Inferentializing Consequence

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The proof of correctness and completeness of a logical calculus w.r.t. a given semantics can be read as telling us that the tautologies (or, more generally, the relation of consequence) specified in a model-theoretic way can be equally well specified in a proof-theoretic way, by means of the calculus (as the theorems, resp. the relation of inferability of the calculus). Thus we know that both for the classical propositional calculus and for the classical predicate calculus theorems and tautologies represent two sides of the same coin. We also know that the relation of inference as instituted by any of the common axiom systems of the classical propositional calculus coincides with the relation of consequence defined in terms of the truth tables; whereas the situation is a little bit more complicated w.r.t. the classical predicate calculus (the coincidence occurs if we restrict ourselves to closed formulas; otherwise $\forall x Fx$ is inferable from Fx without being its consequence). And of course we also know cases where a class of tautologies of a semantic system does not coincide with the class of theorems of any calculus. (The paradigmatic case is the second-order predicate calculus with standard semantics.)

This may make us consider the problem of "inferentializability". Which semantic systems are "inferentializable" in the sense that their tautologies (their relation of consequence, respectively) coincide with the class of theorems (the relation of inferability, respectively) of a calculus? One answer is ready: it is if and only if the set of tautologies is recursively enumarable. But this answer is not very informative, indeed saying that the set is recursively enumerable is only reiterating that it conicides with the class of theorems of a calculus. Moreover, paying due attention to the terms such as "calculus" and "inference" shows us that it is possible to relate them to various "levels", whereby the problem of inferentializability becomes quite nontrivial.

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1 Consequence

Consequence, as the concept is usually understood, amounts to truth-preservation, i.e., A is a consequence of $A_1, \ldots, A_n$ iff the truth of all of $A_1, \ldots, A_n$ brings about the truth of A, i.e., iff any truth valuation mapping all of $A_1, \ldots, A_n$ on 1 maps also A on 1.[1] It is obvious that the "any" from the previous sentence cannot mean "any whatsoever" (of course there *does* exist a function mapping all of $A_1, \ldots, A_n$ on 1 and A on 0!), it must mean something like "any admissible". Hence there must be some concept of admissibility in play: some mappings of sentences of $\{0, 1\}$ will be admissible, others not. But, of course, that if we take the sentences to be sentences of a meaningful language, such a division of valuations is forthcoming: if $A_1, \ldots, A_n$ are *Fido is a dog* and *Every dog is a mammal* (hence $n = 2$), A is *Fido is a mammal*, then the valuation mapping the former two sentences on 1 and the latter one on 0 is not admissible — it is not compatible with the semantics of English.

Hence we assume that any semantics of any language provides for the division of the sentences of the language into true and false, thereby dividing the space of the mappings of the sentences on $\{0, 1\}$ into admissible and inadmissible. (In fact I maintain a much stronger thesis, namely that any semantics can be *reduced* to such a division, but I am not going to argue for this thesis here — I have done so elsewhere, see (Peregrin, 1997).) Thereby it also establishes the relation of consequence, as the relation of truth-preservation for all admissible valuations. If we use the sentences $S_1, S_2, \ldots$ of the language in question to mark columns of the following table using all possible truth-valuations as its rows, we can look at the delimitation of the admissible valuations as striking out rows of the table.

	S_1	S_2	S_3	S_4	$\cdots$
v_1	0	0	0	0	$\cdots$
~~v_2~~	~~1~~	~~0~~	~~0~~	~~0~~	~~$\cdots$~~
v_3	0	1	0	0	$\cdots$
v_4	1	1	0	0	$\cdots$
v_5	0	0	1	0	$\cdots$
~~v_6~~	~~1~~	~~0~~	~~1~~	~~0~~	~~$\cdots$~~
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

[1] See (Peregrin, 2006).

A more exact articulation of these notions yields the following definition:

Definition 5. A *semantic system* is an ordered pair $\langle S, V\rangle$, where S is a set (the elements of which are called *sentences*) and $V \subseteq \{0,1\}^S$. The elements of $\{0,1\}^S$ are called *valuations* (of S). (A valuation will be sometimes identified with the set of all those elements of S that are mapped on 1 by it.) The elements of V are called *admissible valuations of* $\langle S, V\rangle$, the other valuations (i.e. the elements of $\{0,1\}^S \setminus V$) are called *inadmissible*. The relation of *consequence* induced by this system is the relation $\models$ defined as follows

$$X \models A \text{ iff } v(A) = 1 \text{ for every } v \in V \text{ such that } v(B) = 1 \text{ for every } B \in X.$$

2 Varieties of Inference

Now consider the stipulation of an inference, $A_1, \ldots, A_n \vdash A$ (for some elements $A_1, \ldots, A_n$, A of S). Such a stipulation can be seen as excluding certain valuations: namely all those that map $A_1, \ldots, A_n$ on 1 and A on 0. (Thus, for example, the exclusions in the above table might be the result of stipulating $S_1 \vdash S_2$.) Hence if we call the pair constituted by a finite set of elements of S and an element of S an *inferon*, we can say that inferons exclude valuations and ask which sets of valuations can be demarcated by means of inferons.

Definition 6. An *inferon* (*over* S) is an ordered pair $\langle X, A\rangle$ where X is a finite subset of S and A is an element of S. An inferon is said to *exclude* an element v of $\{0,1\}^S$ iff $v(B) = 1$ for every $B \in X$ and $v(A) = 0$. An ordered pair $\langle S, \vdash\rangle$ such that S is a set and $\vdash$ is a finite set of inferons (i.e. a binary relation between finite subsets of S and elements of S) will be called an *inferential structure*. An inferential structure is said to *determine* a semantic system $\langle S, V\rangle$ iff V is the set of all and only elements of $\{0,1\}^S$ not excluded by any element of $\vdash$. A semantic system is called an *inferential* system iff it is determined by an inferential structure.

Now an obvious question is which semantic systems are inferential. But before we turn our attention to it, we will consider various possible generalizations of the concept of inference. First, let a *quasiinferon* differ from an inferon in that its second component is not a single statement, but a finite set of statements. A *quasiinferon* will exclude every valuation that maps every element of its first component on 1 and every element of its second component on 0. (Of course the concept of quasiinferon defined in this way is closely connected with the concept of *sequent* as introduced by (Gentzen, 1934) and (Gentzen, 1936).[2]) Second, let a *semiinferon* differ from an inferon in that its first component is not necessarily finite. A *semiquasiinferon* will be a quasiinferon with both its first and its second component not necessarily finite. Third, let a *protoinferential* structure be an inferential structure with its second component not necessarily finite (and think of the concepts of *protosemiinferential*, *protoquasiinferential* and *protosemiquasiinferential* structure analogously).

In the following definition, we abbreviate the prefixes, which have already become somewhat monstrous:

Definition 7. An element of $\mathrm{Pow}(S) \times \mathrm{Pow}(S)$ is called an *SQI-on over S*. It is called a *QI-on* if it is an element of $\mathrm{FPow}(S) \times \mathrm{FPow}(S)$ (where $\mathrm{FPow}(S)$ is the set of all finite subsets of S), it is called an *SI-on* if it is an element of $\mathrm{Pow}(S) \times S$ and it is called an *I-on* if it is an element of $\mathrm{FPow}(S) \times S$.[3] The ordered pair $\langle S, \vdash\rangle$ where $\vdash$ is a set of SQI-ons (QI-ons, SI-ons, I-ons) will be called a *PSQI-structure* (*PQI-structure*, *PSI-structure*, *PI-structure*). It is called an *SQI-structure* (*QI-structure*, *SI-structure*, *I-structure*) iff $\vdash$ is finite. An SQI-on $\langle X, Y\rangle$ is said to *exclude* an element v of $\{0,1\}^S$ iff $v(B) = 1$ for every $B \in X$ and $v(A) = 0$ for every $A \in Y$. A (P)(S)(Q)I-structure $\langle S, \vdash\rangle$ is said to *determine* a semantic system $\langle S, V\rangle$ iff V is the set of all and only elements of $\{0,1\}^S$ not excluded by any element of $\vdash$. A semantic system is called a *(P)(S)(Q)I-system* iff it is determined by a (P)(S)(Q)I-structure.

Summarizing the concepts introduced in this definition, we have the following table:

$\langle S, \vdash\rangle$ is a...	iff $\vdash$ is a...	$\vdash$ thus being a subset of
I-structure	a finite set of I-ons	$\mathrm{FPow}(S) \times S$
QI-structure	a finite set of QI-ons	$\mathrm{FPow}(S) \times \mathrm{FPow}(S)$
SI-structure	a finite set of SI-ons	$\mathrm{Seq}(S) \times S$
PI-structure	a set of I-ons	$\mathrm{FPow}(S) \times S$
SQI-structure	a finite set of SQI-ons	$\mathrm{Pow}(S) \times \mathrm{Pow}(S)$
PQI-structure	a set of QI-ons	$\mathrm{FPow}(S) \times \mathrm{FPow}(S)$
PSI-structure	a set of SI-ons	$\mathrm{Seq}(S) \times S$
PSQI-structure	a set of SQI-ons	$\mathrm{Pow}(S) \times \mathrm{Pow}(S)$

[2] For an exposition of sequent calculus and its relationship to the more straightforwardly inferential approach as embodied in natural deduction see, e.g., (Negri & Plato, 2001).

[3] Throughout the whole paper we identify singletons with their respective single elements; hence we often write simply v instead of $\{v\}$.

Our aim now is to find criteria of the various levels of inferentializability. Before we state and prove theorems crucial in this respect, we introduce some more definitions.

3 Criteria of Inferentializability

Definition 8. Let U be a set of valuations of a semantic system $\langle S, V\rangle$ (i.e. a subset of $\{0,1\}^S$). $T(U)$ (the set of U-*tautologies*) will be the set of all those elements of S which are mapped on 1 by all elements of U; and analogously $C(U)$ (the set of U-*contradictions*) will be the set of all those elements of S which are mapped on 0 by all elements of U. Let X and Y be subsets of S. The *cluster generated by* X *and* Y, $\mathrm{Cl}[X, Y]$, will be the set of all the valuations that map all elements of X on 1 and all elements of Y on 0. Generally, U is a *cluster* iff it contains (and hence is identical with) $\mathrm{Cl}[T(U), C(U)]$. A cluster U is called *finitary* iff both $T(U)$ and $C(U)$ are finite, it is called *inferential* iff $C(U)$ is a singleton.

Now it is clear that a semantic system $\langle S, V\rangle$ is a PSQI-system iff $\{0,1\}^S \setminus V$ is a union of clusters. (Hence every semantic system is a PSQI-system, for every single valuation constitutes a cluster.) The reason is that a system is a PSQI-system if its inadmissible valuations are determined by a set of SQI-ons and what an SQI-on excludes is a cluster of valuations. If we use specific kinds of SQI-ons, such as SI-ons, we will have a specific kind of clusters, like inferential clusters; and if we allow for only a finite number of SQI-ons, we will have to count with only finite unions. This yields us the facts summarized in the following table:

$\langle S, V\rangle$ is a...	iff $\{0,1\}^S \setminus V$ is a union of...
PSQI-system	clusters
PSI-system	inferential clusters
PQI-system	finitary clusters
SQI-system	a finite number of clusters
PI-system	finitary inferential clusters
SI-system	a finite number of inferential clusters
QI-system	a finite number of finitary clusters
I-system	a finite number of finitary inferential clusters

Theorem 3. *A semantic system $\langle S, V\rangle$ is a PSI-system iff V contains every $v \in \{0,1\}^S$ such that for every $A \in C(v)$ there is a $v' \in V$ such that $T(v) \subseteq T(v')$ and $A \in C(v')$.*

Proof. A semantic system $\langle S, V\rangle$ is a PSI-system system iff $\{0,1\}^S \setminus V$ is a union of inferential clusters. This is to say that it is a PSI-system iff for every $v \in \{0,1\}^S \setminus V$ there is a set $X \subseteq T(v)$ and a sentence $A \in C(v)$ such that no valuation v' such that $X \subseteq T(v')$ and $A \in C(v')$ is admissible. In other words, $\langle S, V\rangle$ is a PSI-system iff for every $v \notin V$ there is a set $X \subseteq T(v)$ and a sentence $A \in C(v)$ such that V does not contain any v' such that $X \subseteq T(v')$ and $A \in C(v')$. By contraposition, $\langle S, V\rangle$ is a PSI-system iff the following holds: given a valuation v, if for every set $X \subseteq T(v)$ and every sentence $A \in C(v)$ there is a $v' \in V$ such that $X \subseteq T(v')$ and $A \in C(v')$, then $v \in V$. This condition can obviously be simplified to: given a valuation v, if for every sentence $A \in C(v)$ there is a $v' \in V$ such that $T(v) \subseteq T(v')$ and $A \in C(v')$, then $v \in V$. □

Theorem 4. *A semantic system $\langle S, V\rangle$ is a PQI-system iff V contains every v such that for every finite $X \subseteq T(v)$ and finite $Y \subseteq C(v)$ there is a $v' \in V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$.*

Proof. A semantic system $\langle S, V\rangle$ is a PQI-system system iff $\{0,1\}^S \setminus V$ is a union of finite clusters. This is to say that it is a PQI-system iff for every $v \in \{0,1\}^S \setminus V$ there are finite sets $X \subseteq T(v)$ and $Y \subseteq C(v)$ such that no valuation v' such that $X \subseteq T(v')$ and $Y \subseteq C(v')$ is admissible. In other words, $\langle S, V\rangle$ is a PQI-system iff for every $v \notin V$ there are sets $X \subseteq T(v)$ and $Y \subseteq C(v)$ such that V does not contain any v' such that $X \subseteq T(v')$ and $Y \subseteq C(v')$. By contraposition, $\langle S, V\rangle$ is a PQI-system iff the following holds: given a valuation v, if for every sets $X \subseteq T(v)$ and $Y \subseteq C(v)$ there is a $v' \in V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$, then $v \in V$. This condition can obviously be simplified to: given a valuation v, if for every finite $X \subseteq T(v)$ and finite $Y \subseteq C(v)$ there is a $v' \in V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$, then $v \in V$. □

We leave out the proof of the following theorem, as it is straightforwardly analogous to the proofs of the previous two.

Theorem 5. *A semantic system $\langle S, V\rangle$ is a PI-system iff V contains every v such that for every finite $X \subseteq T(v)$ and every $A \in C(v)$ there is a $v' \in V$ such that $X \subseteq T(v')$ and $A \in C(v')$.*

Hence we have necessary and sufficient conditions for a semantic system being a PSI-, a PQI-, or a PI-system. Unfortunately, we do not have such conditions for its being an SQI-, an SI-, a QI-, or an I-system. However, we are able to formulate at least a useful *necessary* condition for its being an SQI-system.

Theorem 6. *A semantic system $\langle S, V\rangle$ is an SQI-system only if V contains no v such that for every finite $X \subseteq T(v)$ and finite $Y \subseteq C(v)$ there is a $v' \notin V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$.*

Proof. A semantic system $\langle S, V\rangle$ is a PQI-system iff $\{0,1\}^S \setminus V$ is a finite union of clusters. Hence if it is a PQI-system, there must exist a finite set I and two collections $\langle X^i\rangle_{i\in I}$, $\langle Y^i\rangle_{i\in I}$ of subsets of S so that

$$\{0,1\}^S \setminus V = \bigcup_{i\in I} \mathrm{Cl}[X^i, Y^i].$$

This is the case iff V equals the complement of $\bigcup_{i\in I} \mathrm{Cl}[X^i, Y^i]$, hence iff

$$V = \bigcap_{i\in I} \overline{\mathrm{Cl}[X^i, Y^i]}.$$

But as $\mathrm{Cl}[X^i, Y^i] = \{v : X^i \subseteq T(v) \text{ and } Y^i \subseteq C(v)\}$,

$$\begin{aligned}
\overline{\mathrm{Cl}[X^i, Y^i]} &= \{v : X^i \not\subseteq T(v) \text{ or } Y^i \not\subseteq C(v)\} = \\
&= \{v : X^i \cap C(v) \neq \emptyset \text{ or } Y^i \cap T(v) \neq \emptyset\} = \\
&= \{v : X^i \cap C(v) \neq \emptyset\} \cup \{v : Y^i \cap T(v) \neq \emptyset\} = \\
&= \bigcup_{x\in X^i} \{v : x \in C(v)\} \cup \bigcup_{y\in Y^i} \{v : y \in T(v)\} = \\
&= \bigcup_{x\in X^i} \mathrm{Cl}[\emptyset, \{x\}] \cup \bigcup_{y\in Y^i} \mathrm{Cl}[\{y\}, \emptyset].
\end{aligned}$$

Now using the generalized de Morgan's law saying that

$$\bigcap_{j\in I} \bigcup_{j\in J} Z_i^j = \bigcup_{f\in F} \bigcap_{j\in I} Z_{f(j)}^j$$

where $F = I^J$, we can see that

$$V = \bigcup_{f\in F} \bigcap_{j\in f^+} \mathrm{Cl}[f(j), \emptyset] \cap \bigcap_{j\in f^-} \mathrm{Cl}[\emptyset, f(j)]$$

where F is the set of all functions mapping every $i \in I$ on an element of $f(i)$ of $X_i \cup Y_i$, and f^+, and f^-, respectively, are the sets of all those elements of I that are mapped by f on elements of X_i, and Y_i, respectively. It further follows that

$$V = \bigcup_{f\in F} \mathrm{Cl}[X^f, Y^f]$$

where $X^f = \{f(j) : j \in f^+\}$ and $Y^f = \{f(j) : j \in f^-\}$. As both f^+ and f^- are finite, this means that V is a union of finite clusters. It follows that for every $v \in V$ there are finite sets $X \subseteq T(v)$ and $Y \subseteq C(v)$ such that every valuation v' such that $X \subseteq T(v')$ and $Y \subseteq C(v')$ is admissible. In other words, for every $v \in V$ there are sets $X \subseteq T(v)$ and $Y \subseteq C(v)$ such that V contains every v' such that $X \subseteq T(v')$ and $Y \subseteq C(v')$. By contraposition: given a valuation v, if for every set $X \subseteq T(v)$ and $Y \subseteq C(v)$ there is a $v' \notin V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$, then $v \notin V$. This condition can obviously be simplified to: given a valuation v, if for every finite $X \subseteq T(v)$ and finite $Y \subseteq \mathrm{C}(v)$ there is a $v' \notin V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$, then $v \notin V$. □

4 A Hierarchy of Semantic Systems

Let us introduce some more definitions.

Definition 9. A semantic system $\langle S, V\rangle$ is called

- *saturated* iff V contains every v such that for every $A \in C(v)$ there is a $v' \in V$ such that $T(v) \subseteq T(v')$ and $A \in C(v')$;
- *compact* iff V contains every v such that for every finite $X \subseteq T(v)$ and finite $Y \subseteq C(v)$ there is a $v' \in V$ such that $X \subseteq T(v')$ and $Y \subseteq C(v')$;
- *co-compact* iff V contains no v such that for every finite $X \subseteq T(v)$ and finite $Y \subseteq C(v)$ there is a $v' \notin V$ such that $X \subseteq T(v)$ and $Y \subseteq C(v')$.
- *compactly saturated* iff V contains every v such that for every finite $X \subseteq T(v)$ and every $A \in C(v)$, there is a $v' \in V$ such that $X \subseteq T(v')$ and $A \in C(v')$.

Given these, we can rephrase the theorems we have proved in the following way:

Theorem 7. *A semantic system $\langle S, V\rangle$ is*

- *always a PSQI-system;*
- *a PSI-system iff it is saturated;*
- *a PQI-system iff it is compact;*
- *an SQI-system only if it is co-compact;*
- *a PI-system iff it is compactly saturated;*

Moreover, easy corollaries of the theorems are the following necessary conditions for a system being an SI-, a QI- and an I-system:

Corollary 2. *A semantic system $\langle S, V\rangle$ is*

- *an SI-system only if it is saturated and co-compact;*

- *a QI-system only if it is compact and co-compact;*
- *an I-system only if it is compactly saturated and co-compact.*

The kinds of semantic systems we have introduced can be arranged into the following diagram, where the arrows indicate containment in the sense that an arrow leads from a concept to a different one if the extension of the former includes that of the latter.

$$\begin{tikzcd}
 & \text{SPQI-system} = \text{semantic system } [\Sigma_1] \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{PQI-system } [\Sigma_2] \arrow[d] & \text{SPI-system } [\Sigma_3] \arrow[d] & \text{SQI-system } [\Sigma_4] \arrow[d] \\
\text{PI-system } [\Sigma_5] \arrow[dr] & \text{QI-system } [\Sigma_6] \arrow[d] & \text{SI-system } [\Sigma_7] \arrow[dl] \\
 & \text{I-system } [\Sigma_8] &
\end{tikzcd}$$

Diagram 1

What we are going to show now is that all the inclusions are *proper.* The symbols in brackets following each kind term is the name of a semantic system which will witness the properness. The systems are the following (S is supposed to be an infinite set):

- $\Sigma_1 = \langle S, \{v \in \mathrm{Pow}(S) : T(v) \text{ is finite}\}\rangle$;
- $\Sigma_2 = \langle S, \{\emptyset\}\rangle$;
- $\Sigma_3 = \langle S, \{v \in \mathrm{Pow}(S) : C(v) \text{ is finite}\}\rangle$;
- $\Sigma_4 = \langle S, \mathrm{Pow}(S) \setminus \{S\}\rangle$;
- $\Sigma_5 = \langle S, \{S\}\rangle$;
- $\Sigma_6 = \langle \{A, B\}, \{\{A\}, \{B\}\}\rangle$;
- $\Sigma_7 = \langle S, \{v \in \mathrm{Pow}(S) : C(v) = A\}\rangle$ for a fixed $A \in S$;
- $\Sigma_8 = \langle \{A, B\}, \{\{A, B\}, \{B\}\}\rangle$.

To show that they do fit into the very slots of Diagram 1 where we have put them, let us first give one more definition:

Definition 10. A valuation is called *full* if it maps every sentence on 1. (In other words, the full valuation is S.) A valuation is called *empty* if it maps every sentence on 0. (In other words, the empty valuation is $\emptyset$.)

Σ_1 is not saturated, for V does not contain the full valuation, f, though for every $A \in C(f)$ there is a $v \in V$ such that $T(f) \subseteq T(v)$ and $A \in C(v)$. (As there is no $A \in C(v)$, this holds trivially. It follows that no system not admitting the full valuation is saturated.) Hence it is not a PSI-system. It is not compact, because V does not contain the full valuation, but for every finite subset X of $T(f)$ it contains a v' such that $X \subseteq T(v')$ (whereas $Y \subseteq C(v')$ for every finite subset Y of $C(f)$ holds trivially); hence it is not a PQI-system. Moreover, it is not co-compact, for V contains the empty valuation, whereas as V cannot contain any valuation mapping only a finite number of sentences on 0, there is, for every finite subset Y of S, a $v' \notin V$ such that $Y = C(v')$. Hence it is not an SQI-system.

Σ_2 is a PQI-system, for it is determined by the infinite set of QI-ons $\{\langle\{A\}, \emptyset\rangle : A \in S\}$. However, it is not saturated, for V does not contain the full valuation, hence it is not a P(S)I-system. Also it is not co-compact, for Vcontains the empty valuation, whereas for every finite subset Y of S there is a $v' \notin V$ such that $X \subseteq C(v')$ (whereas that $Y \subseteq T(v')$ for every finite subset Y of $T(f)$ holds trivially); hence it is not a (S)QI-system.

Σ_3 is a PSI-system, for it is determined by the infinite set of SI-ons $\{\langle X, A\rangle : X \subseteq S$ and X is infinite$\}$. However, it is not compact, because V does not contain the empty valuation, but for every finite subset $Y$of S it contains a v' such that $Y = C(v)$; hence it is not a P(Q)I-system. Moreover, it is not co-compact, for V contains the full valuation, whereas for every finite subset $X$of S there is a $v' \notin V$ such that $X = T(v')$, hence it is not an S(Q)I-system.

Σ_4 is an SQI-system, for it is determined by the SQI-on $\langle S, \emptyset\rangle$. However, it is not saturated, for V does not contain the full valuation, hence it is not a (P)SI-system. It is not compact, because V does not contain the full valuation, but for every finite subset $X$of S it contains a v' such that $X = T(v')$; hence it is not a (P)QI-system.

Σ_5 is a PI-system for it is determined by the infinite set of I-ons $\{\langle\emptyset, \{A\}\rangle : A \in S\}$. But it is not co-compact, for V contains the full valuation, whereas for every finite subset X of S there is a $v' \notin V$ such that $X = T(v')$, hence it is not a (S)(Q)I-system.

Σ_6 is a QI-system for it is determined by the finite set of QI-ons $\{\langle\emptyset, \{A, B\}\rangle, \langle\{A, B\}, \emptyset\rangle\}$. But it is not saturated, for the supervaluation of V is the empty valuation, hence it is not a (P)(S)I-system.

Σ_7 is an SI-system for it is determined by the single SI-on $\langle S \setminus \{A\}, A\rangle$. But it is not compact, for V contains, for every finite subset Y of $S \setminus \{A\}$, a v' such that $T(v') = Y$ and $C(v') = A$. Hence it is not a (P)(Q)I-system.

Σ_8 is an I-system, for it is determined by the I-on $\langle\emptyset, \{B\}\rangle$.

5 Consequence revisited

If what we are interested in is the relation of consequence, then our classificatory hierarchy becomes excessively fine-grained. In particular, we are going to show that for every (P)(S)QI-system there exists a (P)(S)I-system with the same relation of consequence. To do this let us define a concept introduced by (Hardegree, 2006):

Definition 11. Let U be a set of valuations of the class S of sentences. The *supervaluation* of U is the valuation such that $T(v) = T(U)$.

The next lemma shows that our Theorem 3 is equivalent to one of Hardegree's results:

Lemma 1. *A semantic system $\langle S, V\rangle$ is a (P)(S)QI-system iff V contains supervaluations of all its subsets.*

Proof. This follows directly from the fact that $\langle S, V\rangle$ is a (P)(S)QI-system iff it is saturated, for it can be easily seen that it is saturated iff V contains supervaluations of all its subsets. □

Lemma 2. *Extending admissible valuations of a semantic system by supervaluations does not change the relation of consequence.*

Proof. Let $\langle S, V\rangle$ be a semantic system and $\models$ the relation of consequence induced by it. Let v be a supervaluation of a subset of V and let $\models^*$ be the relation of consequence induced by $\langle S, V \cup \{v\}\rangle$. Suppose the two relations do not coincide; then there is a subset X of S and an element A of S so that $X \models A$, but not $X \models^* A$. This means that it must be the case that $v(B) = 1$ for every $B \in X$ and $v(A) = 0$, but that every $v' \in V$ such that $v'(B) = 1$ for every $B \in X$ is bound to be such that $v'(A) = 1$. But as v' is the supervaluation of an $U \subseteq V$, elements of U map all elements of X on 1, whereas at least one of them maps A on 0; which is a contradiction. □

This gives us the following reduced version of Diagram 1:

$$\begin{tikzcd}
 & \text{PS(Q)I-system} = \text{semantic system} \arrow[dl] \arrow[dr] & \\
\text{P(Q)I-system} \arrow[dr] & & \text{S(Q)I-system} \arrow[dl] \\
 & \text{(Q)I-system} &
\end{tikzcd}$$

Diagram 2

Hence from the viewpoint of consequence, we have four types of semantic systems:

- Systems that are neither P(Q)I, nor S(Q)I. These are systems of the kind of Σ_1 and Σ_3.
- P(Q)I-systems that are not (Q)I-systems. Examples are Σ_2 and Σ_5.
- S(Q)I-systems that are not (Q)I-systems. Examples are Σ_4 and Σ_7.
- (Q)I-systems. Systems of the kind of Σ_6 and Σ_8.

Consequence as induced by the truth tables of classical propositional logic or by the model theory of the classical first-order predicate logic, of course, fall into the last category. Indeed any logic that has a strongly sound and complete axiomatization must trivially belong here. But even among the semantic systems studied by logicians there are some that fall outside this range ((Tarski, 1936) made this into a deep point — consequence, according to him, cannot be in general captured in terms of inferential rules).

From Diagram 2 we can see that there are two ways to go beyond the boundaries of I-systems: we may either alleviate the requirement of finiteness of antecedents of inferences, or alleviate the requirement of finiteness of the whole relation of inference. The ω-rule, which is often discussed in connection with the formalization of arithmetic, is an example of the former way; the axiom scheme of induction, that comprises an infinity of concrete axioms, is the example of the latter.

For a more specific example, consider the language of Peano arithmetic with the single admissible valuation determined by the intended interpretation within the standard model (let me call this system *true arithmetic*, TA). As it turns out, this system is a PQI-system. Indeed, it can be determined by the PQI-structure whose relation of inference consists of the I-ons of the form $\langle\emptyset, A\rangle$ for every true sentence A plus the QI-ons of the form $\langle\{B\}, \emptyset\rangle$ for every false sentence B. (We know that it is not an I-system, as we know that the truths of TA are not recursively enumerable.) Call the single admissible valuation of the system t.

If we extend the (single-element) set of admissible valuations of TA by the full valuation, it becomes saturated (indeed the supervaluation of every subset of the set of its admissible truth valuations will be admissible: the supervaluation of the empty set as well as the singleton of the full valuation is the full valuation, whereas the supervaluation of the two remaining sets is the valuation t). Hence this system is a PI-system (indeed, it is determined by the PI-structure the relation of inference of which consists of the I-ons of the form $\langle\emptyset, A\rangle$ for every sentence A true according to t plus the I-ons of the form $\langle\{B\}, C\rangle$ for every sentence B false according to t, and every sentence C) but has the same relation of consequence as the previous system.

6 Further steps

I hope to have shown how we can set up a useful framework for a systematic confrontation of proof theory and semantics, especially of inference and consequence; and that I have also indicated that this framework lets us prove some nontrivial and interesting results. However, it should be added that to bring results immediately concerning the usual systems of formal logic, our classificatory hierarchy will have to be made still more fine-grained.

The point is that while we only distinguished between systems that are determined by structures with a finite number of (S)(Q)I-ons (i.e. (S)(Q)I-systems) and those where the finiteness requirement is alleviated (the P(S)(Q)I-systems), we would need to consider systems in between these two extremes. The usual systems of formal logic can be considered as generalizing over inferential (as opposed to *pseudo*inferential) structures in two steps. First, they allow for an infinite number of (S)(Q)I-ons, which are, however, instances of a finite number of schemata. (This is, of course, possible only when we, unlike in the present paper, take into account some structuring of the set of sentences and consequently of the sentences themselves — if we consider the sentences as generated from a vocabulary by a set of rules.) This can be accounted for in terms of *parametric* SQI-ons, or p(S)(Q)I-ons. p(S)(Q)I-systems, then, fall in between (S)(Q)I-systems and P(S)(Q)I-system. Thus for example the semantic system of PA is a p(Q)I-system, for the infinity of its axioms is the union of instances of a finite number of axiom schemas. The semantic system of TA is a pSI-system, for we know that we can have its sound and complete axiomatization if we extend the axiomatic system with the omega-rule, which is, in our terminology, a pSI-on. Second they allow for infinite sets of (S)(Q)I-ons that are generated by a finite number of metainferential rules from sets of instances of finite number of schemata.

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