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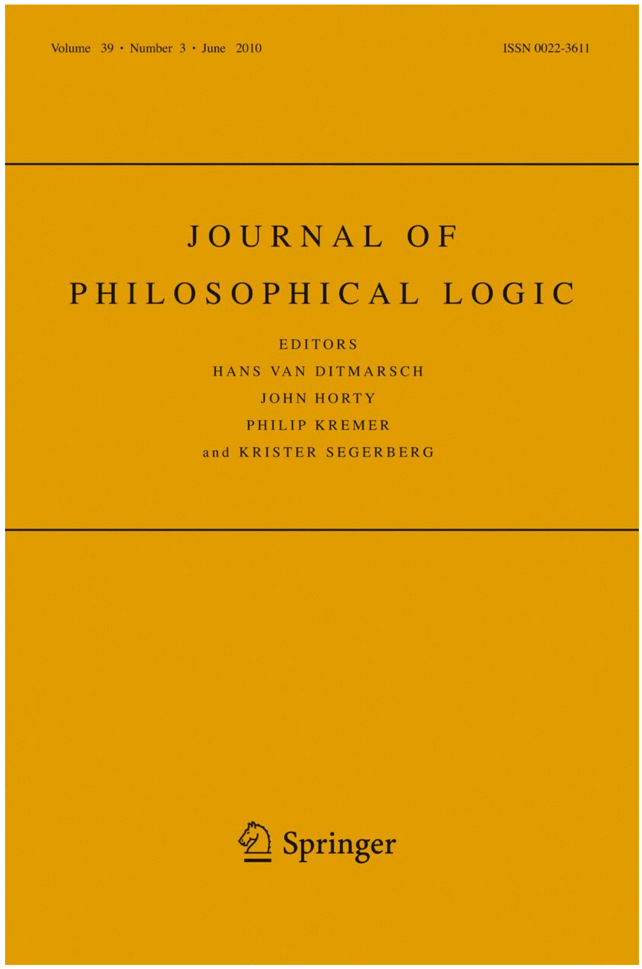

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Inferentializing Semantics

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Abstract The entire development of modern logic is characterized by various forms of confrontation of what has come to be called *proof theory* with what has earned the label of *model theory*. For a long time the widely accepted view was that while model theory captures directly what logical formalisms are *about*, proof theory is merely our technical means of getting some incomplete grip on this; but in recent decades the situation has altered. Not only did proof theory expand into new realms, generalizing the concept of proof in various directions; many philosophers also realized that meaning may be seen as primarily consisting in certain rules rather than in language-world links. However, the possibility of construing meaning as an inferential role is often seen as essentially compromised by the limits of proof-theoretical means. The aim of this paper is to sort out the cluster of problems besetting logical inferentialism by disentangling and clarifying one of them, namely determining the power of various inferential frameworks as measured by that of explicitly semantic ones.

Keywords Inference · Proof theory · Model theory · Inferentialism · Semantics

1 Proving, Inferring & Meaning

Modern logic, throughout its development, has witnessed repeated confrontations of what has come to be called *proof theory* with what has earned the label of *model theory*. The heart of theories of systems of formal logic are usually considered to be proofs of soundness and (in)completeness, which concern the relationship between proof-theoretically delimited *theorems* and model-theoretically delimited *tautologies*.

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Those founding fathers of modern logic for whom logic was primarily a matter of *rules*, people like Frege, Peano or Russell, articulated the basic axiomatic systems of logic, thus putting the concept of *proof* on a firm foundation. Thereafter, Hilbert set up *proof theory* as an ambitious program based on the mathematical investigation of arithmetized proofs, aimed at showing consistency, independence and completeness of axiomatic systems by the perspicuous means of elementary arithmetic, and consequently reducing mathematics to the investigation of the provability of its statements (see [9, 10]). In the nineteen thirties, Gentzen [5, 6] showed that proof theory need not rest on the concept of axiomatic system, and put forward in its stead the systems of natural deduction and sequent calculi (which are not based on inferring sentences from sentences, but rather instances of—multi-conclusion—inference from other such instances). Recently, proof theory has come to be understood as a wholly general investigation of proofs and proof systems, often carried out in ways that are more algebraic than arithmetical (see [2]).

However, the thirties witnessed also the spectacular demonstration of the limitations of proof theory—viz. Gödel's proof of the impossibility of reaching all truths of arithmetic (let alone more complicated systems) by means of proof theory. This led to a revival of semantic methods in logic, due especially to Alfred Tarski [20–23] and his school. (A semantic strand in the development of modern logic had pre-dated Tarski, thanks to logicians who came to logic from algebra, i.e. people like Boole, Schröder and Löwenheim, but it was Tarski who made the presuppositions of this approach explicit.) Subsequent transformations of Tarski's methods have given us what we know today as *model theory.*

Much of the appeal of model theory was that Tarski seemed to have hit upon how to describe directly what proof theory was able to show us at most indirectly—i.e. what the languages of our systems of logic are *about*, what their terms *stand for*. Proof theory, so the story went, captured cases of sentences being consequences of other sentences, but surely a sentence can be a consequence of other sentences only in virtue of what it says, i.e. in virtue of the fact that the terms it consists of stand for something (objects, properties, relations ...). And it was these, allegedly semantically more essential facts that model theory appeared to account for.

All of this generated the widely accepted view that while model theory captures directly what logical formalisms express, proof theory is merely our technical means of getting some incomplete grip on this. Paraphrasing the medieval saying about the relationship between philosophy and theology, we can say that proof theory came to be seen as the handmaiden of model theory. Proofs of the soundness and completeness of logical calculi were relegated to being directly proofs of the *adequacy* of the calculi to what they were about.

However, in recent decades the situation has altered. First, proof theory expanded into new realms, generalizing the concept of proof in various directions. (The most significant direction is connected to the investigation of the so-called substructural logics.[1]) This makes it possible to distinguish between proofs and provability in various weaker and stronger senses. Second, many philosophers realized that meaning may be seen as primarily consisting in certain rules rather than in language-world links. Within the philosophy of language, we witness the rise of *inferentialism*,

[1] See [19].

a doctrine that the meaning of an expression is generally its inferential role (largely due to Brandom; see [1]);[2] a corresponding doctrine, but restricted to logic, had emerged already prior to this (stemming from Gentzen via Dummett and Prawitz) in the context of proof-theory and it has recently yielded what has come to be called *proof-theoretical semantics* (see especially [8, 15]),[3] along with other contributions characterizing themselves explicitly as "inferentialist" [18, 24].

However, the possibility of construing meaning as an inferential role is often seen as essentially compromised by facts pointing out the limitedness of proof-theoretical means. One of the looming objections is that, as Prior [16] putatively showed, logical inferentialism opens the door to the vitiation of semantics by pernicious logical operators. Another objection is that certain logical operators which look pretty natural from the semantic viewpoint cannot be defined inferentially. (This concerns, as noticed already by [3], some of the operators of classical logic.)[4] A third objection stems from the general feeling that inference is not a sufficiently semantic concept to be able to yield true meaning.

The aim of this paper is to help sort out the cluster of problems besetting logical inferentialism by disentangling and clarifying just one of them. My area of focus concerns the relationship between inferential (proof-theoretic, or 'syntactic')[5] methods and methods based explicitly on semantic (model-theoretic) tools, particularly when comparing the former with those of the latter. I will propose a very general framework for 'measuring' inferentializability of semantic systems. I have already foreshadowed such a framework elsewhere (see [13]); but here I want to refine it and to prove some basic facts regarding it.

2 The Framework

I take it for granted that the semantics of a language necessarily involves a division of the truth valuations of its sentences into admissible and inadmissible.[6] (For some sentences, like "Dogs are animals" in English, only valuations mapping them on 1, or only those mapping them on 0, will be admissible; but for many sentences, like "Fido is a dog", there will be valuations mapping them on either of the values.)[7] Hence what I call a semantic system is a set (of sentences) and a set of distributions of the two truth values among them.[8]

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[2] See [14] for an introduction.

[3] Proof-theoretical semantics is the topic of the whole issue of *Synthèse* (148, No. 3, 2006), in which these papers are printed.

[4] Carnap's discovery has been occasionally discussed in the literature; for the most recent exchange see [17] and [11].

[5] It is clear that from the inferentialist viewpoint, calling inference 'syntax' is essentially misleading; indeed the inferentialist *credo* is that inference is just as capable of laying a foundation for *semantics* as methods that are *explicitly* semantic.

[6] Actually I maintain a much stronger thesis, namely that any semantics can be *reduced* to such a division, but I am not going to argue for this thesis here—I have done so elsewhere, see [12].

[7] Though there will not be, for example, any admissible valuation which maps "Fido is a dog" on 1 at the same time as mapping "Fido is an animal" on 0.

[8] This approach to capturing semantics stems from [25]; the framework was later elaborated by [4]. I adapted it in [13].

Definition Let S be a set. The elements of $\{0,1\}^S$ are called *valuations* (*of* S). (We will sometimes identify a valuation with the set of all those elements of S that it maps onto 1.) The pair $<S,V>$, where $V\subseteq\{0,1\}^S$, is called a *semantic system*; elements of S are called *sentences* and elements of V are called *admissible valuations* of the system. The elements of $\{0,1\}^S\backslash V$ are called *inadmissible*.

Hence the definition of a semantic system consists in, besides the delimitation of the set of sentences, the demarcation of the admissible valuations. A way of carrying out such a demarcation is to state that some sentences are inferable from others; for this, in effect, says that some sentences cannot be false provided some others are true. Thus, stating $A_1, ..., A_n \vdash A$ (for some elements $A_1, ..., A_n, A$ of S) can be seen as excluding certain valuations from the set of admissible ones: namely all those that map all of $A_1, ..., A_n$ on 1 and A on 0. If we call the pair of a finite set of elements of S and an element of S an *inferon*, then we can say that inferons exclude valuations and ask which sets of valuations can be demarcated by means of inferons.

Definition An *inferon* (over a set S) is an ordered pair $<X,A>$, where X is a finite subset of S and A is an element of S. The inferon is said to *exclude* an element v of $\{0,1\}^S$ iff $v(B)=1$ for every $B\in X$ and $v(A)=0$. An ordered pair $<S,\vdash>$ such that S is a set and $\vdash$ is a set of inferons over S (i.e. a binary relation between finite subsets of S and elements of S) will be called a *protoinferential structure*; it will be called *inferential* if $\vdash$ is a *finite* set of inferons. A (proto)inferential structure is said to *determine* a semantic system $<S,V>$ iff V is the set of all and only elements of $\{0,1\}^S$ not excluded by any element of $\vdash$. A semantic system is called a (proto)inferential system iff it is determined by a (proto)inferential structure.

An obvious question now is which semantic systems are protoinferential. Many systems can be shown to be protoinferential by simply displaying a protoinferential structure that determines them. However, can we show that a system is *not* protoinferential? We are going to present a theorem that turns out to be helpful in this respect; but before doing so, a few more definitions.

Definition Let U be a set of valuations of a set S (i.e. a subset of $\{0,1\}^S$). $\mathrm{T}(U)$ (the set of *U-tautologies*) will be the set of all those elements of S which are mapped on 1 by all elements of U; and analogously $\mathrm{C}(U)$ (the set of *U-contradictions*) will be the set of all those elements of S which are mapped on 0 by all elements of U. (Where no confusion is likely, I will identify a singleton with its single element; so I will, for example, write $\mathrm{C}(v)$ instead of $\mathrm{C}(\{v\})$.)

Theorem 1 A semantic system $<S,V>$ is protoinferential iff V contains every $v\in\{0,1\}^S$ such that for every finite $X\subseteq\mathrm{T}(v)$ and every $A\in\mathrm{C}(v)$ there is a $v'\in V$ such that $X\subseteq\mathrm{T}(v')$ and $A\in\mathrm{C}(v')$.

Proof A semantic system $<S,V>$ is protoinferential iff there is a set $\vdash$ of inferons over S such that every inadmissible valuation is excluded by at least one element of $\vdash$ and no element of $\vdash$ excludes any admissible valuation of $<S,V>$; hence it is a protoinferential system iff for every $v \in \{0,1\}^S \setminus V$ there is a finite set $X \subseteq \mathrm{T}(v)$ and a sentence $A \in \mathrm{C}(v)$ such that no valuation $v' \in \{0,1\}^S$ such that $X \subseteq \mathrm{T}(v')$ and $A \in \mathrm{C}(v')$ is admissible. In other words, $<S,V>$ is a protoinferential system iff for every $v \notin V$ there is a finite set $X \subseteq \mathrm{T}(v)$ and a sentence $A \in \mathrm{C}(v)$ such that V does not contain any v' such that $X \subseteq \mathrm{T}(v')$ and $A \in \mathrm{C}(v')$. By contraposition, $<S,V>$ is a protoinferential system iff the following holds: given a valuation v, if for every finite set $X \subseteq \mathrm{T}(v)$ and every sentence $A \in \mathrm{C}(v)$ there is a $v' \in V$ such that $X \subseteq \mathrm{T}(v')$ and $A \in \mathrm{C}(v')$, then $v \in V$. □

Hence we have a necessary and sufficient condition for the protoinferentiality of a semantic system. This condition lets us quickly decide of many semantic systems that they are not protoinferential. (Examples are any system that does not admit the valuation mapping every sentence on 1, or any system that admits all but a finite number of valuations.) What may be surprising is that the semantics of classical propositional calculus, as defined by the truth tables of its operators, also does not fulfill this condition. This can be seen when we consider, e.g. the truth table for classical negation: for any sentence A which is neither a tautology nor a contradiction there is an admissible valuation that maps A on 1 and $\neg A$ on 0, as well as one that maps A on 0 and $\neg A$ on 1, but none that maps both A and $\neg A$ on 0, which contradicts the criterion of protoinferentiality stated by the theorem.

3 Consequence

Every semantic system induces a relation of consequence, understood as the relation of truth-preservation.

Definition Let $<S,V>$ be a semantic system. The relation of consequence induced by this system is the relation $\vDash$ defined as follows: if $X \subseteq S$ and $A \in S$, then
$X \vDash A$ iff $v(A) = 1$ for every $v \in V$ such that $v(B) = 1$ for every $B \in X$

The set of sentences of a semantic system plus its consequence relation forms something akin to a protoinferential structure; the difference being that consequence is a relation between (not only finite) sets of sentences and sentences. To be able to analyze this structure from the viewpoint of its ‘inferentiality’, let us generalize our concept of protoinferential structure in the following way:

Definition A semiinferon over a set S is an ordered pair $<X,A>$, where X is a (not necessarily finite) subset of S and A is an element of S. The semiinferon is said to *exclude* an element v of $\{0,1\}^S$ iff $v(B) = 1$ for every $B \in X$ and $v(A)=0$. An ordered pair $<S,\vdash>$ such that S is a set and $\vdash$ is a set of semiinferons (i.e. a binary relation between subsets of S and elements of S) will be called a *protosemiinferential structure*; it will be called a *semiinferential* structure iff $\vdash$ is finite. Such a structure is said to *determine* a semantic system $<S,V>$ iff V is the set of all and only elements of

$\{0,1\}^S$ not excluded by any element of ⊢. A semantic system is called a (proto) semiinferential *system* iff it is determined by a (proto)semiinferential structure.

Now if we take the set of sentences of a semantic system together with its consequence relation, we have a protosemiinferential structure. Hence for every semantic system there is a protosemiinferential structure that is inherent to it, it is the structure *of* the system.

Definition The protosemiinferential structure $<S, \vDash>$, where $\vDash$ is the consequence relation induced by the system $<S,V>$ is called the protosemiinferential structure *of* $<S,V>$.

To obtain a general characterization of protosemiinferential semantic systems, we modify the previous theorem into the form corresponding to one proved by Hardegree [7]:

Theorem 2 A semantic system $<S,V>$ is protosemiinferential iff V contains every v such that for every $A\in \mathrm{C}(v)$ there is a $v'\in V$ such that $\mathrm{T}(v)\subseteq \mathrm{T}(v')$ and $A\in \mathrm{C}(v')$.

Proof A semantic system $<S,V>$ is protosemiinferential iff there is a set of semiinferons such that every inadmissible valuation is excluded by at least one element of the set and no element of the set excludes any admissible valuation. This is to say that it is a protoinferential system iff for every $v\in\{0,1\}^S\backslash V$ there is a set $X\subseteq \mathrm{T}(v)$ and a sentence $A\in \mathrm{C}(\mathrm{v})$ such that no valuation v' such that $X\subseteq \mathrm{T}(v')$ and $A\in \mathrm{C}(v')$ is admissible. In other words, $<S,V>$ is a protoinferential system iff for every $v\notin V$ there is a set $X\subseteq \mathrm{T}(v)$ and a sentence $A\in \mathrm{C}(v)$ such that V does not contain any v' such that $X\subseteq \mathrm{T}(v')$ and $A\in \mathrm{C}(v')$. By contraposition, $<S,V>$ is a protoinferential system iff the following holds: given a valuation v, if for every set $X\subseteq \mathrm{T}(v)$ and every sentence $A\in \mathrm{C}(v)$ there is a $v'\in V$ such that $X\subseteq \mathrm{T}(v')$ and $A\in \mathrm{C}(v')$, then $v\in V$. This condition can be obviously simplified to: given a valuation v, if for every sentence $A\in \mathrm{C}(v)$ there is a $v'\in V$ such that $\mathrm{T}(v)\subseteq \mathrm{T}(v')$ and $A\in \mathrm{C}(v')$, then $v\in V$. □

Hardegree calls a valuation v the *supervaluation* of a set U of valuations iff $\mathrm{T}(v) = \mathrm{T}(U)$; and given this terminology, we can say that a semantic system $<S,V>$ is protosemiinferential iff V contains the supervaluation of each of its subsets. (This follows from the fact that v is a supervaluation of a subset of V iff the following holds: there is a $U\subseteq V$ so that $A\in \mathrm{C}(v)$ if $A\in \mathrm{C}(u)$ for some $u\in U$ and $A\in \mathrm{T}(v)$ if $A\in \mathrm{T}(u)$ for every $u\in U$. And such a subset obviously exists iff for every $A\in \mathrm{C}(v)$ there is a $v'\in V$ such that $\mathrm{T}(v)\subseteq \mathrm{T}(v')$ and $A\in \mathrm{C}(v')$.)

4 Systems that are not Protosemiinferential

The concept of a protosemiinferential system may seem quite general, so the question is whether all the usual systems of logic can be seen as falling into this category. But returning to the example of classical negation we can immediately see that this is not the case even for classical logic.

Let us analyze the example in greater detail. Let $S = \{A,B\}$ and let V consist of the two 'truth-value-swapping' valuations, i.e. the valuations $\{A\}$ and $\{B\}$. What is the (proto)(semi)inferential structure of the system? It is clear that the valid instances of consequence within this system will be the following:

$$
\begin{array}{l}
A \vDash A \\
A, B \vDash A \\
B \vDash B \\
A, B \vDash B
\end{array}
$$

Hence the structure of the system is the inferential structure whose inference relation consists of the following four inferons:

$$
\begin{array}{l}
<\{A\}, A> \\
<\{A, B\}, A\}> \\
<\{A\}, B> \\
<\{A, B\}, B>
\end{array}
$$

It is readily seen that none of these inferons excludes *any* valuation. In other words, this inferential structure does not determine the original semantic system $<\{A,B\}, \{\{A\},\{B\}\}>$, but rather the 'full' system $<\{A,B\}, \{\varnothing,\{A\},\{B\},\{A,B\}\}>$, in which any sentence is a consequence of any others. It is also readily seen that extending the inferential relation would not help; for let us list all the inferons that could be added to the structure and let us list the valuations of S that they would exclude along with each of them:

$$
\begin{array}{ll}
<\varnothing, A> & \varnothing, \{B\} \\
<\varnothing, B> & \varnothing, \{A\} \\
<\{B\}, A> & \{B\} \\
<\{A\}, B> & \{A\}
\end{array}
$$

We can see that no combination of the inferons is capable of excluding the valuation $\{A,B\}$; and also no combination is capable of excluding $\varnothing$ without excluding either $\{A\}$ or $\{B\}$. In other words, no (proto)(semi)inferential structure determines the system $<\{A,B\},\{\{A\},\{B\}\}>$.

Let us make a little digression and note that this indicates that the relation between semantic systems and protosemiinferential structures that results from associating a structure with the system it determines, is many-one: there may be many structures determining the same system (whereas, on the other hand, there are systems determined by no structure). There is, obviously, only one structure that is the structure *of* the system—it is the structure whose relation of inference coincides with the relation of consequence of the system. The fact that there may be more structures *determining* the same system, then, is due to the fact that some (semi)inferons do not exclude any valuation, and hence adding them to a structure determining a semantic system produces *a different structure* determining *the same system*. Only when we add all such (semi)inferons, the structure becomes 'saturated' and it becomes the structure of the system. Hence there exist (semi)inferons which are 'idle' in the sense of not excluding any valuations.

Similarly, the relation which results from associating a system with the structure of the system is many-one; whereas there is only one system that is determined by the structure. Again, additions of some valuations to a system causes no change of the relation of consequence of the system (and hence of the structure of the system) and it is only when all such valuations are present that we gain a system that is 'saturated' and hence determined by its own structure. Thus, besides inferons that are 'idle' in that they do not exclude any valuations, there are valuations that are 'idle' in the sense that their presence/absence does not influence the structure of the system.

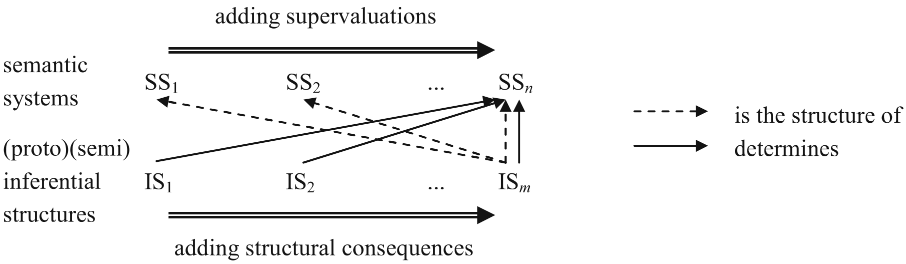
Both the 'idle' inferons and the 'idle' valuations were characterized, in effect, by Hardegree (*ibid*.), who wanted to answer the question which systems are determined by their own structures and, conversely, which structures are the structures of the systems they determine. (This is not directly Hardegree's terminology, but rather the result of translating his results into our conceptual framework.) Hardegree has shown that the former systems are those that are closed w.r.t. supervaluations and the latter structures are those that are closed w.r.t. the following closure conditions:

$X \vdash A$ if A is an element of X

$X \vdash A$ if $X \vdash B$ for every element B of an Y such that $Y \vdash A$

The last result corresponds to the result I presented elsewhere ([13], Theorem 3): namely that a structure is a structure of a semantic system—I called such a structure *truth-preserving* there—iff it complies with the Gentzenian structural rules. (Here, as we count the premises of an inference as a set, rather than a sequence, two of the rules, namely contraction and permutation, are superfluous. The second of Hardegree's rules amounts to cut, whereas the first one is a generalized version of reflexivity, which follows from the reflexivity via expansion and yields expansion together with cut.) Let us call the inferons that are obtainable from some given ones by means of the Gentzenian structural (meta)rules their *structural consequences*.

Hence we can summarize that a valuation is 'idle' in the above sense iff it is a supervaluation of a set of admissible valuations and that a (semi)inferon is 'idle' iff it is a structural consequence of already valid (semi)inferons. Hence the following picture:



Returning from the digression, we can say that what is important for us is that the example of negation showed us that even some very simple semantic systems are *not* protosemiinferential. In particular, we saw that we cannot get classical negation, and hence classical logic, by means of semiinferons. What kind of a further generalization of the concept of protosemiinferential structure would lead us to a concept that would comprise classical logic?

It is easy to see that what would help is to generalize the concept of (semi) inferon from one of single-conclusion to one of multiple-conclusion, along the lines familiar from sequent calculus. Hence we can think of defining a semiquasiinferon as an ordered pair $<X,Y>$, where X and Y are (not necessarily finite) subsets of S; and defining (proto)semiquasiinferentiality of structures and systems accordingly. Then it would be easy to show that the system $<\{A,B\}, \{\{A\}, \{B\}\}>$ is quasiinferential, as it is determined by the structure constituted by the following two quasiinferons:

$$<\{A, B\}, \varnothing>$$
$$<\varnothing, \{A, B\}>$$

This indicates that this last generalization of the concept of inference might be ultimate, i.e. that it might enable us to encompass all kinds of semantics systems.[9] We will prove this in the next section, where we will also investigate the hierarchy of stricter levels of inferentiality.

5 The Hierarchy

Let us now repeat and extend the definitions we have given so far. To prevent our terminology from becoming too cumbersome, we will use abbreviations: P for *proto*, S for *semi*, Q for *quasi* and I for *infer*(*ential*).

Definitions Let S be a set. An element of Pow(S)×Pow(S) is called a SQI-*on over S*. It is called a QI-*on* if it is an element of FPow(S)×FPow(S) (where FPow(S) is the set of all finite subsets of S), it is called an SI-*on* if it is an element of Pow(S)×S and it is called an I-*on* if it is an element of FPow(S)×S. The ordered pair $<S,\vdash>$, where $\vdash$ is a set of SQI-ons (QI-ons, SI-ons, I-ons), will be called a PSQI-*structure* (PQI-*structure,* PSI-*structure,* PI-*structure*). It is called a SQI-*structure* (QI-*structure,* SI-*structure,* I-*structure*) iff $\vdash$ is finite. A SQI-on $<X,Y>$ is said to *exclude* an element v of $\{0,1\}^S$ iff $v(B) = 1$ for every $B \in X$ and $v(A)=0$ for every $A \in Y$. A (P)(S)(Q)I-structure $<S,\vdash>$ is said to *determine* a semantic system $<S,V>$ iff V is the set of all and only elements of $\{0,1\}^S$ not excluded by any element of $\vdash$. A semantic system is called a (P)(S)(Q)I-system iff it is determined by a (P)(S)(Q)I-structure.

[9] The generalization of the concept of inferon allowing for inferons with empty consequents is tantamount to entering the concept of incompatibility; indeed the set A is incompatible iff $A\vdash$. The introduction of inferons with multiple consequents goes beyond this. (See [13].)

All the concepts we have introduced give us the following hierarchy:

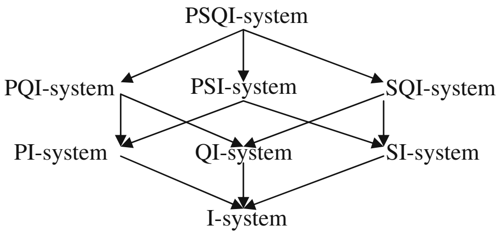


Diagram 1

The arrows indicate containment in the sense that an arrow leads from a concept to another one if whatever falls under the latter falls also under the former. What we are going to prove now is that (i) every semantic system is a PSQI-system, (ii) the arrows in the diagram capture *all* inclusions among the types of semantic systems listed on it, and (iii) all the inclusions are *proper*. To be able to do this we need some more definitions.

Definition Let X and Y be subsets of a set S. The *cluster over S generated by X and Y*, $\mathrm{Cl}_S[X,Y]$, will be the set of all elements of $\{0,1\}^S$ that map all elements of X on 1 and all elements of Y on 0. (Thus U is a *cluster* iff it contains, and hence is identical with, $\mathrm{Cl}_S[\mathrm{T}(U),\mathrm{C}(U)]$.) A cluster U is called *finitary* iff both $\mathrm{T}(U)$ and $\mathrm{C}(U)$ are finite; it is called *inferential* iff $\mathrm{C}(U)$ is a singleton.

Given these definitions, we can formulate the following lemma, the proof of which is straightforward.

Lemma A semantic system $<S,V>$ is

- a PSQI-system iff $\{0,1\}^S\backslash V$ is a union of clusters
- a PQI-system iff $\{0,1\}^S\backslash V$ is a union of finitary clusters
- a PSI-system iff $\{0,1\}^S\backslash V$ is a union of inferential clusters
- a SQI-system iff $\{0,1\}^S\backslash V$ is a union of a finite number of clusters
- a PI-system iff $\{0,1\}^S\backslash V$ is a union of finitary inferential clusters
- a SI-system iff $\{0,1\}^S\backslash V$ is a union of a finite number of inferential clusters
- a QI-system iff $\{0,1\}^S\backslash V$ is a union of a finite number of finitary clusters
- a I-system iff $\{0,1\}^S\backslash V$ is a union of a finite number of finitary inferential clusters

Proof Obvious.

To be able to formulate more useful criteria for categorizing semantic systems into these categories, we need some more concepts:

Definition A semantic system $<S,V>$ is called

- *saturated* iff V contains every $v\in\{0,1\}^S$ such that for every $A\in\mathrm{C}(v)$ there is a $v'\in V$ such that $\mathrm{T}(v)\subseteq\mathrm{T}(v')$ and $A\in\mathrm{C}(v')$

- *compact* iff V contains every $v\in\{0,1\}^S$ such that for every finite $X\subseteq\mathrm{T}(v)$ and finite $Y\subseteq\mathrm{C}(v)$ there is a $v'\in V$ such that $X\subseteq\mathrm{T}(v')$ and $Y\subseteq\mathrm{C}(v')$.
- *co-compact* iff V contains no $v\in\{0,1\}^S$ such that for every finite $X\subseteq\mathrm{T}(v)$ and finite $Y\subseteq\mathrm{C}(v)$ there is a $v'\notin V$ such that $X\subseteq\mathrm{T}(v')$ and $Y\subseteq\mathrm{C}(v')$.
- *compactly saturated* iff V contains every $v\in\{0,1\}^S$ such that for every finite $X\subseteq\mathrm{T}(v)$ and every $A\in\mathrm{C}(v)$, there is a $v'\in V$ such that $X\subseteq\mathrm{T}(v)$ and $A\in\mathrm{C}(v)$.

With the help of this conceptual machinery, we can formulate and prove the following theorem:

Theorem 3 A semantic system $<S,V>$ is

1. always a PSQI-system;
2. a PQI-system iff it is compact;
3. a PSI-system iff it is saturated (this restates Theorem 2);
4. a SQI-system only if it is co-compact;
5. a PI-system iff it is compactly saturated (this restates Theorem 1);
6. a SI-system only if it is saturated and co-compact;
7. a QI-system only if it is compact and co-compact;
8. a I-system only if it is compactly saturated and co-compact.

Proof

1. It is clear that any set consisting of a single valuation is a cluster; hence any set of valuations is a union of clusters.
2. A semantic system $<S,V>$ is a PQI-system iff $\{0,1\}^S\backslash V$ is a union of finitary clusters. This is to say that it is a PSI-system iff for every $v\in\{0,1\}^S\backslash V$ there is a finite set $X\subseteq\mathrm{T}(v)$ and a finite set $Y\subseteq\mathrm{C}(v)$ such that $\{0,1\}^S\backslash V$ contains the whole cluster $\mathrm{Cl}_S[X,Y]$, i.e. iff for every $v\notin V$ there are finite sets $X\subseteq\mathrm{T}(v)$ and $Y\subseteq\mathrm{C}(v)$ such that V does not contain any v' such that $X\subseteq\mathrm{T}(v')$ and $Y\subseteq\mathrm{C}(v')$. By contraposition, $<S,V>$ is a PQI-system iff the following holds: given a valuation v, if for every finite sets $X\subseteq\mathrm{T}(v)$ and $Y\subseteq\mathrm{C}(v)$ there is a valuation $v'\in V$ such that $X\subseteq\mathrm{T}(v')$ and $Y\subseteq\mathrm{C}(v')$, then $v\in V$. But this is clearly the definition of compactness of V.
3. See proof of Theorem 2 (though now it can be brought to a form straightforwardly parallel to that of 2. above).
4. A semantic system $<S,V>$ is a SQI-system iff $\{0,1\}^S\backslash V$ is a finite union of clusters, i.e. iff there is a finite set I and two collections $<X^i>_{i\in I}$ and $<Y^i>_{i\in I}$ of subsets of S so that $\{0,1\}^S\backslash V=\bigcup_{i\in I}\mathrm{Cl}_S[X^i,Y^i]$. This is the case iff V is the complement of $\bigcup_{i\in I}\mathrm{Cl}_S[X^i,Y^i]$, hence iff $V=\bigcap_{i\in I}\overline{\mathrm{Cl}_S[X^i,Y^i]}$. But as $\mathrm{C1}_S[X^i,Y^i]=\{v\mid X^i\subseteq\mathrm{T}(v)\text{ and }Y^i\subseteq\mathrm{C}(v)\}$, $\overline{\mathrm{Cl}_S[X^i,Y^i]}=\{v\mid X^i\nsubseteq\mathrm{T}(v)\text{ or }Y^i\nsubseteq\mathrm{C}(v)\}=\{v\mid X^i\cap\mathrm{C}(v)\neq\varnothing\text{ or }Y^i\cap\mathrm{T}(v)\neq\varnothing\}=\{v\mid X^i\cap C(v)\neq\varnothing\}\cup\{v\mid Y^i\cap\mathrm{T}(v)\neq\varnothing\}=\bigcup_{x\in X^i}\{v\mid x\in\mathrm{C}(v)\}\cup\bigcup_{y\in Y^i}\{v\mid y\in\mathrm{T}(v)\}=\bigcup_{x\in X^i}\mathrm{Cl}_S[\varnothing,\{x\}]\cup\bigcup_{y\in Y^i}\mathrm{Cl}_S[\{y\},\varnothing]$. Hence $V=\bigcap_{i\in I}\left(\bigcup_{x\in X^i}\mathrm{Cl}_S[\varnothing,\{x\}]\right)\cup\bigcup_{y\in Y^i}\mathrm{Cl}_S[\{y\},\varnothing]$

and using the generalized de Morgan law saying that $\bigcap_{i\in I}\bigcup_{j\in Ji} Z_i^j = \bigcup_{f\in F}\bigcap_{i\in I} Z_i^{f(i)}$, where F is the set of all functions mapping elements of I on elements of respective sets J^i, we reach $V = \bigcup_{f\in F}\left(\bigcap_{i\in f^+}\mathrm{Cl}_S[f(i),\varnothing] \cup \bigcap_{i\in f^-}\mathrm{Cl}_S[\{\varnothing, f(i)]\right)$, where F is the set of all functions mapping every element $i\in I$ on an element $f(i)\in X^i\cup Y^i$, and f^+ resp. f^- is the set of all those elements of I that f maps on elements of X_i resp. Y_i. It further follows that $V=\bigcup_{f\in F}\mathrm{Cl}_S\left[X^f, Y^f\right]$, where $X^f=\{f(i)|i\in f^+\}$and $Y^f=\{f(i)\mid i\in f^-\}$. As both f^+ and $f^-$are finite, this means that V is a union of finitary clusters and by the same line of reasoning as employed in the proof of 2. above, we reach the conclusion that <*S*,*V*> is co-compact.

5. See proof of Theorem 1 (though again, now it can be brought to a form straightforwardly parallel to that of 2. above).

6.-8. are obvious consequences of 1.-3. □

Now we will give examples of semantic systems that witness the properness of all the inclusions on Diagram 1. Before we do so, however, some more definitions:

Definitions A valuation is called *full* if it maps every sentence on 1. (In other words, the full valuation is *S*.) A valuation is called *empty* if it maps every sentence on 0. (In other words, the empty valuation is ∅.)

Now let *S* be infinite; and let us consider the following list of semantic systems:

- Σ_1 = <*S*, {*v*∈Pow(S) | T(*v*) is finite}>
- Σ_2 = <*S*,{∅}>
- Σ_3 = <*S*, {*v*∈ Pow(S) | C(*v*) is finite}>
- Σ_4 = <*S*, Pow(S)\{S}>
- Σ_5 = <*S*,{*S*}>
- Σ_6 = <{*A*,*B*},{{*A*},{*B*}} >
- Σ_7 = <*S*, {*v*∈Pow(S) | C(*v*) = *A*}> for some fixed *A*∈*S*
- Σ_8 = <<{*A*,*B*},{{*A*,*B*},{*B*}} >>

Theorem 4

- Σ_1 is a semantic system, and hence a PSQI-system, that is neither a PSI-system, nor a PQI-system, nor an SQI-system.
- Σ_2 is a PQI-system that is neither a PI-system, nor a QI-system.
- Σ_3 is a PSI-system that is neither a PI-system, nor an SI-system.
- Σ_4 is a SQI-system that is neither an SI-system, nor a QI-system.
- Σ_5 is a PI-system that is not an I-system.
- Σ_6 is a QI-system that is not an I-system.
- Σ_7 is an SI-system that is not an I-system.
- Σ_8 is an I-system.

Proof

Σ_1 is not saturated, for V does not contain the full valuation, which is the supervaluation of the empty set, hence it is not a PSI-system. It is not compact, because V does not contain the full valuation, but for every finite subset X of S it contains a v' such that $X = \mathrm{T}(v')$; hence it is not a PQI-system. Moreover, it is not co-compact, for V contains the empty valuation, but as V cannot contain any valuation mapping only a finite number of sentences on 0, for every finite subset Y of S there is a $v' \notin V$ such that $Y = \mathrm{C}(v')$. Hence it is not an SQI-system.

Σ_2 is a PQI-system, for it is determined by the infinite set of QI-ons $\{<\{A\},\varnothing> \mid A \in S\}$. However, is not saturated, for V does not contain the full valuation, hence it is not a P(S)I-system. Also it is not co-compact, for V contains the empty valuation, whereas for every finite subset Y of S there is a $v' \notin V$ such that $Y = \mathrm{C}(v')$; hence it is not a (S)QI-system.

Σ_3 is a PSI-system, for it is determined by the infinite set of SI-ons $\{<X,\varnothing> \mid X \subseteq S$ and X is finite$\}$. However, it is not compact, because V does not contain the empty valuation, but for every finite subset Y of S it contains a v' such that $Y = \mathrm{C}(v')$; hence it is not a P(Q)I-system. Moreover, it is not co-compact, for V contains the full valuation, whereas for every finite subset X of S there is a $v' \notin V$ such that $X = \mathrm{T}(v')$; hence it is not an S(Q)I-system.

Σ_4 is an SQI-system, for it is determined by the SQI-on $<\mathrm{S},\varnothing>$. However, it is not saturated, for V does not contain the full valuation, hence it is not a (P)SI-system. It is not compact, because V does not contain the full valuation, but for every finite subset X of S it contains a v' such that $X = \mathrm{T}(v')$; hence it is not a (P)QI-system.

Σ_5 is a PI-system for it is determined by the infinite set of I-ons $\{<\varnothing,\{A\}> \mid A \in S\}$. But it is not co-compact, for V contains the full valuation, whereas for every finite subset X of S there is a $v' \notin V$ such that $X = \mathrm{T}(v')$; hence it is not a (S)(Q)I-system.

Σ_6 is a QI-system for it is determined by the finite set of QI-ons $\{<\varnothing,\{A,B\}>, <\{A,B\},\varnothing>\}$. But it is not saturated, for the supervaluation of V is the empty valuation, hence it is not a (P)(S)I-system.

Σ_7 is an SI-system for it is determined by the single SI-on $<S\backslash\{A\},A>$. But it is not compact, for V contains, for every finite subset Y of $S\backslash\{A\}$, a v' such that $\mathrm{T}(v') = Y$ and $\mathrm{C}(v') = A$. Hence it is not a (P)(Q)I-system.

Σ_8 is an I-system, for it is determined by the I-ons $<\varnothing,\{B\}>$. □

On the following modification of Diagram 1, each type of semantic system is complemented by an example of a system which is of this type, but of no lower one:

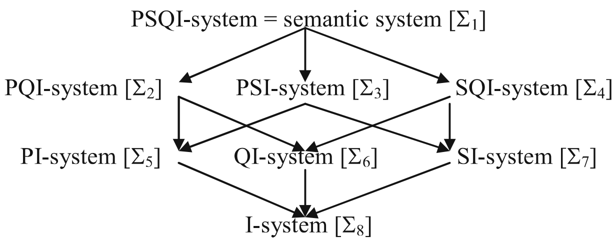


Diagram 1*

6 Structured Systems of Sentences

So far we have been dealing with semantic systems and inferential structures based on unstructured sets of sentences. A more interesting case, however, is when the set of sentences is generated from a basic vocabulary by some grammatical rules, and hence its elements have a grammatical structure. Only in this case we can formulate inferential *rules* logic is interested in.

A plausible explication of the concept of *language* would be a finite collection of finite sets (of words) and a collection of rules generating complex expressions and especially sentences. Sentences thus generated have structures that reflect the ways they are generated from words. If we replace some of the words they consist of by parameters, we obtain *sentence forms*. A sentence form has as many instances as there are sentences that can be obtained from it by substituting words of appropriate categories for all its parameters. In this way, a form may let us address an infinite number of sentences in one sweep.

Here we will adopt a much more simplified explication: we will identify forms simply with arbitrary sets of sentences. Thus we will consider language as a set of sentences plus a set of its subsets considered as forms of sentences. Hence the following definitions:

Definition A *language* is an ordered pair $<S,F>$, where S is a set (the elements of which are called *sentences*) and $F \subseteq \mathrm{Pow}(S)$ (the elements of this set are called *forms* of sentences). An *instantiation* over $<S,F>$ is any function i from F to S such that for every $f \in F$, $i(f) \in f$; $i(f)$ is then called the i-instance of f. A *generalized semantic system* is an ordered pair $<<S,F>,V>$ such that $<S,F>$ is a language and $<S,V>$ is a semantic system. A pair of subsets of $F \cup S$ is called a *parametric SQI-on*, or *pSQI-on* over $<S,F>$; the concepts of pQI-on, pSI-on and pI-on are defined analogously. (It follows that every (S)(Q)I-on over S is a p(S)(Q)I-on over $<S,F>$.) The pair $<<S,F>,\vdash>$, where $<S,F>$ is a language and $\vdash$ is a (finite) set of p(S)(Q)I-ons over $<S,F>$, is called a *p(P)(S)(Q)I-structure*. It is called a *(P)(S)(Q)I-structure*, iff $<S,\vdash>$ is a (P)(S)(Q)I-structure (which means that $\vdash$ consists exclusively of nonparametric (S)(Q)I-ons).

The i-instance of a p(S)(Q)I-on, for an instantiation i, is the (S)(Q)I-on which arises from it by the replacement of forms by their i-instances. The p(S)(Q)I-on excludes those and only those valuations of S that are excluded by some of its instances. A generalized (p)(P)(S)(Q)I-structure is said to *determine* a generalized semantic system $<<S,F>,V>$ iff V is the set of all and only elements of $\{0,1\}^S$ not excluded by any element of $\vdash$. If $<<S,F>,V>$ is determined by a (p)(P)(S)(Q)I-structure, it is called a (p)(P)(S)(Q)I-system. (We will drop the adjective *generalized* when the context makes it clear that we are dealing with generalized structures and systems.)

Considering generalized semantic systems and entering the concepts of p(P)(S)(Q)-systems provides for a substantial refinement of Diagram 1, doubling the number of categories. However, the following simple theorem will provide for its simplification.

Theorem 5 $<<S,F>,V>$ is a p(S)(Q)I-system if it is a (S)(Q)I-system and only if it is a P(S)(Q)I-system, but both the inclusions are proper.

Proof As a (S)(Q)I-on is a special case of a p(S)(Q)I-on, a (S)(Q)I-system is trivially a p(S)(Q)I-system. On the other hand, as a valuation is excluded by a pSQI-on iff it is excluded by one of its instances, a pSQI-on is, in this respect, wholly replaceable by the set of all SQI-ons that are its instances. Hence if we are free to use an infinite number of SQI-ons, we can do any job done by pSQI-ons with SQI-ons.

Now consider the system <<*S*,{*S*}>,{∅}>, where *S* is infinite. It follows from Theorem 4 that it is not an SQI-system. However, it is a p(S)QI-system, for it is determined by the single p(S)QI-on <*S*,∅>. Hence not every SQI-system is a pSQI-system. Consider, in turn, the system <<*S*,∅>,{∅}>. It is a P(S)QI-system, for it is determined by the infinite set of (S)QI-ons {<{*A*},∅> | *A*∈*S*}. But <<*S*,∅>,{∅}> is obviously a SQI-system just in the case <*S*,{∅}> is, which we know it is not. Hence not every P(S)(Q)I-system is a p(S)(Q)I-system. □

Hence the reduced refinement of the Diagram 1 brought about by engaging the parametric versions of (S)(Q)I-ons is the following:

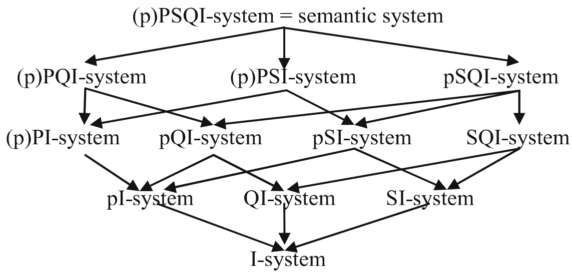


Diagram 2

If what we want to study are *logical* systems, then what we are interested in are *general* rules, and hence systems that allow for *parametric* SQI-ons—i.e. we should concentrate on the kinds of systems with "p" in their category name. Alongside this, we are usually interested in systems that can be determined by a *finite* number of such rules—i.e. we should concentrate on those without "P" in their category name. This leaves the following four highlighted categories in the focus of our attention:

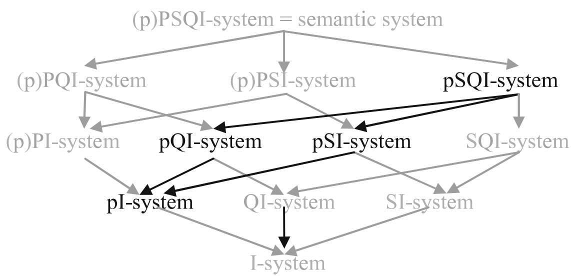


Diagram 2*

Let us consider some, now more realistic, examples of systems which fall into these categories.

Σ_{CPC} is the semantic system $<<S,F>,V>$ of classical propositional logic, where S is the set of wffs of propositional logic, F is a set of sets of instances of corresponding schemata, and V is the set of all those valuations of the set that do justice to all the truth tables of the classical connectives.

Σ_{CPC*} is the semantic system $<<S,F>,V>$ of classical propositional logic, where S and F are as before and V is the set of all those valuations of the set that map all axioms of a system of classical propositional logic on 1 and map a b on 1 whenever there is an a so that a and $a \rightarrow b$ are mapped on 1. (Note that this system, as well as the following one, admits the full valuation; this valuation is then usually excluded 'manually', by introducing the concept of *consistency* and banning inconsistent valuations.)

Σ_{In} is the semantic system $<<S,F>,V>$ of intuitionist propositional logic, where S and F are as before and V is the set of all those valuations of the set that map all axioms of a system of intuitionist logic on 1 and map a b on 1 whenever there is an a so that a and $a \rightarrow b$ are mapped on 1.

Σ_A is the semantic system $<<S,F>,V>$ of standard arithmetic, where S is the set of wffs of Peano arithmetic, F is a set of sets of instances of corresponding schemata and V consists of the single valuation that maps a sentence on 1 iff it is true in the standard model.

Theorem 6

1. Σ_{CPC} is a pQI-system, but not a pI-system
2. Σ_{CPC*} is a pI-system
3. Σ_{In} is a pI-system
4. Σ_A is a pSQI-system, but neither a pSI-system, nor a pQI-system

Proof

1. Σ_{CPC} is a pQI-system, for it is determined by the set of pQI-ons (we assume that the primitive connectives are $\rightarrow$ and $\neg$; a and b are parameters, and we let schemata stand for the sets of their instances): $\{<\{a, a \rightarrow b\},\{b\}>, <\{b\},\{a \rightarrow b\}>, <\emptyset,\{a,a \rightarrow b\}>, <\{a,\neg b\},\emptyset>, <\emptyset,\{a,\neg b\}>\}$. However, it is not saturated: for any atomic a it admits a valuation that maps a on 0 and one that maps $\neg a$ on 0, but no valuation that maps both a and $\neg a$ on 0 is admissible; hence it is not a pI-system.
2. Σ_{CPC*} is a pI-system, for it is determined by the set of pI-ons: $\{<\{a, a \rightarrow b\}, \{b\}>, <\emptyset,\{a \rightarrow (b \rightarrow a)\}>, <\emptyset,\{a \rightarrow (b \rightarrow c) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c))>, <\emptyset, (a \rightarrow b) \rightarrow ((\neg a \rightarrow b) \rightarrow b)>\}$. (To talk more concisely, we will identify an axiom of a logical system with the pI-on that has an empty antecedent and the axiom in the consequent, and also we will identify an inference rule, such as *modus ponens*, with the corresponding pI-on. In this way we can say simply that Σ_{CPC*} is determined by the axiomatic system of classical propositional logic.)
3. Σ_{In} is a pI-system, for it is determined by the axiomatic system of intuitionist propositional logic. (Note that here we have nothing corresponding to the 'non-

axiomatic' version of classical logic. Intuitionist operators do not have any truth tables independent of the axiomatization of the logic.)

4. Σ_A is a pSQI-system, for it is determined by the PQI-ons determining Σ_{CPC} plus the axioms of Peano arithmetic plus the pSI-on $<\{p(n)\}_{n=1,\ldots,\infty},\{\forall xp(x)\}>$. It is not a pSI-system, for it is not saturated (for the same reason as Σ_{CPC}) and it is not a pQI-system, for it is not compact: if p is a predicate such that $\forall xp(x)$ is undecidable, then for every finite set N^* of numerals there is a valuation mapping every $p(n)$ on 1 for every $n \in N^*$ and $\forall xp(x)$ on 0, whereas there is none mapping $p(n)$ on 1 for every $n \in N$ and $\forall xp(x)$ on 0. □

7 Consequence Revisited

If what we are interested in is the relation of consequence, then our classificatory hierarchy is excessively fine-grained. In particular, we will show that a (p)(P)(S)QI-system does not differ in this respect from the corresponding (p)(P)(S)I-system. (Thus, for example the semantic systems Σ_{CPC} and Σ_{CPC^*}, which differ just in this respect, yield the same relation of consequence.)

Lemma Let V, V_1, V_2 be subsets of $\{0,1\}^S$ such that $V_2 = V_1 \cup V$, where every element of V is a supervaluation of a subset of V_1. Then the consequence relations induced by the systems $<<S,F>,V_1>$ and $<<S,F>,V_2>$ coincide.

Proof Let $\models_1$ and $\models_2$ be the consequence relations induced by $<<S,F>,V_1>$ and $<<S,F>,V_2>$, respectively. Let us assume that these relations are different, hence as it is clear that $\models_2 \subseteq \models_1$, we have $\models_1 \nsubseteq \models_2$. Then there is a subset X of S and an element A of S so that $X \models_1 A$, but not $X \models_2 A$. This means that there must be a $v \in V$ so that $v(B)=1$ for every $B \in X$ and $v(A)=0$. As v is the supervaluation of an $U \subseteq V_1$, $\mathrm{T}(u)=\mathrm{T}(U)$, and hence every element of U maps every element of X on 1, and at least one element of U maps A on 0. But then V_1 contains a valuation that maps every element of X on 1 and A on 0, which is a contradiction. □

Definition sv(U) will denote the supervaluation of the set U.

Lemma $\mathrm{sv}\left(\bigcup_{i \in I} U_i\right) = \mathrm{sv}\left(\{\mathrm{sv}(U_i)\}_{i \in I}\right)$.

Proof Let $\mathrm{sv}\left(\bigcup_{i \in I} U_i\right)$ map an element A on 1. Then all elements of every U_i map A on 1, hence all the supervaluations of the U_i's map A on 1, and hence also $\mathrm{sv}(\{\mathrm{sv}(U_i)\}_{i \in I})$ maps A on 1. Let $\mathrm{sv}\left(\bigcup_{i \in I} U_i\right)$ map an element A on 0. Then there is a j so that U_j contains an element that maps A on 0. Hence $\mathrm{sv}(U_j)$ maps A on 0; and hence so does $\mathrm{sv}(\{\mathrm{sv}(U_i)\}_{i \in I})$. It follows that $\mathrm{sv}\left(\bigcup_{i \in I} U_i\right)$ is the same function as $\mathrm{sv}(\{\mathrm{sv}(U_i)\}_{i \in I})$. □

Lemma Let V, V_1, V_2 be subsets of $\{0,1\}^S$ such that $V_2 = V_1 \cup V$, where V consists of the supervaluations of all subsets of V_1. Then $<<S,F>,V_2>$ is saturated.

Proof Take an arbitrary $U_2 \subseteq V_2$; we will show that $\mathrm{sv}(U_2) \in V$. As $V_2 = V_1 \cup V$, there must be sets U_1 and U so that $U_1 \subseteq V_1$, $U \subseteq V$, and $U_2 = U_1 \cup U$. According to the previous lemma, $\mathrm{sv}(U_2) = \mathrm{sv}(\{\mathrm{sv}(U_1),\mathrm{sv}(U)\})$. As V consists of supervaluations of subsets of V_1 and as $U \subseteq V$, there must exist, for every $u \in U$, a $U_u \subseteq V_1$ so that $u = \mathrm{sv}(U_u)$. Hence according to the previous lemma, $\mathrm{sv}\{U\} = \mathrm{sv}\left(\bigcup_{u \in U} U_u\right)$, and as every U_u is a subset of V_1, $\mathrm{sv}\{U\}$ is a supervaluation of a subset U^* of V_1. Hence $\mathrm{sv}(U_2) = \mathrm{sv}(\{\mathrm{sv}(U_1),\mathrm{sv}(U^*)\})$, where both U_1 and U^* are subsets of V_1. But according to the previous lemma, $\mathrm{sv}(U_2) = \mathrm{sv}(U_1 \cup U^*)$ and so $\mathrm{sv}(U_2)$ is the supervaluation of a subset of V_1 and thus is an element of V. □

Theorem 7 For every (p)(P)(S)QI-system there exists a (p)(P)(S)I-system with the same relation of consequence.

Proof Let $<<S,F>,V_1>$ be a (p)(P)(S)QI-system. Let V_2 be V_1 extended by the supervaluations of all its subsets. Then, according to the first of the above lemmas, $<<S,F>,V_2>$ yields the same consequence relation as $<<S,F>,V_2>$, while being, according to the third lemma, saturated. Hence it is a (p)(P)(S)I-system. □

This gives us the following reduced version of Diagram 2:

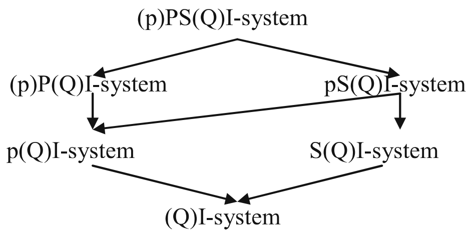


Diagram 3

De-highlighting the less interesting cases as in case of Diagram 2, we are left with only two important ones:

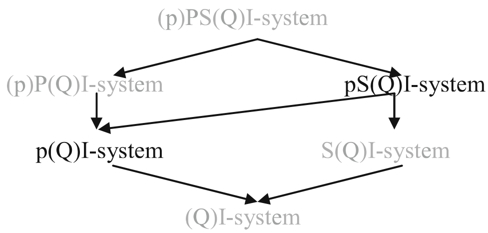


Diagram 3*

The p(Q)I case is exemplified by any fully axiomatized system (however, Σ_{CPC} is also such a system). The pS(Q)I case is exemplified by the standard model arithmetic, the delimitation of which requires the omega rule.

8 Conclusion and directions of further development

I hope to have shown how we can set up a useful framework for a systematic 'measurement' of how powerful various generalizations of the straightforward inferential framework are from the viewpoint of explicitly semantic frameworks; and that I have also indicated that this framework lets us prove some nontrivial and interesting results. There are, however, still many ways in which this framework calls for elaboration. Let me indicate at least three:

1. What axiomatic systems were developed for was not determining the semantic systems in our sense, but rather only capturing the corresponding sets of tautologies. This may lead us to the following concept: a semantic system is a t(p)(P)(S)(Q)-system iff there is a (p)(P)(S)(Q)-system with the same set of tautologies. Thus, for example, the system Σ_{CPC}, though we saw it is not a pI-system, is a tpI-system. (The reason is that the system Σ_{CPC^*}, which is a pI-system, shares the same set of tautologies.) It might be both interesting and useful to fully incorporate the concepts of t(p)(P)(S)(Q)-systems into our hierarchy.
2. The reader may have noticed that I have picked up examples from calculi of propositional logic and then skipped over those of predicate logic directly to a specific theory within predicate logic, namely arithmetic. The reason is that the general languages of predicate logic still escape our hierarchy. For consider all the valuations that are induced by interpretations of the language of the classical predicate calculus in all kinds of model structures. If we restricted ourselves to interpretations which leave no individuals nameless, the situation would be straightforward: it would be enough to add the pI-on $<\{\forall xp(x)\},\{p(n)\}>$ and the pSI-on $<\{p(v)\}_{v\in N},\{\forall xp(x)\}>\}$ to the PQI-ons determining the semantics of classical propositional logic, where N is the set of all names (terms) of the language. (Arithmetic is a special case of this: we have the universe of numbers and have a numeral for each of them.) However, if we allow for all interpretations admitted by the standard model theory for first-order logic, including those that may leave some of the elements of the universe nameless, the situation is much more complicated. The point is that now there are admissible valuations that map all instances of a general statement on 1, but the general statement itself on 0. (This is the case when all the counterexamples are nameless.) However, this does not mean that we can simply omit the corresponding pSQI-on $<\{p(v)\}_{v\in N},\{\forall xp(x)\}>\}$ without a replacement. Some valuations mapping all instances on 1 and the corresponding generalization on 0 must still be rendered inadmissible—for example those with $p(x)$ logically valid. The way to accomplish this that follows the standard strategy employed by the axiomatizations of the first-order predicate calculus would be to include the requirement that if A is mapped on 1 by every admissible valuation, then so is $\forall xA$—but this cannot be directly articulated in terms of our pSQI-ons.

3. Our explication of the concept of language, as we have pointed out, is an essential oversimplification. Whereas to any language in the more adequate sense outlined above (a finite vocabulary plus a finite collection of grammatical rules) there corresponds a language in our oversimplified sense, the converse is not the case. Hence our explication overgenerates and leaves room for amendments.

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