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THE 'NATURAL' AND THE 'FORMAL'

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ABSTRACT. The paper presents an argument against a "metaphysical" conception of logic according to which logic spells out a specific kind of mathematical structure that is somehow inherently related to our factual reasoning. In contrast, it is argued that it is always an empirical question as to whether a given mathematical structure really does captures a principle of reasoning. (More generally, it is argued that it is not meaningful to replace an empirical investigation of a thing by an investigation of its a priori analyzable structure without paying due attention to the question of whether it really is the structure of the thing in question.) It is proposed to elucidate the situation by distinguishing two essentially different realms with which our reason must deal: "the realm of the natural", constituted by the things of our empirical world, and "the realm of the formal", constituted by the structures that we use as "prisms" to view, to make sense of, and to reconstruct the world. It is suggested that this vantage point may throw light on many foundational problems of logic.

KEY WORDS: philosophy of logic, logical form, logical truth, structuralism, mathematical models

1. WHAT ARE STATEMENTS?

In the course of contemplating the nature of necessary truth, Hilary Putnam (for one) considers the feasibility of giving up a claims like

($*$) For all statements p, $\ulcorner(p\&\neg p)\urcorner$ is true.[1]

I think that a good way to illuminate many crucial problems of this centuryþs logic and analytic philosophy is to inquire into the nature of objects quantified over in claims of this kind; i.e. to examine the nature of *statements* (or *thoughts*, when ($*$) is disguised as something like 'it is not possible to think a thought together with its negation') that are dealt with by logicians and analytic philosophers.

Two basic responses are clear: (1) statements can be taken to be some factual objects that exist and can be identified independently of logic, typically sentences of some real language; or (2) the realm of statements can be taken to be constituted by logic.

For simplicity's sake, let us take a claim simpler than ($*$):

($**$) For all true statements p, $\ulcorner\neg p\urcorner$ is false.

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Let us concentrate on an individual instance of (∗∗): let us suppose we know which statement is represented by the sign 'p', and let us consider the sign $\ulcorner\neg p\urcorner$. Which statement does this sign represent? Various answers are possible:

If we subscribe to (1), then we must have a procedure to determine which statement $\ulcorner\neg p\urcorner$ represents (given p is determined) without any recourse to logic. If we see statements as (uniquely determined by) sentences of a factual language, then this procedure could be identified with the application of a specific grammatical construction (say a construction which typically results in a modification of the main verb of p by the particle 'not'). In this case, (∗∗) is clearly a meaningful claim with nontrivial *empirical* content.

On the other hand, if we subscribe to (2), then we have to understand the statement represented by $\ulcorner\neg p\urcorner$ as determined via the stipulation that $\neg p$ is the statement which is true just in case p is false. In this case, statements are understood as objects *constituted* by logical laws; and in this case we shall speak, as usual, about *propositions*. Understood in this way, (∗∗) is evidently trivial – it is no more than the direct consequence of our way of understanding the sign $\ulcorner\neg p\urcorner$. In this sense, (∗∗) obviously has no relevance for our factual reasoning.

However, neither of these answers is satisfactory: neither the claim that the truths of logic are empirical, nor that they have no relevance for our factual reasoning is acceptable. This seems to force a standpoint somewhere in between (1) and (2), resulting in the claim that statements do exist independently of logic and of logical laws (and hence that claims like (∗∗) are nontrivial, that they are more than consequences of our definitions), but that their compliance to the laws of logic is nevertheless not an empirical matter – that these laws express some necessary and eternal relationships between them.

This yields a *metaphysical* (or *ultra-physical*, as Wittgenstein, 1956, §I.8, would put it) conception of logic: according to this, logic *reports facts* about a realm of *non-empirical* things. My point in this paper is that accepting this view can easily engender the trivialization of the most important problems with which logic and analytic philosophy are devised to cope, and leads to an intrinsically corrupted view of the way our language and our knowledge functions. This does not mean that we are not to avoid both the conclusion that the truths of logic are empirical, and that they are mere consequences of our definitions; the point, however, is that the metaphysical conception of logic is of no real help. Thus, what I am going to say should not be taken to imply that we have no alternative other than to grasp logic as a chapter either of descriptive linguistics or of algebra;

I do think logic indeed *is* situated somewhere 'in between', but not in the way suggested by the metaphysical conception.

2. Digression I: Sunspots and headaches

Let me illustrate this point by a little story. Let us imagine a person, call him X, who complains that he often has headaches and claims that his headaches are caused by the occurrence of sunspots. However, what he claims is not simply that he has headaches whenever there are sunspots; he claims that what the sunspots do to him depends on a further factor, namely on the *influence mode* of the sun on him. His thesis is that if this influence mode is positive, then his head aches if and only if there are sunspots, while if it is negative, then his head aches if and only if there are *no* sunspots.

In such a case, the contentfulness of X's claim would clearly depend on how he explains his notion of influence mode. If he specifies it as a matter of, say, his blood pressure – so that the influence would be positive, e.g., if and only if his blood pressure were high – then the claim would clearly be a perfectly meaningful, empirical thesis that could be tested and verified or falsified in a straightforward way. The claims that somebody's head aches if and only if either there are sunspots and his blood pressure is high, or there are no sunspots and his blood pressure is low is something that may, or may not, be true – depending on how things really are.

However, imagine that the only thing X is willing to say about the influence mode is something like "it is that which determines whether sunspots are causing me headaches or rather the other way around". It is clear that in this case the contentfulness of his claim becomes highly suspect. The point is that now his thesis is defensible come what may, and consequently is empty of any real, empirical content. It could never, for instance, happen that there would be sunspots, that the influence of the sun on X would be positive, and yet that he would nevertheless have no headache – for in such a case he would always say that the influence is *not* positive and thereby save his thesis. Thus he could *always* brand the influence mode so that the thesis would keep holding – independently of the real pattern of co-occurrence of sunspots and his headaches.

In this situation X's thesis is merely something like an (implicit) definition of his concept of *influence mode* (a concept which would be not only probably useless, but possibly even harmful: for its very employment would tend to deceive us into believing that there is a real, empirically significant correlation between the sunspots and the headaches). If X insisted that the thesis *is* contentful, we would probably suspect him of either being hopelessly slow-witted, or trying to cheat – for his insistence would look like a foolish, or a treacherous, attempt to conjure up content where there is none. Now my point is that the metaphysical conception of logic (and, indeed, any stipulation of metaphysical reality of this kind) does, in a sense, an analogous thing, namely tries to produce claims which are simultaneously both infallible and contentful.

There is, no doubt, a clear sense in which we, when doing logic and giving philosophical accounts of language, deal with propositions rather than with sentences. When we speak, for instance, about one sentence implying another, we cannot see the sentences as mere syntactic objects. We are obliged to see them as meaning what they normally mean in a certain language (such as English). And when we say: 'meaning what they normally mean in English', we can equally well say 'expressing the propositions they normally express in English'. If we say that *Prague is in Europe* implies *Something is in Europe*, we obviously mean the two sentences as *English* sentences, or, we can say, as sentences expressing the propositions which they normally express in English. This is straightforward and indubitable. The trouble is that if we take the picture of *expressing propositions* (or *thoughts*) at face value, propositions may easily come to acquire a role similar to that of the *influence mode* of the previous anecdote: they may come to make logic and philosophy into an enterprise which is (seemingly) both contentful (in the sense that it can tell us something about our factual reasoning) and infallible.

The starting point of the previous story was X's claim about sunspots causing him headaches. However, his headaches were really *not* co-occurrent with sunspots; and what X did was to posit a 'ghostly entity', the *influence mode*, which immunized his claim from falsification. Now the starting point of logic is the presumption that logical laws apply to our factual reasoning and hence to our factual language, which is the basic medium of our reasoning. But if we saw logical laws as being directly about natural language, then their validity would be a dubious empirical matter: if we saw '$\neg$' as representing an English grammatical construction, then we could never be sure that ($**$) is universally valid. And hence we might come up with a 'remedy' akin to X's move, namely to posit 'ghostly entities', in this case *propositions*, that would immunize logical laws from being falsified: whenever we found a sentence apparently violating ($**$), we would conclude that this merely shows that the sentence does *not* express the proposition which it seems to express after all, and therefore is *not* an instance of ($**$).

Let us imagine *sunspots* and *headaches* as acquiring the values 1 and 0 (in the sense ‘occurring’ resp. ‘non-occurring’). Then what X did could be depicted as replacing the invalid equation

$$\text{sunspots} = \text{headaches},$$

by

$$\text{sunspots} + \text{influence mode} = \text{headaches},$$

whose validity is secured by taking

$$\text{influence mode} = \text{headaches} - \text{sunspots}.$$[2]

Thus, if there are sunspots, but no headaches, the influence mode can always be blamed. The general structure of this trick is this: we would like A to ‘yield’ B ($A = B$), but the fact is that A does not ‘yield’ it ($A \neq B$). Hence we say that A does not ‘yield’ B ‘by itself’, but rather ‘via’ C ($A + C = B$), where C is precisely what is needed to neutralize the disparity of A and B ($C = B - A$). Doing this, though, is turning the original claim into an infallible triviality. However, we can not only substitute sunspots for A, headaches for B and influence mode for C to gain the story of the brave Mr. X; we can also substitute logical laws for A, their real-language instances for B, and the propositions expressed by these instances for C — and we have the metaphysical conception of logic I warn against.

The conclusion, therefore, is that if X’s employment of the influence mode was misleading because it pretended to establish a regularity regardless of whether there really was one, then the introduction of propositions may be similarly misleading – if we take them to establish ‘logical’ regularities independently of whether these really do obtain within our factual language and our factual enterprise of ‘giving and asking for reasons’.[3] The point is not that if we pay insufficient attention to our factual language and our factual reasoning, we might reach logical laws which would be invalid (after all, the ‘law’ *sunspots* + *influence mode* = *headaches* is also not *invalid*), it is rather that such laws may be simply without purpose.

3. THE TWO REALMS

In his *Timaeus*, Plato claims: “Now first of all we must, in my judgment, make the following distinction. What is that which is Existent always and has no Becoming? And what is that which is Becoming always and never is Existent? Now the one of these is apprehensible by thought with the aid of reasoning, since it is ever uniformly existent; whereas the other is an object of opinion with the aid of unreasoning sensation, since it becomes and perishes and is never really existent.”

In this way, Plato became the first of many philosophers to notice that we, humans, somehow come to deal with two essentially different kinds of entities: that besides the ordinary things like dogs, trees, secret service agents or Rolls-Royces, whose properties continually change and which we never know for absolutely certain, we also operate with other kinds of entities, such as categories, geometrical shapes and structures, which are strangely rigid and somehow completely seized by our reason.

The soundness of such a distinction itself is hard to doubt; irrespectively of if and how we cash it out philosophically. It is not my purpose here to address the general philosophical questions regarding the nature and status of the two kinds of entities and of their mutual relationship (in particular I do not aim to answer the traditional philosophical question about the extent to which our categories are ‘within the things themselves’ and that to which they are ‘in the eye of the beholder’). Nevertheless, it is my conviction that the nature of such problems as the one mentioned at the beginning of this paper may be helpfully elucidated by realizing that our understanding is often the result of an interplay of entities of such two radically different kinds: that we often understand the ‘empirical world’ with the help of formal ‘prisms’.

To bring Plato’s high-flown cogitation down to earth, imagine a more mundane situation: imagine that you move to a city wholly unknown to you, and that a friend of yours who has lived there for a long time draws a simple plan of the city for you. The plan contains marks representing some basic orientation points (the City Hall, the theatre, a famous Chinese restaurant, the railway station etc.) and the main streets connecting them. Despite the fact that such a plan is a drastically simplified and compressed picture of the city, it can obviously help you to acquaint yourself with your new environment. You begin to see the streets and buildings via the prism of the plan and thus you start to see them occupying their places within a general layout. And note that what is crucial is the extent of simplification and condensation of the plan: if it were too simplified, it would cease to be a plan *of the city* and hence would be of no use; but at the same time it would also be of no use if it were as complicated and as large as the city itself.

The point I want to make is that many things we employ within our enterprise of coping with our environment play roles analogous to such a map of an unknown city. Mathematical representations are a typical case: when we say that something is a *triangle*, or that something can be captured by a *set of differential equations*, what we usually mean is that it *can be viewed as a triangle or as something governed by the dependencies spelled out by the equations on a certain level of abstraction – i.e. disregarding some amount of discrepancy*. After all, nothing we can find around us is a precise, geometrical triangle, nothing displays a pure mathematical structure.

Although I have no wish to embrace Plato’s general philosophical standpoint, I think it may be helpful to accept Plato’s illuminating terminology. Hence I propose to consider the situation in terms of two essentially distinct ‘realms’ related to our reason. First, there is what we can call ‘The Realm of the Natural’ (RN; the realm of Plato’s *Becoming*) – the realm of the ‘things’ with which we live our lives. In this realm, things and matters can be *found* and *described*, but they are essentially *vague* and*fuzzy* (in the sense that nothing has a pure mathematical structure). Nothing regarding this realm can be *proven* in the mathematical sense. It can be seen as inhabited with things (in the prototypical sense of the word) and events, and prototypically it is the subject of natural science. Contrasting with this, there is what we will call ‘The Realm of the Formal’ (RF; the realm of Plato’s *Being*). Here everything is precisely *defined* and sharply *delimited*; things are *stipulated* and facts about this world can be unambiguously *proved*. The inhabitants of this realm can, with a certain amount of oversimplification, be called *structures*; they are addressed most directly by mathematics.

The metaphysical conception of logic criticized above can now be seen as resting on the assumption that what logic addresses is something belonging both to RF and RN – something which, on the one hand, is rigid and directly susceptible to mathematical treatment, and yet, on the other hand, is a matter of our factual language and our factual reasoning. Truths of logic are taken to be true come what may, i.e. independently of what may or may not happen within the real world, but simultaneously they are assumed to be somehow inherently related to the language we happen to use and to the way we happen to think. Our point, then, is that the inherentness of this relationship is a pernicious illusion; we claim that whether a real thing can be reasonably seen as having this or another structure, or whether a structure can be helpfully ‘projected’ on this or that thing, is always an *empirical* matter. Thus, proving something about a structure from RF can be taken as proving something about a thing from RN *only if it is taken for granted that the thing has this structure* – which is itself something that is beyond a formal proof.

From this viewpoint the metaphysical conception of logic results from the misconstrual of the relationship between RN and RF, namely from neglecting the fact that the roles of these two realms within our coping with the world are quite distinct: RN is the very world with which we are destined to cope, whereas RF is the realm of prisms we employ (maybe *have* to employ, constituted as we are) to 'tame' it and indeed to 'make sense' of it, to understand it. Thus, structures from RF serve as prisms through which we see and understand the world, and which we may employ to explicitly reconstruct its regularities and to point out the 'forms' or 'structures' of things or events.[4]

4. DIGRESSION II: GEOMETRY

It is of essential importance to recognize the difference between dealing with RN via RF and dealing with RF itself, i.e. between addressing reality via the prism of a structure and addressing the structure itself. To provide a vivid illustration of what kind of confusions may arise if this difference is not properly acknowledged, let us return, for a moment, to the beginning of this century, to the time when modern, formal mathematics, as a powerful way of addressing RF, was establishing itself. Some mathematicians, by that time, had begun to see mathematics no longer as the study of some parts or features of reality carried out by analyzing their mathematical structures, but rather as the study of the structures themselves; and they consequently begun to urge that it is only this conception of mathematics that can guarantee that mathematics is truly rigorous. It was in geometry where this process took place most spectacularly – for geometry was the mathematical discipline traditionally most strongly tethered to a specific aspect of reality.

Hence, while geometry traditionally was conceived of as a way of accounting for certain aspect of reality (of RN) by means of analyzing its structure, some mathematicians with the new vision now shifted their attention to the very structure itself (i.e. RF) seeing in it the real subject matter of geometry. Thus, "geometry gradually moved from the study of absolute or perceived space – matter and extension – to the study of freestanding structures" (Shapiro, 1996, p. 149).[5] The new formalistic conception of geometry was most systematically presented in Hilbert's (1899) *Grundlagen der Geometrie*: Hilbert's idea was that all basic concepts of geometry, like *point*, *line*, *plane*, *is situated*, *parallel* etc. are delimited by nothing more than by their mutual relationships, which are spelled out by the axioms of geometry. (As Poincaré, 1900, p. 78, put it, "if one wants to isolate a term and abstract from its relations to other terms, what remains is *nothing*".)

But obviously those mathematicians who did not make this ‘formalistic turn’ and continued to treat mathematical structures as mere tools of an account for reality (rather than the very subject matter of mathematics) were puzzled – for them such proposals were tantamount to making zoological concepts like *dog*, *elephant*, *mammal* etc. wholly independent of any real animals and taking them to be constituted simply by their relations to each other. Gottlob Frege famously protested that what Hilbert calls “axioms” (and what thus, according to Frege, should have been the most indubitable truths) are not truths at all, but just mere definitions, which point out certain structures, but do not say to what these structures are ascribed. However, this, of course, was precisely what Hilbert meant – so no wonder he rejected Frege’s objections as misguided:

Sie schreiben: “. . . Aus der Wahrheit der Axiome folgt, dass sie einander nicht widersprechen”. Es hat mich sehr interessirt, gerade diesen Satz bei Ihnen zu lesen, da ich nämlich, solange ich über solche Dinge denke, schreibe und vortrage, immer gerade umgekehrt sage: Wenn sich die willkürlich gesetzten Axiome nicht einander widersprechen mit sämtlichen Folgen, so sind sie wahr, so existieren die durch die Axiome definirten dinge. . . . Ja, es ist doch selbstverständlich eine jede Theorie nur ein Fachwerk oder Schema von Begriffen nebst ihren nothwendigen Beziehungen zu einander, und die Grundelemente können in beliebiger Weise gedacht werden (printed in Frege, 1976, pp. 66, 67)[6]

For Frege, this was clearly simply preposterous: what he desperately missed was the projection of the ‘free-standing structure’ delimited by Hilbert’s axioms onto reality, a projection which would made it into a prism through which to see real things. Without such a projection, for him no axioms could make sense:

Ich weiss nicht, wie ich mit Ihren Definitionen die Frage entscheiden soll, ob meine Taschenuhr ein Punkt sei. Gleich das erste Axiom handelt von zwei Punkten; wenn ich also wissen wollte, ob es von meiner Uhr gälte, müsste ich zunächst von einem anderen Gegenstande wissen, dass er ein Punkt wäre. Aber selbst wenn ich das z.B. von meinem Federhalter wüsste, so könnte ich noch immer nicht entscheiden, ob meine Uhr und mein Federhalter eine Gerade bestimmten, weil ich nicht wüsste, was eine Gerade wäre (*ibid*., p. 73)[7]

Hilbert’s reply was again rather laconic:

Meine Meinung ist Eben die, dass ein Begriff nur durch seine Beziehungen zu anderen Begriffen logisch festgelegt werden kann. Diese Beziehungen, in bestimmten Aussagen formulirt, nenne ich Axiome und komme so dazu, dass die Axiome . . . die Definitionen der Begriffe sind (*ibid*., 79)[8]

In this way, the dispute soon foundered in deadlock.

Now it is hard to get rid of the impression that *both* parties of this quarrel are at least partly right, each in its own way. And indeed what we claim is that if we distinguish properly between RF itself and RN viewed via RF, we can see that the differences between the standpoints of Frege and Hilbert may consist more in an overaccentuation of different aspects of a common picture than in proposing incompatible pictures.

What Frege (and Russell and other 'realists') demanded was that there be a way to use geometrical axioms and the structure they spell out to address real things like watches, pen-cases etc., that the structure be somehow projected on the real world. This is indeed a reasonable demand, but we should add, on behalf of Hilbert, that *if geometry is to be exercised with mathematical exactitude, then the projection cannot play a real role within the system as such*. However, it is hard to believe that Frege, the depth of whose contributions to the development of modern, exact logic and mathematics is indubitable, would not have seen this. It seems unlikely that the author of the *Begriffsschrift*, which has set the standard of logical regimentation of our judging, would have underestimated exactitude and theoretical precision. He stressed rather that exactitude and precision have real value only when addressing something 'real', something that is not a mere conclusion of our definitions and stipulations.

On the other hand, what Hilbert (and Poincaré and other 'formalists') insisted was that if we want to do with geometry what Peano did with arithmetic, if we want to leave nothing unproved, then we must treat it as an abstract, 'ideal' system whose terms are significant only as its nodes. Again, this is surely true, but on behalf of Frege we must add that *we call something 'geometry' only if it is capable of serving a certain specific purpose*, namely to help us cope with certain spatial aspects of the things which surround us. And again, it is hard to believe that Hilbert would be blind to this; it would be more than difficult to believe that he would accept that *any* system of axioms, say that of Peano arithmetic, would constitute as good a geometry as that which is constituted by Hilbert's own axioms of geometry. I think that the truth is rather that he believed this to be too self-evident to dwell on; and he simply wanted to stress that the matters concerning the projectibility of the geometrical structure on reality are not capable of being included into mathematics itself.

The emerging conclusion is that geometry originated as a matter of addressing, and thereby explicating, certain aspects of RN with the help of a certain prism from the RF. It is consequently a relatively uninteresting, terminological problem whether we should use the term 'geometry' for the abstract prism alone, or for the prism together with the projection. In the first case, we would have to keep in mind that the prism is called so only in virtue of its relevant projectibility; in the second we would have to realize that it is only the prism, not the projection, that can be subjected to 'mathematical' treatment.

From this point of view, the Frege–Hilbert controversy may appear more a misunderstanding than a real disagreement. This, of course, is not to say that there are no substantial differences between the views of the two theoreticians; it is to suggest that some of the differences may be less deep than generally supposed.[9]

5. Mathematical models and reality

The fact that we have to distinguish between an abstract mathematical structure and reality captured via that mathematical structure – between ‘pure’ and ‘applied’ mathematics, somewhat oversimplifying – is, of course, no breath-taking discovery. Probably everybody dealing with mathematics recognizes it on a general level; and those who reflect upon the workings of modern, formal mathematics have sometimes even articulated it with remarkable clarity. Thus Reichenbach (1920, pp. 32–35):

> Der *mathematische Gegenstand* ist durch die Axiome und die Definitionen der Mathematik vollständig definiert. ... Für den *physikalischen Gegenstand* aber ist eine derartige Definition unmöglich. Denn er ist ein Ding der Wirklichkeit, nicht jener konstruierten Welt der Mathematik. . . . Es ist Methode der Physik geworden, eine Größe durch andere zu definieren, indem man sie zu immer weiter zurückliegenden Größen in Beziehung setzt und schließlich ein System von Axiomen, Grundgleichungen der Physik, an die Spitze stellt. Aber was wir auf diese Weise erreichen ist immer nur ein System von verflochtenen mathematischen Sätzen, und es fehlt innerhalb dieses Systems gerade diejenige Behauptung, daß dies System von Gleichungen *Geltung für die Wirklichkeit hat*. Das ist eine ganz andere Beziehung als die immanente Wahrheitsrelation der Mathematik. Wir können sie als eine Zuordnung auffassen: die wirklichen Dinge werden Gleichungen zugeordnet. . . . Nennen wir die Erde eine Kugel, so ist das eine Zuordnung der mathematischen Figur “Kugel” zu gewissen Wahrnemungen unserer Augen und unseres Tastsinns, die wir, bereits eine primitive Stufe der Zuordnung vollziehend, als “Wahrnemungsbilder der Erde” bezeichnen.[10]

Thus, a mathematical term can have two kinds of ‘meanings’: the ‘internal’ meaning which it acquires by denoting a node in a mathematical structure, and possibly also an ‘external’ meaning which it acquires as the consequence of the fact that the structure is somehow projected on reality. The term “sphere” or the numeral “five” represent certain nodes within certain ‘free-standing’ mathematical structures (which have been brought out by the axioms of the corresponding theories, geometry resp. arithmetic), but they can also be seen as representing, or accounting for, something from the non-mathematical world: objects of certain shapes, resp. groups of objects of certain cardinality. The difference is, in our words, whether what we have in mind is merely an element of RF, or rather those elements of RN onto which this formal element is taken to be adequately ‘projectible’.

Unfortunately, although the distinction between a mathematical object and a real object captured by it – i.e. between mathematical model and reality – may be clear on the general level, the fact that in many circumstances we can simply neglect it makes us sometimes disregard it even in cases when its acknowledgment is crucial.

Imagine that you inspect a room, see that it is empty and then you see three, and later four people enter it. You claim: “The room now contains seven people”; and if this claim is challenged, you offer the proof of the fact that three plus four equals seven within Peano arithmetic. Is this really the proof of the claim? This is clearly so only provided your ‘mathematization’ of the problem has been adequate. If you later opened the room and discovered there not seven, but only six people, you would surely not blame Peano arithmetic. You would certainly say that, apparently, one person has left the room without your noticing; so that the correct thing to prove was not $3 + 4 = 7$, but rather $3 + 4 - 1 = 6$.

But perhaps the slack between the ‘mathematical model’ and reality can be dispensed with, perhaps the only thing needed is to make the model *really* adequate? Perhaps we only have to pay attention to achieve the perfect fit? Perhaps, in our case, if we took pains to achieve the perfect fit, the proof of the mathematical theorem would be, *eo ipso*, a proof of the number of people within the room? However, what would such a perfect fit amount to? We have seen that you should have guarded against people entering or leaving the room unnoticed. But this is in no way the only thing that could spoil the fit. Some person in the room might have killed and eaten another person; or some person might have borne a child. Or somebody may be so positioned that it is unclear whether she is still in the room or not. Or it may turn out that the room you were observing does not, in fact, exist, for what you took to be the walls of the room was only an optical illusion. It seems clear that it is never possible to really exclude or even spell out all of these potentialities to secure the ‘perfect fit’. We rather *assume* such fit, but carry somewhere in the back of our minds that there is this assumption, which might turn out to be false. And troubles begin if we forget about it.

This is, of course, not to say that ‘mathematical’ or ‘logical’ models cannot capture reality and help us solve real problems. It would, of course, be absurd to claim that we do not build bridges, planes or cyclotrons with the help of mathematics, or that we cannot, say, build systems of knowledge representation with the help of logic. It is to say that any such ‘mathematization’ necessarily has an empirical ingredient brought about by the fact that the fit of a structure from RF to a problem from RN is bound to be an empirical matter. In practice, this is not particularly problematic, for the fit we can achieve is usually quite sufficient. However, the situation may alter when we turn our attention to foundational questions about the nature of logic, philosophy or mathematics: disregarding the matter here may result into grave misconceptions.

This prompts us to give the following answer to the question about the nature of logic: the truths of logic are necessary and ‘mathematically treatable’ because (and only insofar as) they constitute a system within RF, while they are ‘factual’ and ‘about our reasoning’ because (and only insofar as) this system is projected onto RN and used as a prism to capture the relevant aspects of our reasoning. In other words, they are necessary, for they spell out a constant structure, and they are about our reasoning, for this reasoning can be seen to display this very structure. Their ‘necessity’ and their ‘factualness’ are thus properties of different levels, and it is crucial to hold them in a certain equilibrium: concentrating exclusively on the former puts us in danger of losing the connection with the real world (thus falling into scholastic speculations, or, in the better case, into pure mathematics); while concentrating exclusively on the latter puts us in danger that we shall not be able to really *understand* (for understanding requires an appreciation of regularities, the application of a ‘mathematical’ prism).

6. LOGIC VS. METAPHYSICS

Laying the foundations of modern logic, Frege realized that in order to get a grip on the ‘content’ of sentences (i.e. on the propositions they express), he had to strip them of everything not relevant from the viewpoint of the consequence relation.[11] At this point, finding out which formula was to regiment a given sentence and thereby which proposition the sentence expressed, seemed to be a matter of a certain simplification of syntax. Frege himself, and especially his followers (notably Russell), subsequently concluded that sometimes it is necessary to accept that finding a correct logical regimentation of a sentence (hence locating the proposition expressed by the sentence) might itself be a nontrivial task (this was exemplified, e.g. by the Russellian analysis of sentences containing definite descriptions[12]). A logician or a philosopher who admits the possibility of such a nontrivial gap between a sentence and the proposition it expresses must recognize the possibility of investigating the world of propositions, bypassing sentences expressing the propositions. However, there seems to be a bifurcation of ways of understanding this enterprise.

The ‘metaphysical’ standpoint amounts to the conclusion that it is this investigation that is the ultimate task of logic. After all, propositions are what are really substantial; and the question about which propositions are expressed by which sentences is the business of empirical linguistics. “That my logic does not apply to natural language,” such a theoretician is likely to say, “is not my business – the worse for the language. What I am investigating are propositions; and I do not care which propositions happen to be expressed by sentences of a factual language.” But he would also reject that what he is doing is simply mathematics (an investigation of a certain abstract structure). He would insist that the world of propositions he is addressing is in some intimate way connected with our reasoning. This is a standpoint that Norman Malcolm (1940, p. 197) characterized as follows: “Philosophers and logicians have the idea that when a question as to whether one statement entails another arises, verbal considerations enter only because of ambiguity, and that the *real* question is not a verbal one, but one to be settled by the intellect’s fixing its gaze upon the proposition, *after* the ambiguity has been cleared up”. The trouble with the metaphysical view of logic is not that it accepts propositions, but rather that it accepts the notion that propositions can be investigated by “the intellect’s fixing its gaze upon them.”

The alternative standpoint we advocate amounts to seeing the investigation into the realm of propositions as an enterprise internal to logic – as the analysis of the formal structures logic uses to account for our reasoning. Such an enterprise then has its substantiation not simply in itself; it is substantiated only insofar as the tool it analyzes is a *useful* tool. Thus, while the metaphysical conception of logic simply assumes that the ‘mathematics of propositions’ equals logic because the propositions are somehow inherently related to our reasoning, we urge that it is logic only because, and only insofar as, we are able to use the propositions as the nodes of the prism that helps us understand reasoning.

If this is correct, then the sense of the system of propositions which is susceptible to mathematical treatment should be seen in its capacity to account for the regularities of the way we use language and of the rules implicit to this usage. It does its job only if it can be projected on our factual usage of language and our factual reasoning in such a way that it explicates its substantial regularities and rules. It is the possibility of such a projection which generally substantiates the usage of items from RF outside of mathematics; and as the projection is a matter of the relationship between RF and RN, its existence can never be proved or subjected to formal criteria (for these make sense only inside RF). Thus, to see logic or philosophy as the study of a world of propositions makes real sense only insofar as the world can be seen as parasitic upon our factual games of ‘giving and asking for reasons’.

7. Davidson on Propositions

The warning against promoting the abstract world of propositions (or thoughts in the Fregean sense) to an independently accessible reality is a point which plays, I am convinced, an important role in the writings of several key figures of this century’s philosophy of logic and analytic philosophy. I think that this was precisely what Wittgenstein had in mind when he insisted that the real subject matter of the philosophy of language is constituted by the factual language games we play, and that the mental entities that we tend to see as making our expressions meaningful are better seen as our way of accounting for the games. (“Sieh auf das Sprachspiel als das *Primäre*! Und auf die Gefühle, etc. als auf eine Betrachtungsweise, eine Deutung, des Sprachspiels!”[13] – 1953, §656.)

I also think that this view is central to many of the founding fathers of American analytic philosophy, namely Quine, Sellars, Davidson etc., in their effort to revise the picture of the relationship between language and the world provided by their European predecessors.[14] Quine is so vehemently against seeing sentences as expressing propositions (which, according to him, may so easily lead to what he calls the *museum myth*) that he insists on rejecting the very notion of proposition and of meaning in general. Sellars’ way of rejecting the metaphysics of propositions concentrates in his claim that meanings (and especially propositions) are *inherently functional* entities; that they – in an important sense – do not exist apart from their embodiment.[15] However, the most vivid elaboration of the anti-metaphysical standpoint urged here is offered by Donald Davidson.

Davidson’s claim is that propositions should be construed as the units of measurement we use to characterize certain aspects of the world surrounding us (namely rational beings) in the same sense in which we use meters and kilograms to characterize other aspects of it. Thus, saying that a speaker believes a proposition (or that a sentence expresses the proposition) is like saying that something is five meters long, or that something can be captured by a certain mathematical equation. Davidson (1989, p. 11) claims:

> Just as in measuring weight we need a collection of entities which have a structure in which we can reflect the relations between weighty objects, so in attributing states of belief (and other propositional attitudes) we need a collection of entities related in ways that will allow us to keep track of the relevant properties of the various psychological states. In thinking and talking of the weights we need not suppose there are such things as weights for objects to have. Similarly in thinking and talking about the beliefs of people we needn’t suppose there are such entities as beliefs.

What Davidson denies is that there are such *things* as propositions, which would be, on the one hand, associated with sentences (thus becoming their *meanings*), and which would, on the other hand, come to inhabit people's heads (thus becoming their *beliefs*). However, saying that propositions are not *things* in this sense does not amount to saying that they are nothing at all – it amounts to saying, as we would put it, that they are not to be sought within the RN.

Thus, according to Davidson, propositions do exist in the same way as meters and kilograms do: as nodes within a structure that we cast over a certain part of our world to make it intelligible in the way our reason seeks it. In the same way as it is helpful for us to see, say, a stone as assuming a place on the scales of meters, kilograms etc., it is also helpful to see a sentence as assuming a place within the network of propositions, and to see a rational agent as assuming a place within the network of theories, i.e. sets of such propositions.[16]

8. Foundational problems of logic

It is my conviction that the view urged above can not only provide a clearer insight into the conceptual framework of modern logic and into the nature of the entities this framework employs, but can also throw some new light on some of the most frequently discussed foundational problems of logic. Let us briefly review some of the cases where carefully distinguishing RF and RN may, I believe, be enlightening.

Before turning to genuine problems, let me mention an instance of the rare case where the relationship between RF and RN comes to the open to such an extent that it usually does not cause any serious confusion. *Church's thesis* states explicitly that a formal concept, namely *recursivness* (or, equivalently, *Turing-computability*, *representability in lambda-calculus* etc.), matches a natural one, namely *computability in the intuitive sense*.[17] It is clear that Church's thesis cannot be proved, for we can prove the equivalence only of two *formal* concepts (like recursivness and Turing-computability); we cannot prove that a formal concept does capture an informal one. On the other hand, the thesis provides an excellent example of how we can obtain an informal, but compelling justification for a thesis of this kind: if we find out, as we have, that all, or almost all, independently developed formal concepts purporting to capture a given informal one come to the same, there are good grounds for concluding that they do capture it successfully. (Note however, that such a justification, unlike a formal proof, does not *guarantee* that any further disagreement is bound to be a matter of misunderstanding – in fact there continue to be people who undoubtedly understand the issue very well and who nevertheless do challenge Church’s thesis.[18])

A case where the relationship between the RF and RN plays a less perspicuous role is the problem of the significance of *Gödel’s incompleteness proof*. This result is often interpreted as a stunning discovery stating ‘the limits of human reason’, the ‘inscrutability of mathematical truth’ or, in Roger Penrose’s words, the “unalgorithmicity of the mind”.[19] However, the perspective urged here leads to the conclusion that such interpretations should only be accepted with caution.

Was Gödel’s proof a *formal* proof (of the kind of that of the equivalence of recursivness and Turing-computability), or was it a finding of something *factual* (about something like computability in the intuitive sense, i.e. about what we humans, as a matter of fact, can do)? It seems that the first is the case: Gödel’s proof appears to be a mathematical matter which establishes the theorem proved with the certainty, which guarantees that everybody who does not believe simply does not understand it. However, if this is the case and if our above conclusions are right, then Gödel’s result must concern merely an abstract mathematical structure and be thus confined to RF.[20] It can tell us nothing whatsoever about anything factual, such as how our human reason, as a matter of fact, works, or how our mathematical practices may or may not proceed.

On the other hand, if we took Gödel’s result as saying something about RN, then we would have to give up the idea that it is a proof in the mathematical sense, clearing away any possibility of doubt. We have seen that the competence of a mathematical proof within RN is always only conditional: the applicability of the proof to factual matters always depends crucially on the adequacy of the relevant projection, which is itself beyond any proof. In other words, we can interpret Gödel’s result as being about something from our real world only to the extent to which the thing in question can be adequately ascribed the structure which Gödel’s proof concerns directly. This is to say that it can be taken as being about what we humans do when we count and when we do what we call *arithmetic* only if we take this activity of ours to display the very structure which is envisaged by formal Peano arithmetic. Of course, we have various kinds of convincing reasons to think that it does display it; never, though, can we have a formal proof.

The problem is that the shining of Gödel’s result derives, at least partly, precisely from the fact that it is taken to be a *formal* proof of something *factual*, namely of how we humans do or can think – and this, if our above conclusions are right, is simply not possible. To be formally provable (to be ‘mathematically certain’) and to say something about our real, human world (to ‘refer to reality’) are mutually exclusive properties: a formal proof, as we have seen, is the matter of RF, it can directly concern neither RN, nor a projection of RF onto RN. (As Einstein, 1983, p. 28, put it: "So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.") We may *either* take Gödel's proof to be a proof in the strong sense, but then we cannot take it as directly addressing 'human arithmetic capacities' (let alone human reason as a whole), *or* insist that it is about 'real' arithmetic, but then we cannot see it as a proof in the strict mathematical sense.

Another foundational problem I will mention is the *theory of truth* as established by Alfred Tarski[21]: in this case, too, the failure to distinguish between 'the natural' and 'the formal' is likely to cause much confusion. It is often claimed that the status of Tarskian T-sentences, i.e. sentences like

(T) 'Snow is white' is $\mathbf{true_T}$ if and only if snow is white,

is essentially problematic, for if we take $\mathbf{true_T}$ to be the predicate introduced by Tarski's theory, then there is a sense in which we can say that (T) is a truth of logic – for it follows from nothing else but the principles of logic (including, possibly, set theory) and definitions. This means that (T) must be uninformative in the way logical truths are, it has to be "true in every possible world" (see Putnam, 1985, p. 63). However, this seems to be in contradiction with Tarski's declared intention to address the pre-formal concept of truth. Some theoreticians (e.g. Putnam) conclude that Tarski managed to develop a logical theory but failed to tell us anything about truth; others (e.g. Etchemendy, 1988) suggest that what is missing from Tarski's theory is the stipulation that a sentence is $\mathbf{true_T}$ if and only if it is true (what Davidson, 1990, calls the "truth axiom").

I think the best way to describe what is going on here is to say that what Tarski was after was to point out a formal structure capable of serving as a reconstruction of our language with its truth-predicate. Tarski's theory is a *formal* theory in that it constitutes a system within which some statements can be proven to be 'logical truths' (notably those of the shape of (T)); but it is also a theory *of truth* in that it can be projected onto our natural language and thus helps us understand the functioning of the predicate *true* and the concept expressed by it. (As Davidson, 1990, p. 314, stresses, the Tarskian theory of the "formal properties" of the concept of truth must be supplemented by an indication of "how a theory of truth can be applied to particular speakers or groups of speakers".) If we see Tarski's theory as pointing out an item within RF, then his $\mathbf{true_T}$ is a formal predicate governed by its formal definition; but if we see it as using this item for the purpose of capturing an item from RN, then we may see it as an explication of our informal concept of truth. However, formulating a "truth axiom" stipulating that Tarski's formal system is adequate to that which it is devised to capture is no more meaningful than stipulating Church’s thesis – adding axioms can make items from RF into larger items of RF only, it can never pin them down to items of RN. The point is that an axiom, being by its nature a matter of RF, can never guarantee that a formal theory is adequate to which it is supposed to be a theory of. This is always necessarily a matter of practical assessment and evaluation by human subjects, who may, or may not, find it useful for their enterprises of coping with the RN. Nothing can belong *both* to RF *and* to RN (be *both* a stipulation *and* a ‘phenomenon’); nevertheless, an item from RF may turn out to be a helpful prism to observe an item from RN.[22]

Another cluster of problems which might be clarified by adopting the vantage point urged here centers on the concept of *semantic interpretation*. An interpretation is a mapping of a system of items (‘expressions’) onto another system of items (‘denotations’, ‘meanings’). Within logic, semantic interpretations of this kind are studied by model theory. It was Montague (1974, p. 188) who voiced the claim that from the point of view of semantics, there is no real difference between natural language and formal languages; he was also one of the first to demonstrate how to apply model-theoretic notions to natural languages in an interesting way, and thus he laid the foundations of what is nowadays called *formal semantics*.

There are two essentially different ways of interpreting a formal language (i.e. a certain system of items of RF). We can map it either (i) on another system of items within RF, or (ii) on a system of items within RN. In the case (i) we do not leave the RF, i.e. the province of mathematics; so even the mapping itself is a formal object which can be studied in a mathematical way. This is exactly what model theory does: it addresses mappings of certain kinds of formal structures (‘formal languages’) onto another kind of formal structures (‘model structures’). In case (ii), interpretation amounts instead to what we have called *projection*: it is the means of perceiving a part, or a feature, of RN through the prism of the structure from RF (typically perceiving a factual language as a formal structure); and in this way it also licenses us to see the structure from RF as being ‘about’ something factual, as being a structure ‘of’ something factual (e.g. our natural language).

It is important to fully appreciate the depth of the difference between these two kinds of mappings; and to realize that calling them both *interpretations* may even be misguiding[23]. If we say that a formal language, a mere system of strings of items, has to be interpreted in order to become a language worth its name, what this should typically mean is that an element of RF has to be projected onto RN in order to become useful for the purposes of explicating factual matters.[24] However, it is often assumed that *the same* can be achieved by mapping the formal language onto a *formal system of denotations*, by furnishing the language with a formal, set-theoretic interpretation. But a formal interpretation makes a formal language into merely a more complex formal system; it can never make it really 'meaningful'.[25]

What is possible, and sometimes indeed useful, is to take a formal mapping of a formal language (an item of RF) onto a formal system of denotations (another item of RF) as a 'picture' of a 'natural' mapping of a natural language (an item of the RN) onto the system of meanings of the expressions of the language (another item of RN). Personally, I think that this is the only viable sense in which we can take 'formal semantics' as a theory of natural language (see Peregrin, 1997). However, this can be also deeply misleading and hence dangerous: it may suggest that natural language is a set of labels stuck on pre-existing things; and this is a view that is essentially problematical (in fact, to see language thus involves falling for what Quine calls the museum myth – see Peregrin, 1995; 1998).

9. DIGRESSION III: THE NATURE OF LINGUISTIC THEORY

The last issue discussed in the previous section is connected to problems concerning the very nature of a theory of language. As this is a deep and interesting problem and as it can, I believe, throw some further light on the conceptual framework introduced here, let us make a short excursion into linguistics (and its philosophy) and say a few words about it. (Elsewhere I have discussed it at length – see Peregrin, 1998.)

The question to be answered is this: is linguistic theory about individual speakers (be it about their minds, language faculties, behavior or whatever), or about a realm of abstracta? As a starting point, let us take Katz and Postal's (1991) paper, where the authors urge the replacement of Chomskyan "conceptualism" with "realism". The "conceptualist" view which Katz and Postal challenge (and which they ascribe to Chomsky) is that "grammars and grammatical theory describe a psychological reality" (p. 517); their own "realistic" view takes "natural languages to be abstract objects rather than concrete psychological or acoustic ones" (p. 515). Although I think Katz's and Postal's criticism is basically sound, I am persuaded that the two standpoints do not exclude each other to the extent to which they would seem to do so – that if we look at the dispute from the proper angle, we may see it at least partly as a terminological matter.

On the one hand, there is a straightforward sense of 'realism' in which every minimally plausible semantic theory trivially has to be realistic. It is hard to believe that anybody, even the most diehard mentalists and conceptualists, would claim that semantics is a matter of describing some mental (neural) particulars within the head of an individual speaker – for this would be no theory of English (nor of any other language), but rather the theory of some features of a particular person. Even if we accept the assumption that semantics is a matter of particulars of such a kind, we simply have to assume that these particulars can somehow be equated across speakers; that they have some properties which make them treatable as tokens of recurrent types. So the linguist must talk about some non-particulars – be they construed as cross-subjective type-identities of particulars, or some abstract entities borne by these identities. In any case, talk about meaning is in a clear sense a talk about *types*, not about *tokens*; and semantics is – in this sense – *inevitably* (and trivially) realistic.

On the other hand, even the most diehard realist has to assume that there are some contingent facts that elicit which meaning an individual expression has. We do not discover meanings by an ‘intellectual trip’ into a realm of abstracta where we could see them attached to expressions; but rather by observing and recording certain concreta. It is the occurrence of certain particular events or entities (be it the occurrence of certain utterances of speakers, or the occurrence of certain contents within the heads of speakers) which establishes the meaning of an expression. Therefore, both the conceptualist and the realist apparently must agree that meanings are abstracta (universals) which are in a certain sense determined by (parasitic upon) certain concreta (particulars).

So, if the only thing that realism claimed were that semantics is a matter of abstracta rather than of concreta, of types rather than of tokens, then realism would seem to be unobjectionable. And if the only thing which conceptualism asserted were that abstracta make no sense unless they are in the sense outlined ‘parasitic’ upon concreta, then it too would be hardly objectionable. Hence, such modest conceptualism and modest realism might even coincide – if we accept that our knowledge (in general) arises out of apprehending some particular arrays of occurrences as displaying universal structures, out of seeing items of RN via items from RF. The only clash is then, again, a terminological one: whether this situation justifies us in saying that linguistics is about the particular occurrences, or about the universal structures. This is a legitimate subject for a quarrel, but not for one with great significance.

Troubles begin when conceptualism or realism ar taken as claiming something more. Conceptualism sometimes seems to claim that the theoretician of language has no use of abstract entities whatsoever, whereas realism sometimes appears to claim that that these abstract entities are accessible in a direct way, wholly bypassing their concrete embodiments.

Extreme conceptualism thus disregards the fact that to understand is to discern a pattern, a structure, to see tokens as tokens of types; whereas extreme realism forgets that the abstract structures we discern are interesting only insofar as they are the structures *of that which we have set out to study*.

Thus, I think that this kind of quarrel can be again largely clarified by pointing out that in making a theory such as the theory of our linguistic performance we usually address something from RN by means of something from RF. We address the potential infinity of concrete utterances of speakers by means of a certain structure; we 'capture' the former by the latter. This is to say that if we ask what it is that linguistic theory is about, then there are, just like in the case of geometry discussed above, *two* different kinds of answers available, corresponding to two different senses of "about". In the first sense, linguistics is about the part of RN which it addresses (i.e. about certain 'concreta'); in the second sense, it is about the part of RF which is capable of providing an adequate reconstruction the part of RN in question (i.e. about certain abstracta). The quarrel between 'realists' and 'conceptualists' thus may again turn out to be more a misunderstanding than a real discrepancy.

10. Conclusion

The main thesis of this paper, the usefulness of distinguishing between the 'natural' and the 'formal' should not be read as a metaphysical pronouncement. The talk about the two 'realms' should be read not as a report of a (re)discovery, but rather as a vivid way of making the point that something may be susceptible to a *more geometrico* treatment only if it is a thing we can somehow completely seize by our reason, not one of the things we *encounter* within the world of our everyday experience. This point was duly made long ago by Brouwer (1907, p. 76), who stressed that people err when they think that they "could reason logically about other subjects than mathematical structures built by themselves". We can *prove* things only about entities we ourselves *stipulate*, not about entities we *encounter*.

However, this is not to say that the 'formal' entities we stipulate cannot help us comprehend and grasp the 'natural'. On the contrary, they are essentially important, for the imposing of *structures* (which is what they basically are) is our fundamental means of theoretically coping with our environment (or indeed of handling it, predicting its behavior etc.). We can, nevertheless, never restrict ourselves to the structures alone: we have to constantly check and assess whether they are fully adequate to that to which they have been ascribed, whether they are sufficiently helpful as prisms through which to look. We can never eradicate a ‘pragmatic factor’: any mathematical theory of anything outside of mathematics, in order to be helpful, has not only to be in an internal order (be consistent), but also has to fit adequately to that of which it purports to be a theory. And the second requirement always involves an assessment from the viewpoint of some of our specifically human interests.[26]

In particular, when we return to logic, it is often helpful to reconstruct our reasoning, our language and our thought as various formal or mathematical structures; however, this helps us only insofar as we remember that these structures are nothing more (and, indeed, also nothing *less*) than our way of getting a grip on that which we thus reconstruct.

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NOTES

[1] See Putnam (1994, p. 250; I have replaced quotes by the more appropriate Quinean quasiquotes). The author’s claim is that mathematical and logical necessity is a matter of our not being able to give up, and indeed not being able make any sense of the falsity of, such claims.

[2] Where the value 0 of the *influence mode* is to be understood as representing positive influence, while the other two possible values as representing the negative one.

[3] A phrase due to Brandom (1994).

[4] Cf. Stekeler-Weithofer (1994).

[5] Shapiro’s paper also contains a more detailed discussion of the Frege–Hilbert controversy outlined below and of its historical context.

[6] “You write: ‘ . . . From the truth of the axioms it follows that they do not contradict one another’. I was very interested to read this particular sentence of yours, because for my part, ever since I have been thinking, writing and lecturing about such matters, I have been accustomed to say just the reverse: if the arbitrarily posited axioms are not in mutual contradiction with the totality of their consequences, then they are true – the things defined by the axioms exist. . . . . Yes, it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes.”

[7] “I do not know how to use your definitions to decide the question whether my pocket watch is a point. Already the first axiom treats of two points; thus if I wanted to know whether it is valid for my watch, I would first have to know about some other object that it is a point. But even if I knew this, e.g., about my pen-case, I could still not decide whether

my watch and my pen-case determine a straight line, for I would not know what a straight line is."

[8] "It is my opinion just that a concept can be fixed logically only by its relations to other concepts. These relations, formulated in certain statements, I call axioms, thus arriving at the new view that axioms . . . are the definitions of the concepts."

[9] This distinction is also closely connected to that between the two ways of understanding mathematics as the "science of structure" discussed by Shapiro (1996), namely between the "*ante rem*" structuralism and the "*in re*" or "eliminative" structuralism. The former consists in seeing mathematics as directly addressing structures from RF, while the latter sees it as addressing items from RN as instances of these structures and sees the structures as not independent, but rather only as parasitic on their instances.

[10] "The *mathematical object* is fully defined by the axioms and definitions of mathematics. . . . Such a definition is, however, not possible for the *physicalistic object*, for it is a thing of the real world, not of the constructed world of mathematics. . . . It has become the method of physics to define one magnitude through others, in that it is related to magnitudes lying ever more in the background, finally putting the system of axioms, of the basic equations of physics, on the top. However, what we reach in this way is still only a system of entangled mathematical sentences, and this system does not contain the assertion that the system of equations *is valid for reality*. This is a quite different relation than the immanent truth-relation of mathematics. We can grasp it as an assignment: real things are assigned to the equations. . . . If we call the Earth a sphere, then it is the assignment of the mathematical figure 'sphere' to certain perceptions of our eyes and our taste, which we denote, thereby accomplishing a primitive level of the assignment, as 'perception of the Earth'." (My translation.)

[11] See Frege (1879, p. IV).

[12] See Russell (1905).

[13] "See the language game as the *primary*! And the feelings etc. as a way of dealing with, or of accounting for, the language game!"

[14] Cf. Peregrin (1999a).

[15] Cf. Brandom (1994).

[16] Cf. Hofman (1995).

[17] See, e.g., Boolos and Jeffrey (1974, p. 20).

[18] See, e.g., Hintikka and Mutanen (1998).

[19] See Penrose (1990).

[20] To highlight the formal character of the proof, we can characterize Gödel's incompleteness proof, e.g., as follows. Let **A** be an alphabet (a finite set of objects) and let us call **L** the set of all strings over **A**. If $a, b \in$ **L**, let $a^\frown b$ denote the concatenation of a and b (i.e. the string which arises out of appending b to a) and let $\oplus$ denote some binary operation defined on the basis of concatenation ($a \oplus b$ might be, for example, substituting b for a given symbol within a). Let **M** be a subset of the powerset of **L**, i.e. a set of subsets of **L**. We define the relation $=_M$ among the elements of **L** in such a way that $a =_M b$ if and only if every $m \in$ **M** contains either both a and b, or neither a nor b. If $x \in$ **A**, then by the *x-variant* of an $a \in$ **L** we shall call the string $x^\frown a$, i.e. such a string which arises out of a by the prefixation of x. A subset of **L** will be called *x-open*, if it contains no string together with its x-variant; and we shall call it *x-saturated*, if it contains x-variants of all such strings from **L** which it does not contain. Now it clearly holds that if, for some $x \in$ **A**, some $a \in$ **L** and every $b \in$ **L**, $a \oplus b =_M x^\frown b \oplus b$, then no x-open set from **M** contains $a \oplus a$ (and hence $x^\frown a \oplus a$); thus no x-open set from **M** is x-saturated. Now we can see Gödel's proof as the

proof of the fact that a particular structure fulfills the premises of this general theorem: that if we take **L** to be the language (the set of wffs) of Peano arithmetic, **M** the set of consistent theories in this language containing the axioms of Peano arithmetic, x the negation-sign, and $\oplus$ the appropriate kind of substitution of the numeral expressing the Gödel’s number, then there will indeed be an $a \in \mathbf{L}$ so that for every $b \in \mathbf{L}$, $a \oplus b =_M x^\frown b \oplus b$; hence no x-open (= consistent) theory is x-saturated (= complete).

[21] See Tarski (1932; 1944). See also Peregrin (1999b).

[22] For a similar argumentation see García-Carpintero (1999), who argues, in effect, that it is precisely this what is constitutive of an *explication* (in the Carnapian and Quinean sense).

[23] One of the bad habits of contemporary ‘formal semantics’ is to confuse the two senses of *interpretation*. See Stekeler-Weithofer (1986) for a discussion.

[24] As Brandom (1994, p. 144) puts it, “it is only in so far as it is appealed to in explaining the circumstances under which judgments and inferences are properly made and the proper consequences of doing so that something associated by the theorist with interpreted states or expressions qualifies as a semantic interpretant, or deserves to be called a theoretical concept of content”. Brandom’s way of reflecting the distinction we stress here is distinguishing between what he calls *formal* and *philosophical* semantics (where the former is an enterprise internal to what we call RF, while the latter’s concern is, in our terms, to explicate a relevant portion of RN, perhaps with the help of some tools from RF).

[25] This ambiguity of the concept of interpretation also engenders the ambiguity of concepts which are based on it, especially of the concepts of *soundness* and *completeness*. Formal soundness and completeness amounts to capturing all and only sentences universally valid w.r.t. a given class of model structures; natural soundness and completeness means an exhaustive capturing of a pre-formal range of truths. The former, not the latter, can be subject to mathematical proof – and the proof of the former is not a proof of the latter (*pace* Kreisel, 1967). See also Peregrin (1995, §4.9).

[26] It may be illuminating to invoke the good old Kantian dualism of *Verstand* and *Vernunft* here: the adequacy assessment, we can say, is always a matter not of the calculating *Verstand*, but rather of the understanding *Vernunft* inseparable from the ability to perceive things from the distinctively human visual angle. Cf. Stekeler-Weithofer’s (1992) attempt to explain Hegel’s criticism of Kant in these terms.

REFERENCES

Boolos, G. S. and Jeffrey, R. C. (1974): *Computability and Logic*, Cambridge University Press, Cambridge.

Brandom, R. (1994): *Making it Explicit*, Harvard University Press, Cambridge, MA.

Brouwer, L. E. J. (1907): Over de grondslagen der wiskunde, Thesis, Amsterdam; reprinted in A. Heyting (ed.), Brouwer: *Collected Works I: Philosophy and Foundations of Mathematics*, North-Holland, Amsterdam, 1975.

Davidson, D. (1989): What is present to the mind?, in J. Brandl and W. L. Gombocz (eds), *The Mind of Donald Davidson*, Rodopi, Amsterdam, pp. 3–18.

Davidson, D. (1990): The structure and contents of truth, *J. Philos.* **87**: 279–328.

Einstein, A. (1983): *Geometry and Experience*, Sidelights on Relativity, Dover, New York (translation of the German original ‘Geometrie und Erfahrung’ from 1921).

Etchemendy, J. (1988): Tarski on truth and logical consequence, *J. Symbolic Logic* **53**: 51–79.

Frege, G. (1879): *Begriffsschrift*, Nebert, Halle; English translation in van Heijenoort (1971), pp. 1–82.

Frege, G. (1976): *Wissenschaftlicher Briefwechsel* (ed. by G. Gabriel *et al.*), Meiner, Hamburg; English translations in Frege: *Philosophical and Mathematical Correspondence*, Blackwell, Oxford, 1980.

García-Carpintero, M. (1999): The explanatory value of truth theories embodying the semantic conception, in Peregrin (1999b), pp. 185–212.

Gödel, K. (1931): Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, *Monatsh. Math. Phys.* **38**: 173–198.

van Heijenoort, J. (ed.) (1971): *From Frege to Gödel: A Source Book in Mathematical Logic*, Harvard University Press, Cambridge, MA.

Hilbert, D. (1899): *Grundlagen der Geometrie*, Teubner, Leipzig.

Hintikka, J. and Mutanen, A. (1998): An alternative concept of computability, in J. Hintikka (ed.), *Language, Truth and Logic in Mathematics* (*Selected Papers 3*), Kluwer Acad. Publ., Dordrecht.

Hofman, A. (1995): On the nature of meaning and its indeterminacy: Davidson's view in perspective, *Erkenntnis* **42**: 15–40.

Katz, J. J. and Postal, P. M. (1991): Realism vs. conceptualism in linguistics', *Linguistics and Philosophy* **14**: 515–554.

Kreisel, G. (1967): Informal rigour and completeness proofs, in I. Lakatos (ed.), *Problems in the Philosophy of Mathematics*, North-Holland, Amsterdam; reprinted in J. Hintikka (ed.), *The Philosophy of Mathematics*, Oxford University Press, Oxford, 1969, pp. 78–94.

Malcolm, N. (1940): Are necessary propositions really verbal?, *Mind* **49**: 189–203.

Montague, R. (1974): *Formal Philosophy: Selected Papers of R. Montague* (ed. R. Thomason), Yale University Press, New Haven.

Penrose, R. (1990): *The Emperor's New Mind*, Vintage, London.

Peregrin, J. (1995): *Doing Worlds with Words*, Kluwer Acad. Publ., Dordrecht.

Peregrin, J. (1997): Language and its models, *Nordic J. Philos. Logic* **2**: 1–23.

Peregrin, J. (1998): Linguistics and philosophy, *Theoretical Linguistics* **24**: 245–264.

Peregrin, J. (1999a): The pragmatization of semantics, in K. Turner (ed.), *The Semantics/Pragmatics Interface from Different Points of View*, Elsevier, Amsterdam, pp. 419–442.

Peregrin, J. (ed.) (1999b): *Truth and its Nature (if Any)*, Kluwer Acad. Publ., Dordrecht.

Poincaré, H. (1908): Science and method (English translation), in *Foundations of Science*, Science Press, New York, 1921, pp. 359–546.

Putnam, H. (1985): A comparison of something with something else, *New Literary History* **17**: 61–79.

Putnam, H. (1994): Rethinking mathematical necessity, in J. Conant (ed.), *Words and Life*, Harvard University Press, Cambridge, MA.

Quine, W. V. O. (1960): *Word and Object*, MIT Press, Cambridge, MA.

Reichenbach, H. (1920): *Relativitätstheorie und Erkenntnis Apriori*, Springer, Berlin.

Russell, B. (1905): On denoting, *Mind* **14**: 479–493.

Sellars, W. (1963): *Science, Perception and Reality*, Routledge, New York.

Shapiro, S. (1996): Space, number and structure: A tale of two debates, *Philos. Math.* **4**: 148–173.

Stekeler-Weithofer, P. (1986): *Grundprobleme der Logik*, de Gruyter, Berlin.

Stekeler-Weithofer, P. (1992): *Hegels Analytische Philosophie*, Schöningh, Paderborn.

Stekeler-Weithofer, P. (1994): Ideation und Projektion, *Deutsche Zeitschrift für Philosophie* **42**: 783–798.

Tarski, A. (1932): Der Wahrheitsbegriff in den Sprachen der deduktiven Disziplinen, in *Akademischer Anzeiger der Akademie der Wissenschaften in Wien* 69; English translation ‘The Concept of Truth in Formalized Languages’ in Tarski (1956), pp. 152–278.

Tarski, A. (1944): The semantic conception of truth, *Philosophy and Phenomenological Research* **4**: 341–375.

Tarski, A. (1956): *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford.

Wittgenstein, L. (1953): *Philosophische Untersuchungen*, Blackwell, Oxford; English translation *Philosophical Investigations*, Blackwell, Oxford, 1953.

Wittgenstein, L. (1956): *Bemerkungen über die Grundlagen der Mathematik*, Blackwell, Oxford; English translation *Remarks on the Foundations of Mathematics*, Blackwell, Oxford, 1956.

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