongs to the numbers of provable formulas ($\vdash_{PA}$ **Pr**[**g**]) implies the provability of the negation of ***G*** ($\vdash_{PA} \neg$***G***)

(baa) is the case because of the way **Pr** is constructed;

(bab) follows from (aa);

Now (bb) is the case because

(bba) the axioms of PA are true (of the standard model) and its inference rules are truth-preserving.

And we could go into greater details.

Disregarding all simplifications, this *looks* as a proof - if anything does! Hence it would seem we *are*, after all, capable of proving ***G***. Of course, it is not a proof in the sense of *proof* considered by Gödel; hence in order not to cause a confusion, we will call it a *demonstration*. But could it not be, then, that the sense of the term *proof* considered by Gödel is suspect?

6. Proving the unprovable?

It is a plain fact that we can demonstrate the truth of ***G***, and hence prove it in the intuitive sense of the word. To be sure, we are not able to do it wholly *within PA*, for the demonstration requires to step out of PA and look at it, as it were, 'from outside'. Is the demonstration of ***G*** that convinces us that ***G*** is true, 'unformalizable' (and thus perhaps achievable somehow exclusively within the medium of human mind)? Not really. In fact, we can consider various kinds of (quasi)formal versions of demonstrations of ***G***.

1. First, let us notice that we can prove ***G*** in PA+{***G***}. This is, of course, an utterly trivial observation; but it is worth stating, for it points out that there is no one true formula of arithmetic which could not be proved in an extension of PA. In other words, we can prove *any* true arithmetical formula in *an* axiomatic system of arithmetic. (The only problem is that there is, of course, no system in which we can prove *every* such proposition.)

2. More substantially, we can see that every true arithmetical formula could be proven in a system with a 'quasirule', such as the ω-rule:

$F[0], F[1], F[2], \ldots \vdash \forall x F[x].$

The point is that if we accept this as a rule of proof, we can arguably wholly imitate the definition of the truth-in-the-standard-model in terms of provability, and hence reach the coincidence of truth and provability[10].

[10] See also Peregrin (2006).

(To avoid misunderstanding let us return to our description of Gödel's proof. There we remarked that if it were the case that ¬**Pr**(**n**) were provable for all numbers *n* of unprovable formulas, the unprovability of ***G*** would yield the provability of ***G***, whereas the ω-rule now seem to guarantee that ¬**Pr**(**n**) is provable for all numbers *n* of unprovable formulas. So do we not end up in a contradiction? No; for adding the ω-rule to our rules of proof would compromise the original predicate **Pr**. It would no longer be the case that **Pr**(*n*) would be provable for every number *n* of a provable formula; and, more importantly, no predicate which would do better that **Pr** in this respect would be available. Gödel's proof is based on the assumption that provability is *finitary* and hence capturable by the means of PA; and once we elevate this restriction, it is no longer reproducible.)

It is, to be sure, quite dubious to call a system with the ω-rule a system of *rules of proof*: we cannot *apply* the ω-rule in the way in which we apply what we standardly think of as rules of proof (for due to the infinity of its premises we would be never able to check its applicability). However, it is one thing to say that this rule is not practically applicable and it is another thing to say that to find out that something is derivable with its help is a 'non-mechanical', 'nonalgorithmic' or 'exclusively human' matter.

3. Even more importantly, it seems that we could prove every true arithmetical formula in a 'quasiaxiomatic' system, which would have not only rules, but also metarules, rules for deriving rules from rules. This seems to be, in effect, the proposal of Dummett (1963):

> The only way to *explain* quantification over natural numbers is to state the principles of recognizing as true a statement which involves it; Gödel's discovery amounted to the demonstration that the class of these principles cannot be specified exactly once for all, but must be acknowledged to be an indefinitely extensible class.

This is to say that though the 'rules' underlying the demonstration of the truth of ***G*** are available only *after* we have the system of axioms and rules of PA and therefore it cannot be among them, once we accept that the system is open and extendable, there is no reason not to take these 'rules' as its extension and in this sense as its part.

One way to give this idea a formal shape is to reflect on the fact that we would be able to prove ***G*** if we were able to prove the consistency of PA (for the implication from the consistency to ***G*** is a theorem of PA). Whereas we cannot prove this consistency within PA, we can extend PA with its postulation. This was proposed and studied by Feferman (1962; 1991) under the title of *reflection principles.* Of course that then the office of the Gödel formula shifts from ***G*** to a different formula, but we can repeat the procedure and so continue indefinitely; whereby *every* Gödel formula is proved at *some* stage. (The price to be payed for there not being a Gödel formula for the theory as a whole is, of course, again that it cannot be finitary.)

7. What is the moral we should draw from Gödel's result?

Is there a crucial moral we should draw from Gödel's result? The train of thought leading from the result to the 'impossibility of mechanization of human thought', which we sketched in the beginning of this paper, seems to be based on the assumption that this moral concerns the confrontation of 'syntax' (provability) and 'semantics' (model-theoretic truth); in particular that

No axiomatic system can prove all truths.

In this way, Gödel's result seems to point out the fact that 'syntax' can never catch up with 'semantics'. However, is this characterization adequate? Gödel, as we saw, did not prove anything concerning *all axiomatic systems* – what he was analyzing was exclusively systems of *arithmetic*. Hence should we not rather state

No axiomatic system of arithmetic can prove all truths about natural numbers?

Expressed thus, it would seem that what his result concerns is the confrontation of 'syntax' *of arithmetic* with 'semantics' *of arithmetic*. But on closer inspection, even this might not be *the* moral. For as we have seen, Gödel's proof can be also seen as not concerning directly truth vs. proof, but rather the fact that no theory that is about natural numbers (i.e. is sound is w.r.t. the standard model, **N**) can reveal all truths about them, and hence can be guaranteed to exclude anything that is not a natural number (i.e. is semantically complete w.r.t. **N**). Hence what about

No axiomatic system sound w.r.t. N is complete w.r.t. N?

Now it would seem that the result is a matter exclusively of 'semantics' of arithmetic. But there is still another way to characterize Gödel's result, a way which may be the most faithful to the very approach of Gödel:

No consistent axiomatic system of arithmetic is syntactically complete,

Construed in this way Gödel's result appears as a matter exclusively of 'syntax' of arithmetic.

I think that this exegetical exercise indicates that we should be wary of seeing Gödel's result as bringing about a simple moral. Gödel's proof is a piece of mathematics (a masterpiece, for that matter!), whose significance surely *does* outrun the boundaries of mathematics; but does not do so in a way which would be transparent and which could be identified without a deep understanding of the matter.

8. Conclusion

What Gödel's result shows beyond doubt is that there is a notion of proof beyond that codified by the Hilbertian concept of axiomatic system. We can envisage this in terms of the contrast between 'syntax' (proof theory) and 'semantics' (model theory); however, there are also ways of approaching the whole matter purely proof-theoretically, as a contrast between stricter and looser concepts of proof. Hence we should be very careful when using Gödel's result to underpin breathtaking theses about the nature of human reason.

We must also bear in mind that though ***G*** undoubtedly has a significance that is internal to the axiom system of PA, it does not directly follow that it has also an 'external' significance – *viz.* that it articulates a meaningful statement of 'intuitive' mathematics. The latter kind of significance is a matter of the relationship between a formal model and the non-formal phenomenon it aims to capture (see Peregrin, 2000). But this is a complicated problem which I must leave for another occasion.

References

Dummett, M. (1963): 'The Philosophical Significance of Gödel's Theorem', *Ratio* V, 140-155; reprinted in Dummett: *Truth and other Enigmas*, Duckworth, London, 1978.

Feferman, S. (1962): 'Transfinite recursive progressions of axiomatic theories', *Journal of Symbolic Logic* 27, 259–316.

Feferman, S. (1991), 'Reflecting on Incompleteness', *Journal of Symbolic Logic* 56, 1–49.

Feferman, S. (2006): 'Review of *Incompleteness. The proof and paradox of Kurt Gödel,* by Rebecca Goldstein', *London Review of Books,* vol. 28, no. 3.

Gaifman, H. (2006): 'Naming and Diagonalization, from Cantor to Gödel to Kleene', *Oxford Journal of the IGPL* 15, 709-728.

Gödel, K. (1930): 'Die Vollständigkeit der Axiome des logischen Funktionenkalküls', *Monatshefte für Mathematik und Physik* 37, 349-360.

Gödel, K. (1931): 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatshefte für Mathematik und Physik* 38, 173-198.

Lucas, J. R. (1961): 'Minds, Machines and Gödel', *Philosophy* 36, 112-127.

Penrose, R. (1989): *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics*, Oxford University Press, New York.

Peregrin, J. (1997): 'What does one need when she needs 'higher-order' logic?', *Proceedings of LOGICA'96*, Filosofia, Praha.

Peregrin, J. (2006): 'Consequence & Inference', Kolman, V. (ed.): *Miscellanea Logica: Truth and Proof*, FF UK, Prague.

Peregrin, J. (2000): 'The 'Natural' and the 'Formal'', *Journal of Philosophical Logic* 29, 75-101.

Potter, M. (2001): 'Was Gödel a Gödelian Platonist?', *Philosophia Mathematica* (*series* III) 3, 331-346.

Rosser, J.B. (1936): 'Extensions of some Theorems of Gödel and Church', *Journal of Symbolic Logic* 1, 87-91.