LOGIC AS "MAKING IT EXPLICIT"

Jaroslav Peregrin

Logical operators in the inferentialist perspective

In considering the very possibility of deviant logic, we face the following question: what makes us see an operator of one logical system as a deviant version of an operator of another system? Why not see it simply as a different operator? Why do we see, say, intuitionist implication as an operator 'competing' with classical implication? Is it only because both happen to be called implications?[1]

It is clear that if we want to make cross-systemic comparisons, we need an 'Archimedean point' external to the systems compared. Some logicians and philosophers, including Quine (1986), come close to saying that no such Archimedean point is available, and hence that there can be no deviant logics, for if two operators are governed by different axioms, then they are simply two different operators (or, if you prefer, operators with different meanings). From this viewpoint, intuitionist implication is no less different from classical implication than, say, intuitionist or classical conjunction.

This conclusion is indeed plausible if we consider logical calculi simply as algebraic structures[2]; however, things are different if we see them as a means of accounting for something that is already 'there' before we establish the structures and is to be explicated by them. From such a perspective, two operators of different systems may be variants of the same operator in virtue of the fact that they are both means of capturing the same pre-theoretical item.

What might these items be? Sometimes it seems that logicians tacitly assume that there are some mythical archetypes of implication, conjunction etc., located somewhere in some Platonic heaven or somehow underlying the *a priori* structures of our mind, which logic tries to capture (for better or worse). However, when it comes to the comparison of the concrete outcomes of our logical efforts, say the classical and the intuitionist implications with the archetypal Implication, the latter can never be materialized so distinctly as to be of any help.

A more constructive proposal is that the operators are related to elements or constructions of our language. For example, both intuitionist and classical implication can be seen as means of reconstructing the natural language connective "if ... then ...". Clearly, the properties of classical implication far from totally coincide with those of "if ... then ..." (as far as this is possible to judge at all, because of the notorious vagueness of the latter). Hence should we say that classical implication is a modification of "if ... then ...", or rather that it is something different which replaces it?

It is also possible, of course, to say that operators are devised to capture some pre-conceived semantic items, such as truth-functions or Boolean operators over sets of possible worlds. However, this would turn logic into pure mathematics, which it is surely not -- logic is

[1] See Haack (1996) for a thorough discussion of these questions.

[2] It seems that even so there might be a way of accommodating the intuition that two operators of two different systems are two variations on a single theme -- if we had a way to judge relative similarity of structures and consequent relative similarity of places (or roles) within the structures. Intuitively, this should be possible, but I know of no feasible development of this idea.

something essentially related to the ways we actually argue, justify and prove, and is to aid us with classifying the arguments, proofs etc. as right or wrong.

Here I would like to consider a possibility inspired by Bob Brandom's (1994; 2000) notion of logic as a means of "making it explicit". According to Brandom, what underlies both human language and logic are inferences. Human language is structured in such a way that commitments to some claims bring about commitments to other claims -- i.e. that the former ones entail the latter ones and hence the latter are correctly inferable from the former. In fact, Brandom claims that there is a still deeper layer constituted by the concept of incompatibility.

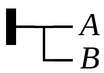
Brandom also claims that what we see as logical vocabulary is first and foremost a means of making explicit the proprieties implicit in our using language. Before we have "if ... then", the inference from "This is a dog" to "This is a mammal" can only be implicitly endorsed (or, as the case may be, violated), but once this connective is at hand, this inference can be explicitly expressed in the form of a claim, *viz*. "If this is a dog, then this is a mammal" (and with a more advanced logical vocabulary perhaps further transformed into "Every dog is a mammal"), and hence discussed w.r.t. its 'appropriateness' (and perhaps in the end rejected). Thus logical vocabulary helps us make the rules that are intractably implicit in our practices explicit and discussable[3].

Hence in this paper I would like to look at the usual logical operators as a means of explicitating inferential structure. I will claim that the need of such an explicitation leads to a set of 'inferentially native' operators, which, within various logical systems, can mutate into various forms.

Historical remarks

It is instructive to see how logical constants were understood in some of the classics of modern logic and analytic philosophy. Thus, introducing his conditional sign, Frege (1879, p.5) writes

> Wenn *A* und *B* beurtheilbare Inhalte bedeuten, so gibt es folgende vier Möglichkeiten:
> 1) *A* wird bejaht und *B* wird bejaht;
> 2) *A* wird bejaht und *B* wird verneint;
> 3) *A* wird verneint und *B* wird bejaht;
> 4) *A* wird verneint und *B* wird verneint.



[3] Of course that as Brandom's proposal is not utterly unprecedential -- seeing connectives as means of articulating facts about inferability is to a certain extent common within some proof-theoretic approaches to logic, especially within the Gernan constructivistic tradition (Lorenzen and Schwemmer, 1975).

> bedeutet nun das Urtheil, *dass die dritte dieser Möglichkeiten nicht stattfinde, sondern eine der drei andern*.[4]

This indicates that the sign Frege defines, and the natural language "if ... then" which it is clearly supposed to 'regiment', has a *metalinguistic content*: it expresses a fact about its content clauses being asserted and/or denied.

The situation is similar in respect to quantifiers. Frege (*ibid.*, p.19) writes

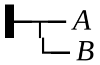
> In dem Ausdrucke eines Urtheils kann man die rechts von $\vdash$ stehende Verbindung von Zeichen immer als Funktion eines der darin vorkommenden Zeichen ansehen. *Setzt man an die Stelle dieses Argumentes einen deutschen Buchstaben, und giebt man dem Inhaltsstriche eine Höhlung, in der dieser selbe Buchstabe steht, wie in*
>
> $\vdash\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})$
>
> *so bedeutet dies das Urtheil, daß jene Function eine Thatsache sei, was man auch als ihr argument ansehen möge*.[5]

In this way, a quantified sentence is not a 'regimentation' of a sentence of the object language, but rather again a shortcut for a *metalinguistic* statement: a statement about the results of replacing a part of an object language sentence by various (suitable) expressions. Hence to say that some object-language expressions, such as "something" or "everything", can be regimented by such quantifiers, is to say that these expressions are means of 'internalizing' certain metalinguistic pronouncements.

Russell (1905, p. 480) takes, in this respect, a very close train of thought:

> ... *everything* and *nothing* and *something* (...) are to be interpreted as follows:
>
> *C*(everything) means '*C*(*x*) is always true';
>
> *C*(nothing) means '"*C*(*x*) is false" is always true';
>
> *C*(something) means 'It is false that "*C*(x) is false" is always true.'

[4] "If *A* and *B* stand for contents that can become judgments, there are the following four possibilities: (1) *A* is affirmed and *B* is affirmed; (2) *A* is affirmed and *B* is denied; (3) *A* is denied and *B* is affirmed; (4) *A* is denied and *B* is denied. Now



stands for the judgment that *the third of these possibilities does not take place*, *but one of the three others does*."

[5] "In the expression for a judgement, the complex symbol to the right of $\vdash$ may always be regarded as a function of one of the symbols that occur in it. *Let us replace this argument with a Gothic letter, and insert a concavity in the content-stroke, and make this same Gothic letter stand over the concavity, e.g.:*

$\vdash\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a})$

This signifies the judgement that the function is a fact whatever we take its argument to be." (Elsewhere I have discussed the consequences of this Fregean notion of quantification for the concept of variable -- see Peregrin, 2000.)

Hence again, sentences with "everything", "nothing" etc. are taken to express *metalinguistic pronouncements*; and their presence in a language thus enables us to say *in the language* what holds *about the language* and what is otherwise only expressible *within a metalanguage*.

'Inferentially native' operators

Suppose we have an inferentially structured language, i.e. a set S of sentences with a relation $\vdash$ which is a relation between finite sequences of elements of S and elements of S. Hence

$A_1,...,A_n \vdash A.$

means that the statement A is (correctly) inferable from the sequence $A_1,...,A_n$ of statements. We will employ the letters A, A_1, ..., A_n, B, C ..., to stand for statements, whereas X, Y, Z, ... to stand for finite sequences thereof.

What would it take to make an inference, such as

$A \vdash B,$

explicit in the structure? We would need a statement which *says* that B is inferable from A. But what does it take for a statement of such a structure to *say* this? Presumably to be true iff B is inferable from A. But the relation $\vdash$ is unchanging and hence the explicitating claim would be true necessarily; and what is the closest approximation to necessary truth within the structure is clearly inferability from nothing.

Let us assume that we have a function, call it d (for *deductor*), mapping pairs of statements on statements in such a way that

(*) $A \vdash B$ iff $\vdash d(A,B)$.

In this way, to claim $d(A,B)$ (as a necessary truth) would be to claim that B is inferable from A. Splitting (*) into its two halves and using "/" as a shorthand for "only if" we have

DED: $A \vdash B \;/\; \vdash d(A,B)$
CODED: $\vdash d(A,B) \;/\; A \vdash B$

However, d allows us to say that a statement is inferable from another statement, but not yet that it is inferable from a *sequence* of statements. To achieve this, we would need either a way to amalgamate a sequence of statements into a single statement, or, alternatively, a deductor which would be applicable recursively.

The former way amounts to a function a (for *amalgamator*) such that

$X,A,B,Y \vdash C$ iff $X,a(A,B),Y \vdash C,$

or split into the two halves

AMLG: $X,A,B,Y \vdash C \;/\; X,a(A,B),Y \vdash C$

DEAMLG: $X,a(A,B),Y \vdash C$ / $X,A,B,Y \vdash C$.

The latter way amounts to strengthening the condition (*) to

(**) $X,A \vdash B$ iff $X \vdash d(A,B)$,

i.e.

DED*: $X,A \vdash B$ / $X \vdash d(A,B)$
CODED*: $X \vdash d(A,B)$ / $X,A \vdash B$.

Hence from the viewpoint of explicitating the relation $\vdash$ we may think of two 'native operators', *d* and *a*.

However, according to Brandom, the relation of inference rests on a more primitive concept, namely that of incompatibility: to say that *A* is correctly inferable from *X* is to say that whatever is incompatible with *A* is incompatible with *X*. Hence if we denote the fact that the sequence *X* is incompatible as

$\perp X$,

we can write

$X \vdash A$ iff for every *Y* and *Z* it is the case that if $\perp Y,A,Z$ then $\perp Y,X,Z$.

From the other side, the best approximation of incompatibility achievable once we work with inference only is in terms of the "ex falso quodlibet" rule, hence

$\perp X$ iff for every *A*, $X \vdash A$.

Anyway, what we might also need is to explicitate the relation of incompatibility. Let us consider a unary function *c* (for *complementator*) such that

$\perp A$ iff $\vdash c(A)$.

However, this again allows us merely to say that a statement is incompatible, whereas we would want to be able to say, more generally, that a *sequence* of statements is. This could be done if we were able not only to express that a statement is incompatible, $\perp A$, but also that a statement is incompatible *with a sequence X*, i.e. $\perp X,A$. So what we may want is to generalize the stipulation that *A* is incompatible iff *c*(*A*) is inferable (from nothing) to the stipulation that *A* is incompatible *with X* iff *c*(*A*) is inferable *from X*. Hence we have

$\perp X,A$ iff $X \vdash c(A)$,

i.e.

COMP: $\perp X,A$ / $X \vdash c(A)$,
DECOMP: $X \vdash c(A)$ / $\perp X,A$.

If we emulate incompatibility in terms of inference, we have

COMP*: $X,A \vdash B$ for every B / $X \vdash c(A)$,
DECOMP*: $X \vdash c(A)$ / $X,A \vdash B$ for every B.

Hence it seems that we can think of at least three kinds of 'inferentially native' operators: deductors, amalgamators and complementators. Our goal now is to see to what extent the usual operators of the common logical calculi can be seen as their embodiments.

Theories and provability

Logic reconstructs what our knowledge tends to result into as *theories*. A theory T (such as the Peano arithmetic, PA) can be seen simply as a set of sentences of a language. But it seems that if a theory is to be producible, learnable and manageable by us, finite human beings, then it must somehow be 'given by finite means'. The standard way is to see it as generated from a finite set A of axioms. (Thus, PA is seen as based on the axioms of first-order logic and the well-known extralogic axioms.) Hence we assume

$T = Cn(A)$,

where Cn is a way of getting T from A. What, more concretely, is Cn? It is a function mapping sets of statements on the sets of their consequences (intensively studied especially by Tarski, 1930 and elsewhere). It is usually seen as induced by a relation between finite sequences (or sets) of statements and statements. Hence, to be more precise we should write

$T = Cn_I(A)$,

where I is a relation between finite sequences of statements and statements and

$Cn_I(A) = \{s \mid \langle X,s \rangle \in I$ for some finite sequence X of statements from $A\}$.

Now the trouble with I, if construed in this way, is that it suffers from the same shortage as the original A: namely it is an *infinite* set (of ordered pairs). Hence we may try to repeat the same move as before, namely to construe it as generated from a finite collection of 'axioms'.

An ordered pair consisting of a finite set of statements and a statement will be called an *inference*; and an *inference relation* will be a set of inferences. Thus, our I is an inference relation. Now we assume that

$I = Cn_{I'}(R)$,

where R is a finite set of 'inferential rules' (for PA, for example, it consists of *modus ponens* and *generalization*). However, at this point we face the question of the nature of our I' (and we may begin to worry whether we have not started an infinite regress).

The standard solution would run as follows: I is the relation which holds between a sequence of statements X and a statement S just in case we can *prove* S from X using the inferential rules from R. This is to say that I holds between X and A iff there is a finite sequence of statements ending with A and such that each of its elements is either an element of X or there is a rule from R which allows us to derive the statement from statements occurring earlier in the sequence. And hence $Cn_{I'}$ is the functor which takes us from a given R to the relation of 'provability by means of R'.

Can we make the nature of I' explicit? Yes, indeed. It is easy to see that a relation $\vdash$ of provability by means of a given collection of rules satisfies the Gentzenian structural rules:

(REF) $A \vdash A$
(EXT) $X,Y \vdash B$ / $X,A,Y \vdash B$
(CON) $X,A,A,Y \vdash B$ / $X,A,Y \vdash B$
(PERM) $X,A,B,Y \vdash C$ / $X,B,A,Y \vdash C$
(CUT) $X \vdash A$; $Y,A,Z \vdash B$ / $Y,X,Z \vdash B$

More importantly, it is precisely the closure of R w.r.t. these rules that amounts to the relation of 'provability by means of R'[6]. For let A be provable from X by means of the rules of R. Then there is a sequence $A_1,...,A_n$ of statements such that $A_n = A$ and every A_i is either an element of X or is inferable by a rule of R from statements which are among $A_1,...,A_{i-1}$. If n=1, then there are two possibilities: either $A \in X$ and then $X \vdash A$ in force of REF and EXT; or A is a consequent of a rule of R with a void antecedent, and then $\vdash A$ and hence $X \vdash A$ in force of EXT. If n>1 and A_n is inferable from some $A_{i_1},...,A_{i_m}$ (where $i_1<n,..., i_m<n$) by a rule from R, then $A_{i_1},...,A_{i_m} \vdash A$, where $X \vdash A_{i_j}$ for j=1,...,m. Then $X,...,X \vdash A$ in force of CUT, and hence $X \vdash A$ in force of PERM and CON.

Inferential structures

To be able to address the explicitation of inference more rigorously, we need to establish a more rigorous framework. Hence the following definitions:

Let S be a set ('of statements'). An *inference over S* is an ordered pair whose first constituent is a finite sequence of elements of S and whose second is an element of S. A *metainference over S* is an ordered pair whose first constituent is a finite sequence of inferences over S and whose second is an inference over S. If P is a set ('of parameters'), then a *(P-)inferential pattern over S* is an inference over S in which some elements of S are replaced by elements of P. An *instance of an inferential pattern over S* is any inference over S

---

[6] Note that if we start investigating substructural logics, thereby denying that I' is bound to be of this kind, we *are* on the verge of the infinite regress hinted at above. If I' be arbitrary, then we should at least require that it be 'finitely based'. i.e. that $I' = Cn_{I''}(R')$ for some I'' and R'. The natural thing would be to take I'' as constituted by the Gentzenian rules, but if we rejected this for I', it seems we might as well reject it for I'' ...

which can be obtained from the pattern by a systematic replacement of the elements of *P* by elements of *S*. A (*P*-)*metainferential pattern over S* is a metainference over *S* with some elements of *S* in its constituents replaced by those of *P*. An *instance of a metainferential pattern over S* is any metainference over *S* which can be obtained from the pattern by a systematic replacement of the elements of *P* by elements of *S*.

An *inferential structure* is the ordered triple $F = <S,I,M>$, where *S* is a set, *I* is a finite set of inferential patterns over *S*, and *M* is a finite set of metainferential patterns over *S*. The notion of *inference valid in F* is defined in the following obvious recursive way:

(i) an instance of an inferential pattern from *I* is valid;
(ii) if $<<I_1,..., I_n>,I>$ is an instance of a metainferential pattern from *M* and all of $I_1,..., I_n$ are valid, then so is *I*;
(iii) nothing else is an inference valid in *F*.

If $<X,A>$ is an inference valid in *F*, then we shall write $X \vdash_F A$. Hence in terms of the notation of the previous section,

$$\vdash_F \equiv_{\text{Def.}} Cn_M(I).$$

We will leave out the subscript accompanying $\vdash$ where no confusion would be likely to arise. We will say that *A* is a *theorem of F* if $<<>,A>$, where $<>$ is the empty sequence, is an inference valid in *F*.

We will say that an inferential structure is *standard* if *I* contains the pattern

(REF) $A \vdash A$

and *M* consists of the metapatterns

(EXT) $X,Y \vdash A \,/\, X,B,Y \vdash A$
(CON) $X,A,A,Y \vdash B \,/\, X,A,Y \vdash B$
(PERM) $X,A,B,Y \vdash C \,/\, X,B,A,Y \vdash C$
(CUT) $X,A,Y \vdash B;\ Z \vdash A \,/\, X,Z,Y \vdash B$

We will say that it is *substandard* if *M* consists of a subset of these patterns.

As we saw in the previous section, standard structures are important in that they explicate the intuitive notion of 'provability by means of a collection of rules'. Now an addition of an operator to a standard inferential structure is to consist in the addition to the structure of some inferential patterns characteristic of the operator. Hence our next task is to try to turn the metainferential patterns we used to characterize our native operators into inferential ones.

Deductor

Let us start with the deductor. It is easily seen that CODED* is equivalent to the *modus ponens* inferential pattern

MP: $d(A,B),A \vdash B$.

The proof is as follows (we underline the structural rules employed within the proofs to emphasize on which of them the individual results depend):

CODED* ⇒ MP

1. $d(A,B) \vdash d(A,B)$	<u>REF</u>
2. $d(A,B),A \vdash B$	from 1 by CODED*

MP ⇒ CODED*

1. $X \vdash d(A,B)$	assumption
2. $d(A,B),A \vdash B$	MP
3. $X,A \vdash B$	from 1 and 2 by <u>CUT</u>

However, things are not that easy with DED*: there does not appear to be an inferential pattern which would be equivalent to it. But there is an indirect way of securing its validity by establishing inferential patterns: The point is that a way of making DED* valid is to stipulate its validity for the instances of *I* and to secure that the validity is preserved by all the instances of *M*. If X is $A_1,...,A_n$, then let us write $d(X,B)$ as a shorthand for $d(A_1, d(A_2, ..., d(A_n,B)))$; if X is the concatenation of the sequences Y and Z, then $d(Y,Z,B)$ will mean $d(X,B)$; and let us interpret $d(X,B)$ as B if X is the empty sequence. Then what is necessary and sufficient for REF, EXT, CON, PERM and CUT to preserve the validity of DED* is obviously the following:

(1) $\vdash d(A,A)$
(2) $\vdash d(X,Y,B)$ / $\vdash d(X,A,Y,B)$
(3) $\vdash d(X,A,A,Y,B)$ / $\vdash d(X,A,Y,B)$
(4) $\vdash d(X,A,B,Y,C)$ / $\vdash d(X,B,A,Y,C)$
(5) $\vdash d(X,A)$; $\vdash d(Y,A,Z,B)$ / $\vdash d(Y,X,Z,B)$

Given (4), we can reduce (2), (3) and (5) to the simpler (2'), (3') and (5'):

(2') if $\vdash B$, then $\vdash d(A,B)$
(3') if $\vdash d(A,A,B)$, then $\vdash d(A,B)$
(5') if $\vdash d(X,A)$ and $\vdash d(A,B)$, then $\vdash d(X,B)$

However, (2'), (3'), (5') as well as (4) are still *meta*patterns; so we are as yet no better than with DED* itself. Nevertheless in contrast to DED*, the validity of these metapatterns can be obviously established by stipulating the patterns

(2'') $B \vdash d(A,B)$
(3'') $d(A,A,B) \vdash d(A,B)$
(4') $d(X,A,B,Y,C) \vdash d(X,B,A,Y,C)$
(5'') $d(X,A), d(A,B) \vdash d(X,B)$

and hence by

(2*) $\vdash d(B,d(A,B))$
(3*) $\vdash d(d(A,A,B),d(A,B))$
(4*) $\vdash d(d(X,A,B,Y,C),d(X,B,A,Y,C))$
(5*) $\vdash d(d(X,A),d(d(A,B),d(X,B)))$

Now it can easily be shown that (4*) can be reduced to (4**), whereas (5*) to (5**):

(4**) $\vdash d(d(A,B,C),d(B,A,C))$
(5**) $\vdash d(d(C,A),d(d(A,B),d(C,B)))$

and that (1) and (4**) then follow from (2*), (3*) and (5**).

In this way we reach the axiomatization of what Hilbert and Bernays (1939, Supplement III) called *positive logic* and which constitutes the implicative part of the intuitionist propositional calculus. It is this system which thus appears to constitute the 'native standard logic of implication'. (The purely implicative axioms of the classical propositional calculus are the same, but as the addition of the classical axiom(s) for negation are not conservative over them, the class of purely implicative theorems of the classical propositional calculus is more inclusive.)

Now assume that we are interested not in standard, but in some substandard kinds of systems. Suppose we suspend one of the structural rules, say CON. As the result, we will have only

$\vdash d(A,A)$
$\vdash d(B,d(A,B))$
$\vdash d(d(C,A),d(d(A,B),d(C,B)))$

In this way we reach the implication of what is sometimes called BCK logic[7], and also what constitutes the purely implicative part of Wajsberg's (1931) axiomatization of Łukasiewicz's (1930) three-valued calculus.

Suppose we leave only REF and CUT and suspend all the other structural rules. Then MP still replaces CODED*; and we only have to strike out those of (1)-(5) which correspond to the suspended rules. If we suspend all save REF and CUT, we have

$\vdash d(A,A)$
$d(X,A), d(Y,A,Z,B), \vdash d(Y,X,Z,B)$

Amalgamator

Now let us consider the amalgamator -- it is not difficult to see that within a standard inferential structure, AMLG becomes equivalent to ICN whereas DEAMLG to ECN1 plus ECN2:

[7] See, e.g., Došen (1993).

ICN: $A,B \vdash a(A,B)$
ECN1: $a(A,B) \vdash A$
ECN2: $a(A,B) \vdash B$

This can be proved as follows:

DEAMLG $\Rightarrow$ ICN
1. $a(A,B) \vdash a(A,B)$ — REF
2. $A,B \vdash a(A,B)$ — from 1 by DEAMLG

ICN $\Rightarrow$ DEAMLG
1. $X,a(A,B),Y \vdash C$ — assumption
2. $A,B \vdash a(A,B)$ — ICN
3. $X,A,B,Y \vdash C$ — from 2 and 1 by CUT

ECN $\Rightarrow$ AMLG
1. $X,A,B,Y \vdash C$ — assumption
2. $a(A,B) \vdash A$ — ECN1
3. $a(A,B) \vdash B$ — ECN2
4. $X,a(A,B),a(A,B),Y \vdash C$ — from 1, 2 and 3 by CUT
5. $X,a(A,B),Y \vdash C$ — from 4 by CON

AMLG $\Rightarrow$ ECN1
1. $A \vdash A$ — REF
2. $A,B \vdash A$ — from 1 by EXT
3. $a(A,B) \vdash A$ — from 2 by AMLG

(the proof of AMLG $\Rightarrow$ ECN2 is analogous)

Now if we have also the deductor, we have a way of making all inferential patterns, with the exception of *modus ponens*, explicit, i.e. of reducing them to *axioms*. (And, in fact, we *must* do so in order to keep DED* valid.) As for the ones constitutive of the amalgamator, i.e. ICN, ECN1 and ECN2, they are transformed into

$\vdash d(A,d(B,a(A,B)))$
$\vdash d(a(A,B),A)$
$\vdash d(a(A,B),B)$

What about nonstandard structures? If we suspend any structural rule other than PERM, the equivalence between AMLG+DEAMLG and ICN+ECN1+ECN2 will fail. However, as long as REF and CUT hold, the concept of amalgamator will coincide with what has come to be called *fussion* within the theory of substructural logics (Restall, 2000):

IF: $X \vdash A$; $Y \vdash B$ / $X,Y \vdash a(A,B)$
EF: $X \vdash a(A,B)$; $Y,A,B,Z \vdash C$ / $Y,X,Z \vdash C$

The proofs of the equivalence to the definition of amalgamator are as follows:

DEAMLG $\Rightarrow$ IF

1. $a(A,B) \vdash a(A,B)$	REF
2 $A,B \vdash a(A,B)$	from 1 by DEAMLG
3. $X \vdash A$	assumption
4. $Y \vdash B$	assumption
5. $X,Y \vdash a(A,B)$	from 3, 4, and 1 by CUT

AMLG $\Rightarrow$ EF

1. $X \vdash a(A,B)$	assumption
2. $Y,A,B,Z \vdash C$	assumption
3. $Y,a(A,B),Z \vdash C$	from 2 by AMLG
4. $Y,X,Z \vdash C$	from 1 and 3 by CUT

EF $\Rightarrow$ AMLG

1. $X,A,B,Y \vdash C$	assumption
2. $a(A,B) \vdash a(A,B)$	REF
3. $X,a(A,B),Y \vdash C$	from 2 and 1 by EF

IF $\Rightarrow$ DEAMLG

1. $X,a(A,B),Y \vdash C$	assumption
2. $A \vdash A$	REF
3. $B \vdash B$	REF
4. $A,B \vdash a(A,B)$	from 2 and 3 by IF
5. $X,A,B,Y \vdash C$	from 4 and 1 by CUT

Complementator

The situation is more complicated with the complementator, whose definition given by COMP* and DECOMP* cannot be turned into axioms so straightforwardly. However, we can reduce DECOMP* to

(DECOMP**) $c(A), A \vdash B$:

The proof is as follows:

DECOMP** $\Rightarrow$ DECOMP*

1. $X \vdash c(A)$	assumption
2. $X,A \vdash B$	from 1 and DECOMP** by CUT

DECOMP* $\Rightarrow$ DECOMP**

1. $c(A) \vdash c(A)$	REF
2. $c(A), A \vdash B$	from 1 by DECOMP*

Moreover, in the presence of a deductor, we can reduce COMP* to:

(COMP**) $d(A,B), d(A,\mathrm{c}(B)) \vdash c(A)$

This can be proven in the following way:

COMP** ⇒ COMP*

1. $X,A \vdash B$	from the assumption
2. $X \vdash d(A,B)$	from 1 by DED*
3. $X,A \vdash c(B)$	from the assumption
4. $X \vdash d(A,\mathrm{c}(B))$	from 3 by DED*
5. $X,X \vdash \mathrm{c}(A)$	from 1, 3 and COMP** by <u>CUT</u>
4. $X \vdash \mathrm{c}(A)$	from 5 by <u>PERM</u> and <u>CON</u>

COMP* ⇒ COMP**

1. $d(A,B),A \vdash B$	MP
2. $d(A,c(B)),A \vdash c(B)$	MP
3. $d(A,c(B)),A, d(A,B),A \vdash C$	from 1, 2 and DECOMP** by <u>CUT</u>
4. $d(A,B), d(A,c(B)),A \vdash C$	from 3 by <u>PERM</u> and <u>CON</u>
5. $d(A,B), d(A,c(B)) \vdash c(A)$	form 4 by COMP*

Now with the help of the deductor, we recast DECOMP** and COMP** as

$\vdash d(c(A),d(A,B))$
$\vdash d(d(A,B),d(d(A,\mathrm{c}(B)),c(A)))$

Conclusion

Let us summarize the inferences we have amassed in trying to make the (standard) inference relation explicit:

$A,d(A,B)) \vdash B$
$\vdash d(B,d(A,B))$
$\vdash d(d(A,A,B),d(A,B))$
$\vdash d(d(C,A),d(d(A,B),d(C,B)))$
$\vdash d(A,d(B,a(A,B)))$
$\vdash d(a(A,B),A)$
$\vdash d(a(A,B),B)$
$\vdash d(c(A),d(A,B))$
$\vdash d(d(A,B),d(d(A,\mathrm{c}(B)),c(A)))$

It is easy to see that they make up an axiomatization of the intuitionist propositional calculus. This indicates that w.r.t. standard inferential structures, it is intuitionist logic which is *the* logic of inference.

What about classical logic? As is well known, it can be derived from intuitionist logic by the addition of an axiom which, in our notation, reads

$$\vdash d(c(c(A)),A)$$

Is there an inferentialist substantiation of this axiom? I do not see one (which, of course, does not mean that none exists[8]). However, this axiom appears to provide for an 'alignment' of the intuitionist calculus, which leads to a considerable simplification.

What about other logical systems? We have seen that for them, deductors, amalgamators and complementators may be embodied in operators of different kinds than are those of classical ones. (From our viewpoint, if the deduction theorem for a calculus fails, then the upshot is that the implication of the calculus is not its appropriate deductor -- for deduction is a *constitutive* feature of the deductor.)

References

Brandom, R. (1994): *Making It Explicit*, Harvard University Press, Cambridge (Mass.).

Brandom, R. (2000): *Articulating Reasons*, Harvard University Press, Cambridge (Mass.).

Došen, K. (1993): 'A historical introduction to substructural logics', in K. Došen and P. Schröder-Heister, eds.: *Substructural Logics*, Clarendon Press, Oxford.

Haack, S. (1996): *Deviant Logic, Fuzzy Logic*, University of Chicago Press, Chicago.

Hilbert, D. and P. Bernays (1939): *Grundlagen der Mathematik*, Springer, Berlin.

Lorenzen, P. & Schwemmer, O. (1975): *Konstruktive Logik, Ethik und Wissenschaftstheorie*, Bibliographisches Institut, Mannheim.

Łukasiewicz, J. (1930): 'Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls', *Compt. Rend. Soc. d. Sc. Let. Varsovie, Cl. III* 23, No. 51-77.

Milne, P. (2002): 'Harmony, Purity, Simplicity and a "Seemingly Magical Fact"', *Monist* 85, 498-534.

Peregrin, J. (2000): 'Variables in Natural Language: Where Do They Come From?', in *Variable-free Semantics* (eds. ed. M. Böttner & W. Thümmel), Secolo, Osnabrück, 46-65.

Prawitz, D. (1965): *Natural Deduction*, Almqvist & Wiksell, Stockholm.

Quine, W.V.O. (1986): *Philosophy of Logic*, Harvard University Press, Cambridge (Mass.).

Restall, G. (2000): *Introduction to Substructural Logics*, Routledge, London.

Russell, B. (1905): 'On denoting', *Mind* 14, 479-493.

Tarski, A. (1930): 'Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften I', *Monatshefte für Mathematik und Physik* 37; English translation in Tarski (1956), 60-109.

Tarski, A. (1956): *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford.

Wajsberg, M. (1931): 'Aksjomatyzacja trójwartościowego rachunku zdan', *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, cl.iii* 24, No. 126-145.

Department of Logic, Institute of Philosophy
Academy of Sciences of the Czech Republic
Jilská 1, 110 00 Praha 1, Czech Republic
peregrin@ff.cuni.cz
www.cuni.cz/~peregrin

[8] See, example the discussion given by Milne (2002).