Logic and *nothing else*

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1. Suppose we were to challenge a mathematician unfamiliar with the problems of modern logic to delimit natural numbers. Very probably his delimitation would run something like this:

(1) 0 (or 1) *is a natural number*.
(2) *A successor of a natural number is a natural number*.
(3) *Nothing else is a natural number*.

Clauses (1) and (2) guarantee the inclusion of all 'intuitive' natural numbers, and (3) guarantees the exclusion of all other objects. Thus, in particular, no nonstandard numbers, which would follow *after* the intuitive ones are admitted (nonstandard numbers are found in nonstandard models of Peano arithmetic, in which the standard natural numbers are followed by one or more 'copies' of integers running from minus infinity to infinity)[1].

What is problematic about this delimitation? I suspect that its hypothetical proponent would see its weakest point in the unexplained concept of successor. However, we logicians know better (or at least some of us are convinced that we do): it is clause (3) which harbours *the* neuralgic spot, by dint of resisting any reasonable logical formalization to the point of appearing utterly void!

There is, of course, no problem with regimenting (1) - we only need an individual constant **0** and a unary predicate constant **N** (the one whose meaning we are interested in) and we postulate

(1) $\mathbf{N}(\mathbf{0})$

Equally of course, (2) is straightforward. We need an additional unary function constant **S** and we have

(2) $\forall x(\mathbf{N}(x) \rightarrow \mathbf{N}(\mathbf{S}(x)))$.

(A problematic point here is the presumption that the contestants **0** and **S** acquire content either by some means external to the axiomatic system, or, and this is more usual, by being co-defined together with **N**. If the latter is the case, then our axioms seem to be insufficient for

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[1] See Gaifman (2004) for a general exposition of 'nonstandardness'.

the task. However, current orthodoxy seems to be that if we extend our set of axioms by the standard axioms for addition and multiplication, we can confer the appropriate meanings on all the extralogical constants involved in mutual interdependence.)

But (3) is a much harder nut. How can we regiment the *nothing else*? The problem appears to be that this term is anaphoric – it refers to everything and anything other than what is included among natural numbers in force of (1) and (2). *Prima facie*, it would seem that the logical structure of (3) is

$$\forall x\,(\neg P(x)\rightarrow\neg\mathbf{N}(x)),$$

i.e., by contraposition,

$$(*)\ \forall x(\mathbf{N}(x)\rightarrow P(x)),$$

where $\neg P$ is the property of *being something else than what is specified by* (1) *and* (2) – hence P is a property which, according to (1) and (2), a natural number *does* have. And what exactly is this property?

Well, as (1) states that zero is a natural number and (2) says that a successor of a natural number is a number, we may consider it to be the property of being either zero or a successor of a natural number, which would yield the regimentation of (3) as

$$(3\text{-}1)\ \forall x(\mathbf{N}(x)\rightarrow((x{=}0)\vee\exists y\,(x{=}\mathrm{S}(y)\wedge\mathrm{N}(y)))$$

However, it is readily seen that this axiom is unable to exclude the nonstandard numbers: it is clear that, as in the standard, so also in every nonstandard model, every number other than zero has a predecessor. (The crux of the matter is that the nonstandard numbers do not copy just standard natural numbers, but all the whole numbers, starting from minus infinity and hence they have no beginning.) This indicates that the strength of the *nothing else* clause has evaporated somewhere along the regimentative way from (3) to (3-1).

Hence let's try another route. Perhaps the *nothing else* of (3) should be regimented in terms of consequences of (1) and (2) – perhaps nothing is a number unless its being a number follows from (1) and (2). However, it is clear that

$$(3\text{-}2)\ \forall x(\mathbf{N}(x)\rightarrow((\mathbf{N}(0)\wedge\forall y(\mathbf{N}(y)\rightarrow\mathbf{N}(\mathbf{S}(y))))\rightarrow\mathbf{N}(x))),$$

which might appear to say that nothing is a number unless its being a number is implied by (1) and (2), would not work. As is readily seen, this formula is tautologous. The reason, obviously, is that we regimented the concept of consequence in terms of material implication. What we, thus, would seem to need is

$$(3\text{-}2')\ \forall x(\mathbf{N}(x)\rightarrow((\mathbf{N}(0)\wedge\forall y(\mathbf{N}(y)\rightarrow\mathbf{N}(\mathbf{S}(y))))\Rightarrow\mathbf{N}(x))),$$

where $\Rightarrow$ is some kind of implication which is 'stricter' than the material one. Would it be possible to exclude the non-standard numbers by means of such an axiom? The answer obviously depends on the semantics we give to '$\Rightarrow$' What comes to mind, of course, is strict implication of the Lewisian kind and subsequent modal logics; but there is an uncountable number of modal logics and, consequently, an uncountable number of versions of strict implication – so can we pinpoint one which would help us?

It would seem that what we need is a strict implication such that, expressed in terms of its Kripkean semantics, it would not necessarily take us from a world containing natural numbers to a world containing also non-standard natural numbers. This is a rather mind-boggling requirement: can different possible worlds contain different sets of natural numbers? And if so, can we pinpoint our required version of implication in a way that would not be blatantly circular?

Since it is hard to see any ground on which to answer such questions, rather than adopting modal logic, we should perhaps remain within the framework of the classical one, but go *meta* instead. The point is that the variant of '$\Rightarrow$' we need *does* have a sense independent of our considerations of natural numbers, namely representing logical entailment. Perhaps, therefore, we should relinquish efforts to express (3) as an axiom and accommodate it as a stipulation on the metalevel. For example, we may stipulate

x is a natural number if '$\mathbf{N}(x)$' logically follows from (1) and (2).

Given the Tarskian explication of logical consequence this yields us

x is a natural number if every model of (1) and (2) is a model of '$\mathbf{N}(x)$'.

This can further yield us, by way of generalization,

(3-3) only the minimal model of (1) and (2) is a model of arithmetic.

This is reminiscent of how Hilbert (1903) amended his axiomatization of geometry – the list of axioms he gives there (though not its modified version presented in the later edition of the same book) contains the following 'axiom' (p. 25): "The elements of geometry form a system which is incapable of being extended, provided that we regard the five groups of axioms as valid." This amounts to the exclusion, by *fiat*, of all but the maximal model of his geometry; whereas (3-3) amounts to the exclusion, by the same kind of *fiat*, of all but the minimal one.

However, could we not have something reasonably seen as a *regimentation* of (3)? What (1) and (2), to which (3) refers, seem to say is that to be a natural number is to have a property N fulfilling them. This suggests that *to be a natural number* is to have a property of this kind, i.e. to be an x such that

$$\exists N((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y))))\wedge N(x)).$$

Substituting this into (*) for P, gives us

(3-4) $\forall x(\mathbf{N}(x)\rightarrow\exists N((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y))))\wedge N(x)))$.

But as is readily seen, this claim follows from (1) and (2); hence this regimentation would render (3) as empty. So unless we are ready to admit that (3) says nothing at all, we must find out what is wrong with our regimentation.

So let us try to find a different entering wedge. We have concluded that according to (1) and (2), being a natural number amounts to having a property N such that $N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y)))$. There may be many such properties, and (1) and (2) therefore present only a *partial* specification of what it takes to be a natural number. Once the specification is completed, it may turn out that having some of these properties does *not* entail being a natural number. The only thing we know for sure at this point is that if an object has *all* of them, it cannot escape being a natural number (for definitely there is at one of these properties that *does* entail it), whichever of the properties is picked up as *the* constitutive property of a natural number by a completion of (1) and (2).

Now how do we complete (1) and (2)? Our way of doing this is (3). From the vantage point sketeched above, what (3) says, namely that that nothing is a number unless is *must* be - in force of (1) and (2) - a natural number, may be read as saying that nothing is a natural number unless it has *all* the properties doing justice to $N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y)))$. Hence from this vantage point is seems that the property we should consider is

$\forall N((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y)))\rightarrow N(x))$.

Let us substitute this into (*):

(3-4') $\forall x(\mathbf{N}(x)\rightarrow\forall N.((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y)))\rightarrow N(x)))$.

Moving the quantifier $\forall N$ to the front, we get

$\forall N\forall x(\mathbf{N}(x)\rightarrow((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y))))\rightarrow N(x)))$,

which is then obviously equivalent with

$\forall N\forall x((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y))))\rightarrow(\mathbf{N}(x)\rightarrow N(x)))$.

Now moving the quantifier $\forall x$ past the antecedent of $\rightarrow$ (which does not have a free occurrence of x) yields us

(I) $\forall N((N(\mathbf{0})\wedge\forall y(N(y)\rightarrow N(\mathbf{S}(y))))\rightarrow\forall x(\mathbf{N}(x)\rightarrow N(x)))$, which is obviously nothing else than a version of the (second-order) axiom of induction.

This brings us, I believe, to an important moral. It may seem that the second-order version of induction is problematic because it requires us to understand the notion of *all* subsets of an infinite domain; and this in turn may lead us to conclude that only the first-order version makes *clear* sense. However, now we see that the second-order version serves, perhaps, as an (indirect) regimentation of a natural language claim that appears to be utterly perspicuous – the claim (3). Maybe, then, the moral we shouls draw is that grasping the (standard) second-order semantics does not presuppose understanding what *all* subsets of an infinite set amount to, but merely presupposes understanding claims like (3).

2. This further brings us to the general problem of the relationship of a logical regimentation to what it regiments. Many authors seem to construe logical regimentation of a sentence as a kind of an instruction for verifying the sentence. Hence, if a sentence is regimented by a formula starting with $\forall x$, it must be verified by going through the universe. Take the popular *raven paradox*, brought forward by Hempel (1943): as *All ravens are black* is regimented as $\forall x(\boldsymbol{raven}(x)\rightarrow\boldsymbol{black}(x))$ which is equivalent to $\forall x(\neg\boldsymbol{black}(x)\rightarrow\neg\boldsymbol{raven}(x))$, the sentence gets confirmed by finding any non black entity that is not a raven. There may be more ways of attacking the paradox, but the above considerations suggest a head-on one: why should we think that a logical form tells us anything about the process of confirmation?

But if logical form is not a matter of confirmation or verification, what is it good for? The answer, I believe, is given (not only) by Davidson (1970, p. 140): "To give the logical form of a sentence is to give its logical location in the totality of sentences, to describe it in a way that explicitly determines what sentences it entails and what sentences it is entailed by"[2]. This suggests that a logical form *does* tell us *something* concerning verification: it may suggest some easier-to-verify claims from which the given claim follows. However, it has nothing to do with verification in the sense of confrontation of language and the world. In particular, the presence of the quantifier does not tell us to initiate a search through a given set.

Logical form, then, reveals inferential significance[3]. What is the inferential significance of the general quantifier? It is clear that $\forall x.\mathrm{F}[x]$ licenses all claims of the form $\mathrm{F}[x]$ with any grammatically acceptable expression of the language in question in the place of x; i.e. that any such sentence is *inferable from* $\forall x.\mathrm{F}[x]$. This seems relatively uncontroversial.

Far more vexing is the question of what licenses the claim $\forall x.\mathrm{F}[x]$ itself, i.e. what this claim is inferable from. It is obvious that in its normal construal, all the claims of the form $\mathrm{F}[x]$ would not suffice. Even were we to admit 'inferences' with an infinite number of premises (which, needless to say, would be problematic in itself), an inference from all the sentences of the above shape to $\forall x.\mathrm{F}[x]$ would still not be acceptable as valid. This is the problem of the induction axiom of arithmetic – if we restrict ourselves to its substitutional instances, we have, in effect, the first-order version of Peano arithmetic with its nonstandard

[2] See my discussion of the problem of logical form in Peregrin (2001, Chapter 10).

[3] See Peregrin (*ibid.*) for a general discussion of this *inferentialist* standpoint.

models. It is precisely this that directs us towards thinking that we must exactly specify the domain in order to construe the quantifier objectually.

And, indeed, we can do this – the trouble, however, is that in reality this perhaps only leads us to an illusion of explanation. Logicians have already pointed out that to say ‘the domain of quantification of the second-order quantifier of Peano arithmetic is the set of all sets of natural numbers’ is tantamount to saying next to nothing, for the phrase "all subsets", though quite clear for finite sets, is so unclear for their infinite cousins that using it to define the domain of quantification is little improvement on leaving the domain simply unspecified.

But there is, I think, an alternative: $\forall x.\mathrm{F}[x]$ may be construed not as claiming that $\mathrm{F}[x]$ holds for *every x* (of an antecedently specified domain), but that it holds for *any x* (i.e. for any kind of object anybody might point out). This construal avoids presupposing a fixed domain restricting the range from which the instances of x can be drawn. However, one may wonder, how can I claim that I can show something about every x, if I have no inkling what objects may come into consideration? Well, certainly I may claim that all humans are mortal without knowing all the humans there are; I know it because it is a matter *of principle* (let us now avoid the discussion of the extent to which this *in principle* must be related to *analytic*). Hence I can be entitled to $\forall x.\mathrm{F}[x]$ even on such an ‘open-ended’ reading of the universal quantifier.

Now if we forget about the assumption that any object we are considering when making a general claim must be drawn from a pre-established domain (which is the result of the indoctrination of the standard apparatus of semantics of modern logic), (3) reads *whatever object that is not bound to fulfill* (1) *and* (2) *you can manage to point out, it is not a number*. I think this is the natural reading. But in fact, as (1) and (2) do not articulate criteria for individual object’s being numbers, but rather for (structured) sets of such objects, this should be modified to *whatever set of objects you can give me that is not bound to fulfill* (1) *and* (2) *contains something that is not a number*. And this seems to lead precisely to the induction formula reached above.

3. To summarize: The way from first-order to second-order logic (and especially arithmetic) is often thought of as tortuous (if not vicious) – to be preying upon the obscure concept of the set of all subsets of an infinite set. However, in practise we do achieve the second-order effect by means of the utterly perspicuous *nothing else*. I have tried to indicate that if the workings of the *nothing else* are logically unpacked, it indeed turns out to involve a second-order quantification. However, I think that this should not lead us to conclude that the *nothing else* is in fact much more complicated than it would seem to be, but rather to conclude that second order quantification may be less vitiated by unclarity than often supposed. I opt for this conclusion because I do not see logical analysis as necessarily leading us from 'appearance' (an expression's uninformative surface) to 'reality' (its true logical form); I see it as a matter merely of depicting logical contents from fresh visual angles.

References

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