Why were two theories (Matrix Mechanics and Wave Mechanics) deemed logically distinct, and yet equivalent, in Quantum Mechanics?

Abstract

A recent rethinking of the early history of Quantum Mechanics deemed the late 1920s agreement on the equivalence of Matrix Mechanics and Wave Mechanics, prompted by Schrödinger's 1926 proof, a myth. Schrödinger supposedly failed to achieve the goal of proving isomorphism of the mathematical structures of the two theories, while only later developments in the early 1930s, especially the work of mathematician John von Neumman (1932) provided sound proof of equivalence. The alleged agreement about the Copenhagen Interpretation, predicated to a large extent on this equivalence, was deemed a myth as well.

If such analysis is correct, it provides considerable evidence that, in its critical moments, the foundations of scientific practice might not live up to the minimal standards of rigor, as such standards are established in the practice of logic, mathematics, and mathematical physics, thereby prompting one to question the rationality of the practice of physics.

In response, I argue that Schrödinger's proof concerned primarily a domain-specific ontological equivalence, rather than the isomorphism. It stemmed initially from the agreement of the eigenvalues of Wave Mechanics and energy-states of Bohr's Model that was discovered and published by Schrödinger in his First and Second Communications of 1926. Schrödinger demonstrated in this proof that the laws of motion arrived at by the method of Matrix Mechanics could be derived successfully from eigenfunctions as well (while he only outlined the reversed derivation of eigenfunctions from Matrix Mechanics, which was necessary for the proof of isomorphism of the two theories). This result was intended to demonstrate the domain-specific ontological equivalence of Matrix Mechanics and Wave Mechanics, with respect to the domain of Bohr's atom. And although the full-fledged mathematico-logical equivalence of the theories did not seem out of the reach of existing theories and methods, Schrödinger never intended to fully explore such a possibility in his proof paper. In a further development of Quantum Mechanics, Bohr's complementarity and Copenhagen Interpretation captured a more substantial convergence of the subsequently revised (in light of the experimental results) Wave and Matrix Mechanics.

I argue that both the equivalence and Copenhagen Interpretation can be deemed myths if one predicates the philosophical and historical analysis on a narrow model of physical theory which disregards its historical context, and focuses exclusively on its formal aspects and the exploration of the logical models supposedly implicit in it.

Introduction

Recently, based on a careful scrutiny of the key arguments pursued by physicists at the beginning of the Quantum Revolution, several philosophers have characterized some of the essential agreements between these physicists as unsubstantiated and unjustified.

To cite perhaps the most notable example, in the late 1920s, the community of quantum physicists agreed on the equivalence of the two competing formal accounts of quantum phenomena, namely, V. Heisenberg's Matrix Mechanics and E. Schrödinger's Wave Mechanics.

Early on, these accounts had been perceived to be substantially different in terms of the mathematical techniques they employed. The Matrix Mechanics was an algebraic approach employing the technique of manipulating matrices. The Wave Mechanics, in contrast, employed differential equations and had a basic partial differential wave equation at its heart.

In addition, the formalisms were initially applied to *two distinct sets of experimental results*. The Matrix Mechanics was deemed successful in treating the appearance of spectral lines and later was found to be successful (to some extent) in experiments with electron scattering. For the Wave Mechanics, its initial applicability to light interference experiments was extended to include the account of the energy values in experiments with hydrogen atoms.

And finally, the ontological commitments arising from the formalisms were at odds with each other. Heisenberg's approach stressed the discrete properties of the observed phenomena, such as the occurrence of spectral lines of different intensities, and attempted to reduce them to essentially corpuscular properties. Schrödinger perceived the field-like continuity of some key micro-physical phenomena (e.g., those related to the double-slit experiments), as they were accounted for by Wave Mechanics, as its main advantage over the old quantum mechanics.

It is not easy to determine to what extent each of these contrasting aspects was responsible for the general understanding that the two theories were irreconcilable. Be that as it may, because of this widespread belief, when the argument for their supposed equivalence was first conceptualized and published by Schrödinger in 1926, it was seen as a major breakthrough – it predicated the development of quantum mechanics.

Recently, however, F.A. Muller (1997a, 1997b) has deemed this equivalence a myth. Muller argues that the initial agreement concerning the equivalence was based on the misconception that both empirical and mathematical equivalence were successfully demonstrated, and that only later developments in the early 1930s, especially the work of mathematician John von Neumann (1932), provided sound proof of the mathematical equivalence, as opposed to the more famous proof provided by Schrödinger or similar attempts by others (Eckart; Dirac; Pauli).

If this re-evaluation tells the true story, it implies that the wide agreement among physicists on the equivalence of two formalisms in the 1920s, on which further developments of the theory were critically predicated, was an unjustified, indeed, an irrational act of faith (or myth, as Muller labels it) on the part of the physics community.

Even the so-called Copenhagen Interpretation of Quantum Mechanics, which has dominated the field since the 1930s, *and which stemmed from the new Quantum Mechanics, largely predicated on the alleged equivalence*, was debunked by the same rethinking of the history of the debate over the foundations of quantum theory (Beller, 1999), and was deemed another myth (Howard, 2004). Thus, presumably, the agreement on the interpretation that argued for the synthesis predicated on both the Wave Mechanics and Matrix Mechanics (initially Niels Bohr's interpretation), and which was thought to have had successfully countered the arguments for the exclusive commitment to continuity based on Wave Mechanics on the one hand, and the discontinuity based on Matrix Mechanics on the other, was forced on the community by the Göttingen group (Beller, 1999) and/or constructed as a myth by subsequent deliberate or semi-deliberate misinterpretations of the history (Howard, 2004). In any case, focusing on the agreement on the mathematico-logical equivalence favors such views. If the mathematico-logical equivalence (i.e., isomorphism) of the two theories was proved in the 1920s, then the physical theory as such did not favor Copenhagen Interpretation over the other two interpretations (Schrödinger's and Heisenberg's), at least not in any straightforward way. But then it becomes rather puzzling how could have such a wide agreement on the Copenhagen Interpretation been justified (if the agreement was reached at all). And if the agreement on the mathematico-logical equivalence was unjustified, as Muller claims, the distinctness of the competing theories could hardly offer a powerful argument for the Copenhagen Interpretation, that won overwhelmingly against the arguments for both wave-mechanical and matrix mechanical approach.

If such analysis is correct, it provides considerable evidence that, in its critical moments, the foundations of scientific practice might not live up to even minimal standards of rigor, as such standards are established in the practice of logic, mathematics, and mathematical physics, thereby prompting one to question the rationality of the practice of physics. Following Muller's line of attack, one might argue that only the efforts of a few able logicians, mathematicians, and mathematical physicists, keen on developing rigorous mathematical models of phenomena, and logical analysis of such models, have a chance of saving science from this charge of (possibly unavoidable) malpractice and messy development predicated on myths and unjustified agreements. Perhaps only in rare moments of lucidity, thanks to these champions of rationality, can we find commendable rational principles at work in science. Furthermore, those pursuing philosophical concerns about the nature of the physical world should draw their insights exclusively from the theory as it is defined at such rare moments.

So did the philosophers finally get it right, or have they missed something crucial in their analysis of scientific practice in the case of Quantum Mechanics? I will argue the latter.

More specifically, I will argue that rationality in physics, and possibly in science more generally, appears elusive in the key moments (and consequently, the rational pursuit essentially is exclusively reserved for the abstraction of logical modeling and the analysis of natural phenomena), only if we premise our analysis of actual scientific practice on narrow models of scientific knowledge. These models, such as that of P. Suppes (1957, 1960), used in the above-outlined analysis of the equivalence case, reduce the conceptual and historical analysis to the aspects of scientific knowledge having to do with the mathematical-logical analysis of the formalisms (such as Matrix Mechanics and Wave Mechanics), which, although indispensable in some aspects of scientific practice, may not be necessary in the establishment of its rational procedures. Such a narrowly-focused analysis is bound to miss some key aspects of the physicists' arguments, embedded as they are in historical and philosophical contexts, contexts which must be unraveled if one is to do justice to the physicists' thinking.

With respect to equivalence, I will argue that the kind of equivalence pursued at a later stage by Von Neumann, and which allegedly represents a moment of lucidity in the overwhelming messiness of the development of Quantum Mechanics, was a very narrowly focused refinement of the previous agreement on the initial concept of equivalence. Although it is true that Schrödinger failed to provide a full-fledged proof of logical equivalence, for the reasons that Muller points out, his paper contained only a preliminary attempt to do so. Judging by its structure, its content, and the historical context in which it appeared, *Schrödinger's proof concerned a domain-specific ontological equivalence, the domain being Bohr's atom.* Bohr's complementarity and Copenhagen Interpretation captured a more substantial convergence of, the subsequently revised (in light of the experimental results), theories. Furthermore, even the full-fledged logical equivalence of the theories did not seem out of the reach of the existing theories and methods, although Schrödinger never intended to fully explore such a possibility in his proof paper.

Section 1: The alleged myth of the equivalence

Muller (1997a, 36) argues, “The Equivalence Myth is that matrix mechanics and wave mechanics were mathematically and empirically equivalent at the time when the equivalence proofs appeared and that Schrödinger (and Eckart) demonstrated their equivalence” (although Schrödinger’s proof was more elaborate and influential than Eckart’s). Thus, the argument goes, Schrödinger (1926a) attempted to prove the mathematical equivalence of Matrix Mechanics and Wave Mechanics by demonstrating their *isomorphism* (the *explananas* of Schrödinger’s overall argument), in order to explain their allegedly established empirical equivalence (*explanandum*) (Muller 1997a, 49). Yet, Muller argues, contrary to the widespread belief at the time (and subsequently), Wave Mechanics and Matrix Mechanics were neither proven mathematically equivalent by Schrödinger, nor were they empirically equivalent.

The incorrect view that Wave Mechanics and Matrix Mechanics were empirically equivalent, Muller argues, stems from an overlooked fact that the two could and should have been treated as empirically distinct in light of the available knowledge. That the electron charge densities were smeared was overlooked, and this “made it conceivable to perform an *experimentum crucis* by charge density measurements” (Muller 1997a, 38) Moreover, the empirical agreement between Wave Mechanics and Matrix Mechanics hinted at by Schrödinger (1926a) on the first page of his paper concerns two cases that are insufficient as evidence of the purported empirical equivalence. The first case was a rather tenuously relevant (to the empirical equivalence thesis) case of coinciding energy values for the hydrogen atom and “the few toy systems” (Muller 1997a, 49), and the second was the quantisation of orbital angular momentum. I will say more about both cases shortly.

If Schrödinger's goal was to prove isomorphism of Wave Mechanics and Matrix Mechanics, the equivalence at stake should be characterized as mathematico-logical equivalence, since labeling it merely 'mathematical equivalence' could refer to the employment of mathematical techniques have no clear logical pretensions or consequences.[1] Muller's idea of the equivalence at stake is much stronger than this. He states that since "the essence of a physical theory lies in the mathematical structures it employs, to describe physical systems, the equivalence proof, including part of Schrödinger's intentions, can legitimately be construed as an attempt to demonstrate the isomorphism between the mathematical structures of matrix mechanics and wave mechanics" (Muller 1997a, 38).

There are three different reasons for the supposed failure of the mathematico-logical (or let us call it simply logical) equivalence. The first reason is that the absence of a state-space in Matrix Mechanics prevented *the direct mutual translation of sentences of Wave Mechanics and Matrix Mechanics*. A related second reason is that the language of Matrix Mechanics could not refer to space, charge-matter densities, or eigenvibrations,[2] "because Matrix Mechanics did not satisfy (in the rigorous model-theoretic sense) any sentence containing terms or predicates referring to these notions" (Muller 1997a, 39). Finally, the most substantial reason was the failure of what Muller labels "Schrödinger-equivalence" – an attempted (Muller believes) proof of a "softer" equivalence than the related one which required a full-fledged logical proof – a failure which was due to the unjustified assumptions regarding the so-called "the problem of the moments" of a function (and this was allegedly resolved in Von Neumann's proof). (Muller 1997b)

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[1] Schrödinger's statements about "mathematical equivalence" are ambiguous. See footnote 10.

[2] See the explanation of eigenvibrations on pp. 16-17.

Section 2: The empirical evidence in early Quantum Mechanics

But was there really a myth of *empirical equivalence*? If so, was it an explanandum of Schrödinger's overall argument?

It is hard to argue for the existence of such a myth without assuming an oversimplified portrayal of the relevance of empirical evidence in the early days of Quantum Mechanics. Schrödinger's (1926a, 45) expression concerning the agreement "with each other" of Wave Mechanics and Matrix Mechanics "with regard to the known facts," employed at the beginning of his proof paper, reflects, at least in Muller's view, the claim of the full-blown empirical agreement.[3] This is a convenient characterization if one aims at constructing the full-fledged empirical equivalence as an explanandum of Schrödinger's (supposed) overall explanation. It is then easy to demonstrate its failure, as Muller does, for example, by pointing to the incapability of Wave Mechanics to account for the line intensities. (Muller 1997a, 54)

But the expression could also reflect the view that although there was some compelling agreement between the two, it was not firmly established. As such, it was not the only, or perhaps the decisive motive for devising the proof. In fact, Schrödinger never committed himself to a strong view of empirical equivalence, and it is actually very unlikely that anybody else believed in the full-blown empirical equivalence at the time.

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[3] It turns out, as I will argue later on, that in order to properly analyze this Schrödinger's statement, the passage should be read in its entirety. "Considering the extraordinary differences between the starting-points and the concepts of Heisenberg's quantum mechanics and of the theory which has been designated "undulatory" or "physical" mechanics, and has lately been described here, it is very strange that these two new theories agree *with one another* with regard to the known facts, where they differ from the old quantum theory. I refer, in particular, to the peculiar "half-integralness" which arises in connection with the oscillator and the rotator." (1926a, 45)

As a matter of fact, some experiments were considered crucial, as they were conceived and performed to decide between the opposing views of micro-physical systems. A set of such experiments concerned the problem of smeared charge densities, the (alleged) lack of which is cited by Muller as evidence of unjustified agreement on the empirical equivalence.

Schrödinger's early wave-mechanical treatment of the atom as a charge cloud (instead of an electron as a particle, orbiting around the nucleus – Bohr's early model) did not at first accurately account for radiation of the atom (while Bohr's model did), given that only certain energy states were observed in spectroscopic experiments. The electric density of the cloud differed from place to place but remained permanent. Thus, in order to account for the radiation in corresponding energy states of the atom, Schrödinger introduced the idea of vibrations of the charge cloud in two or more different modes with different frequencies (i.e., the eigenvibrations accounted for by eigenvalues of the wave equation). As a consequence, the radiation is emitted in the form of *wave-packets* of only certain energies, corresponding to Bohr's frequency conditions. Since Schrödinger assumed that the classical electromagnetic theory accounts for the atom radiation, a number of different radiations could be emitted by the atom as the wave-packet of certain energy expands in space.[4] In the introduction to his proof, Schrödinger (1927a, 45) refers to the case of the oscillator, a special case of this Wave Mechanics treatment of radiation.[5]

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[4] Perhaps surprisingly, this assumption was not at odds with what Bohr believed shortly before Schrödinger developed his own view, as Bohr was advocating the Bohr-Kramers-Slater theory that made a very similar assumption about the way the atom radiated energy.

[5] See footnote 3.

The consequences of Schrödinger's theory, which contradicted Bohr's early view of radiation, were probed experimentally by a series of crucial experiments (Compton and Simon, 1925; Bother and Geiger, 1926).[6] Thus, at the time of writing the proof paper, Schrödinger, as well as others, knew that despite the initial agreement of his theory with Bohr's results with respect to the energy states and radiation, the issue could be addressed further by directly probing "individual radiation processes" that would, in turn, indirectly test the plausibility of the assumption about the vibrations of the atom. Schrödinger was cautioned but was not entirely convinced until 1927 (Mehra and Rechenberg 1982, 138) that the results of these new experiments unequivocally demonstrated the discontinuous nature of matter–energy micro-interactions, as Bohr had claimed. Thus, the issue had been addressed experimentally but remained unresolved at the time of the appearance of the proof.

Nor could the experiments concerning the related issue of quantisation of the orbital angular momentum (referred to as the 'rotator case' by Schrödinger at the beginning of the paper (1927a, 45)) have contributed to the presumed (by Muller) agreement on the empirical equivalence. By introducing the quantised angular momentum of electron, Bohr's model predicted correctly the spectral lines (i.e., Balmer lines) that corresponded to the allowed rotational frequencies of the electron. Heisenberg started with the discrete values of the spectral lines and developed matrices accounting for them. Schrödinger admitted (1926b, 30) that his Wave Mechanics was not capable of accounting for Balmer lines as straightforwardly as Matrix Mechanics did. Yet he presumed this to be a mere technical advantage (Schrödinger 1926a, 57), and the equivalence proof set out to demonstrate this. Schrödinger doubted (and offered his

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[6] See also (Stuewer, 1975) and (Perovic, 2006).

reasons in the proof for this doubt) that this particular success of Heisenberg's approach necessarily reflected a substantial (epistemological or ontological) advantage of Matrix Mechanics, as it was not clear whether the spectral lines indicated the nature of *individual* corpuscular-like interactions of radiation with the matter (i.e., with the spectroscope), or whether they were the consequence of the way wave-packets, not individual corpuscles, interacted with the matter. This issue was also addressed by the above-mentioned scattering experiments and earlier by Ramsauer's (1921) experiments.

The experiments in these two cases, although perceived to be crucial by both the experimentalists (Compton) and those interested in their theoretical implications, did not immediately prompt discarding either approach, if for no other reason than the physicists were simply unsure at the time, how exactly to apply the newly developed formalisms to particular experiments (Heisenberg in Mehra and Rechenberg, 1982, 151). Even a superficial look at the correspondence among them shows that they continued discussing the application of the formalisms and the meaning of such application well into the late 1920s.

Also, it is misleading to say that the coinciding energy values for the hydrogen atom and "a few toy systems," as Muller calls them, were perceived as key evidence of the empirical equivalence of Matrix Mechanics and Wave Mechanics. In fact, these "toy systems" were directly based on Bohr's model of the atom, and Schrödinger's initial major interest concerned the agreement between energy values arrived at by Wave Mechanics (Schrödinger, 1926b; 1926c) and those predicted by Bohr's theory. (Jammer, 1989, 275) This agreement prompted Schrödinger to think about the connection with Heisenberg's Matrix Mechanics. Therefore, the initial agreement between Bohr's model and Schrödinger's Wave Mechanics, that I will discuss shortly, is an essential element of the motivation for the proof.

All these considerations were on going while Schrödinger and others were devising their proofs. More importantly, neither Schrödinger nor anybody else was certain whether or to what extent either of the two formalisms fully accounted for the observed properties of micro-physical processes, *nor whether either was indispensable*. As Jammer (1989, 210) puts it, Matrix Mechanics and Wave Mechanics were "designed to cover the same range of experiences" but it was not firmly established in 1926 that either did so.

Thus, there was considerable agreement with the facts of Wave Mechanics and Matrix Mechanics. This prompted the question about the possibility of a substantial equivalence (both empirical and mathematical). This, in turn, encouraged the construction of new crucial experiments, pushing the limits of the applicability of existing formalisms to them. Given this, the use of the phrase "the two new theories agree with one another with regard to the known facts" was a conditional statement – as both the continuation of the sentence ("where they [Wave Mechanics and Matrix Mechanics] differ from the old quantum theory"), and the subsequent sentence (which tempers the claim by revealing a clearly theoretical consideration behind the mention of the factual connection)[7] indicate.[8] The intention was much more tenuous than the full-fledged empirical evidence demands. And the motivation for the proof (or explanandum) should not be reduced to the meaning of the phrase treated independently from the context of both the proof paper and the experimental and theoretical knowledge of the time. What

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[7] See footnote 14.

[8] See the entire sentence in the footnote 3.

Schrödinger had in mind was not a ‘myth’ of full-fledged empirical equivalence that should be explained. Rather, he wished to show that the factual state of affairs indicated the possibility of domain-specific equivalence, stemming from the agreement of eigenvalues and Bohr’s energy levels, as we will see shortly, which could be revealed by fairly simple manipulations of both methods.

Section 3: Was Schrödinger’s proof, a proof of logical equivalence?

It is tempting to define the goal of Schrödinger’s proof as a single goal. Although there might be a single most important goal, the text reveals the complexity and hierarchy of Schrödinger’s intentions.[9]

In a passage that precedes the actual proof, Schrödinger states that “[i]n what follows the very intimate *inner connection* between Heisenberg’s quantum mechanics and my wave mechanics will be disclosed” (Schrödinger, 1926a, 46). He continues, “From the formal mathematical standpoint, one might well speak of the *identity* of the two theories” and concludes the paragraph by saying, “The train of thought in the proof is as follows.”

An initial reading of this passage might suggest that the author is about to provide a full-blown mathematico-logical proof and that one should judge the effort based on this assumption. Even if Schrödinger’s intentions were different, or at least diverse in terms of the proof’s goals, the mention of the equivalence from “the mathematical standpoint” might urge one to accept such a narrow interpretation. It is possible, however, and as I will argue, quite likely that a rather different key goal is referred to by another phrase used in the passage, namely, the reference to “the intimate connection” between Matrix

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[9] A similar complexity can be revealed in other proofs devised at the time as well.

Mechanics and Wave Mechanics, and that the *subsequent* phrase, “mathematical standpoint,” refers to a distinct issue treated separately in the proof.

We should certainly not rely on this one passage. It does not help, though, that another passage that mentions the goals and the nature of the proof is also quite ambiguous (Schrödinger 1926a, 57-58).[10] Nor does it help that Schrödinger’s attitude with regards to proving the equivalence appears to change significantly over time. In a letter to Wien, dated March 1926, he writes that “both representations are – from the purely mathematical point of view – totally equivalent.” (Mehra and Rechenberg, 1982, 640) Yet in his second Communication, he states that Matrix Mechanics and Wave Mechanics “will supplement each other” (Schrödinger, 1926c, 30)[11] pointing out the advantages of each over the other, rather than noting their similarities. Moreover, as Jammer (1982, 273) points out, the physical and mathematical equivalences that Schrödinger (1926a, 58) mentions, are quite possibly distinct, although we can hardly determine, based on the text of the proof alone, whether or to what extent Schrödinger believed this and what exactly such a view would imply.

A textual analysis of the relevant passages that explicitly state the goals of the proof, although necessary, can go only so far. In order to determine, first, what the real intentions, and possible achievements, of the proof were, and second, how they were

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[10] His adamant statement that “they are completely equivalent from the mathematical point of view, and it can only be a question of the subordinate point of convenience of calculation” (p. 57) certainly overstates the case as the passages that follow this sentence suggest a much more moderate discussion of the formalisms and only a preliminary discussion of logical equivalence. In this context, it is hard to see how the phrase could be interpreted to refer to the isomorphism.

[11] “I am distinctly hopeful that these two advances will not fight against one another, but on the contrary, just because of the extraordinary difference between the starting-points and between methods, that they will supplement one another and that the one will make progress where the other fails.” (Schrödinger, 1926c, 30)

perceived by others, we must judge the text within the historical context in which it was written.

The proof was not motivated by empirical considerations alone. Possibly more important was agreement with Bohr's model of the atom. It prompted articulation of the key step in the proof: the construction of matrices based on the eigenfunctions. As Gibbins says, "Schrödinger in 1926 proved the two theories … equivalent," albeit ontologically, not empirically, "*at least as far as the stationary, or stable-orbit, values for dynamical variables were concerned*" (Gibbins, 1987, 24).

As a matter of fact, both Matrix Mechanics and Wave Mechanics were constructed against the background of Bohr's model and were attempts to improve and, finally, to replace it. While Bohr's model had been changing since its inception, the importance of stationary (permitted) energy states in understanding quantum phenomena remained intact.[12] And as we will see, it became clear to what extent this core of the model remained insightful once Matrix Mechanics and Wave Mechanics were fully developed and the proofs of their equivalence devised.

Bohr's correspondence rules were indispensable guidelines for the construction of Matrix Mechanics in its early phase. As a matter of fact, Matrix Mechanics was envisioned as an improved version of Bohr's method. From Heisenberg's point of view, after he developed Matrix Mechanics, Bohr's method was a useful, albeit rough, first approximation. Matrix Mechanics emerged as a fully independent method once Heisenberg joined efforts with Born and Jordan (Born, Heisenberg, and Jordan, 1926; Jammer, 1989, 221). Commenting on this, Lorentz optimistically notes in 1927, "The fact

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[12] According to Bohr's early model (Bohr, 1913) electrons of the atom can occupy only certain orbital states characterized by appropriate energy levels.

that the coordinates, the potential energy, etc., are now represented by matrices shows that these magnitudes have lost their original meaning, and that a tremendous step has been taken towards increasing abstraction." (Lorentz in D'Abro, 1951, 851)

Pauli's application of Matrix Mechanics to the hydrogen atom illustrated the independence of the method in a similar fashion. (Mehra and Rechenberg, 1982, 656-657) Yet Pauli realized that the fundamental assumptions concerning quantum phenomena, as approached from the point of view of Matrix Mechanics, are in agreement with Bohr's model and that, in this sense, the two might not be as different as they are in terms of methodology.

An insight concerning the relation of Wave Mechanics and Bohr's model, very similar to that of Pauli's concerning Matrix Mechanics, motivated Schrödinger to write the proof paper. In order to understand this, it is critical to take into account that the agreement of Wave Mechanics and Bohr's model (i.e., its core concerning stationary states) precedes the agreement of Wave Mechanics and Matrix Mechanics.

If one replaces the parameter E in Schrödinger's equation: $\Delta\psi + 8\pi^2 m_o/h^2\,[E - E_{pot}\,(x, y, z)]\,\psi = 0,$ with one of the so-called eigenvalues, En, the equation will have a solution (thus becoming one of the eigenfunctions for a given eigenvalue).[13] The solution determines the amplitude of the de Broglie wave (stemming from his compromise between corpuscular mechanics and the theory involving continuity), while the eigenvalue (i.e., the energy) determines the frequency of the wave – that is, the chosen eigenvalue and the corresponding eigenfunction determine the mode of (eigen) vibration.

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[13] Mathematically speaking, if a differential equation (such as Schrödinger's equation) contains an undetermined parameter, and it admits solutions only when particular values (eigenvalues or proper values) are assigned to the parameter, the solutions of the equation are called eigenfunctions.

Now, Schrödinger's solution of the hydrogen atom eigenvalue equation of his first and second communication of 1926 (Schrödinger, 1926b, 1926c) resulted unexpectedly in Bohr's energy levels. Or more precisely, as Bohr, who understood the importance of the insight, stated in 1927, "The proper vibrations of the Schrödinger wave-equation have been found to furnish a representation of electricity, suited to represent the electrostatic properties of the atom in a stationary state" (Bohr, 1985, V.6, 96).[14]

This insight made a great impression on Schrödinger. The newly discovered agreement raised a deeper question concerning an apparently discontinuous nature of the system imposed on an essentially continuous approach of Wave Mechanics by quantum conditions. Others were equally impressed: Wentzel immediately set out to examine this agreement with a new Wave Mechanics approximation method (Jammer, 1989, 275-6).

Wave Mechanics had already emerged as methodologically independent from Bohr's account, and Schrödinger states this explicitly in the first section of the proof: "…we have a continuous field-like process in configuration space, which is governed by a single partial differential equation, derived from a principle of action. This principle and this differential equation replace the equations of motion and the quantum conditions of the older 'classical quantum theory'."(1926a, 45). *However, in light of this newly*

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[14] In (1926b, 8) Schrödinger starts from the wave mechanical assumptions and derives the expression $-E_l = m\,(e^2)^2 / 2K^2\boldsymbol{l}$ where "the well known Bohr energy-levels, corresponding to the Balmer lines, are obtained, if the constant ***K***, introduced in for reasons of dimensions, we give the value $K = h / (2\pi)$, from which comes $-E_l = 2\pi^2 m\,(e^2)^2 / h^2\boldsymbol{l}^2$." In (1926c, 27-28), at the end of the discussion of the case of the rotator, Schrödinger generalizes the expression of an earlier derived wave function (div grad $\psi - (1/u^2)\,\ddot{\psi}$) in the following way: "For it is possible to generalize by replacing div grad ψ by $f\,(qk)$ div $\{[1 / f\,(qk)]$ grad $\psi\}$, where f may be an arbitrary function of the q's, which must depend in some plausible way on E, $V(qk)$, and the coefficients of the line elements." Later on, he comments on the agreement between energy values in Bohr's theory and eigenvalues (discussed on p. 26), emphasizing the advantage of his approach: "… the quantum levels are *at once* defined as the *proper values* of equation (18) [wave equation], which *carries in itself its natural boundary conditions.*" (p. 29) The entire argument for the advantage of the wave-mechanical approach in the second Communication was predicated on this agreement.

obtained agreement, it was not obvious that Wave Mechanics's independence, like that of Matrix Mechanics, was not merely a methodological independence.

Schrödinger was well aware of all this, and it guided the development of the equivalence proof. The central issue of the proof was ontological, rather than the logical. Arguably, it was an attempt, motivated by Wave Mechanics's agreement with Bohr's model, to demonstrate the ontological significance of Wave Mechanics's assumptions (i.e., their non-*ad hoc* nature), and its epistemological significance, doubted (by Heisenberg and others in the Göttingen school, and perhaps Schrödinger himself at first) because of its inapplicability to the spectral line intensities. In other words, given that *Wave Mechanics and Bohr's model agreed with respect to the eigenvalues and stationary energy states, the question was whether Wave Mechanics and Matrix Mechanics agreed with respect to eigenvalues and, thus, to stationary states as well.*

Section 4: The proof of the domain-specific ontological equivalence – as far as eigenvalues/stationary states go

Although the above-stated central goal of Schrödinger's proof may seem disappointingly modest, one should bear in mind that the importance of elucidating the nature of the "intimate connection" between Matrix Mechanics and Wave Mechanics was only superficially apparent at the time, and might have been unsuccessful, as the independence of the two theories could have turned out to be more fundamental.[15] In any

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[15] Already in (1926c) while discussing the rotator case, he notes the agreement between Matrix Mechanics and Wave Mechanics, with respect to the quantum energy levels: "Considering next the proper values, we get … $E_n = (2n+1)/2\ h\nu_0$; $n = 0, 1, 2, 3, \ldots$ Thus as quantum levels appear so-called "half-integral" multiples of the "quantum of energy" peculiar to the oscillator, *i.e.* the *odd* multiples of $h\nu_0/2$. The intervals between

case, Schrödinger's expression of the "intimate connection" between Matrix Mechanics and Wave Mechanics, rather than his reference to the "mathematical equivalence" of the two, indicates the central goal of the proof. That Schrödinger "compared Wave Mechanics and Matrix Mechanics," as M. Bitbol (1996, 68) labels the endeavor, was far more important than his attempted mathematico-logical proof.

The very *structure of the proof* is best explained if the proof were intended to offer further insight into the agreement between Bohr's model and Wave Mechanics, by constructing suitable matrices from eigenfunctions, thereby demonstrating the "intimate connection" between Wave Mechanics and Matrix Mechanics, and thus, indirectly showing the (ontological) significance of their agreement with Bohr's (revised) model.[16]

Schrödinger's paper can be divided into four parts – the introduction, which I have just discussed, and the three parts of the actual proof.[17]

Part 1 of the proof establishes the preliminary connection between Matrix Mechanics and Wave Mechanics. Very early on, Schrödinger emphasizes the limitations placed on his attempt (i.e., quantum conditions). And he explicates the background conditions of the Matrix Mechanics that originate from Bohr's model (i.e., with stationary states and the correspondence rules). He states: "I will first show how to each function of the position and momentum-co-ordinates there may be related a matrix in such a manner, that these matrices, in every case, satisfy the formal calculating rules of Born and Heisenberg (among which I also reckon the so-called 'quantum condition' or 'interchange rule')" (1926a, 46). (Briefly stated, the idea was that the interchange rules –

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the levels, which alone are important for the radiation, are the same in the former theory. It is remarkable that our quantum levels are *exactly* those of Heisenberg's theory." (p. 31)

[16] It was almost certainly understood by others this way, as I will argue shortly.

[17] Parts 1 and 2 do not correspond exactly to the original paragraphs of the paper, whereas Part 3 pretty much corresponds to the last paragraph.

that were, initially at least, a condition that stemmed from Bohr's model – correspond to the analysis of the linear differential operators used in Wave Mechanics.)

Thus, since Born-Heisenberg's matrix relation $\mathbf{pq} - \mathbf{qp} = (h/2\pi i)\mathbf{1}$ corresponds to the Wave Mechanics relation $[(h/2\pi)(\partial/\partial q)]\, q\psi - q\, [(h/2\pi i)(\partial/\partial q)]\, \psi = (h/2\pi i)\, \psi$, a differential operator $F[(h/2\pi i)(\partial/\partial q), q]$ can be associated with the function of momentum and position $F = F(p, q)$. If the phase velocity functions, $u_k = u_k(q)$, in the configuration space of the position q form a complete orthonormal set, then an equation $F_{jk} = \int u^*_j\, [F, u_k]\, dq$, can be derived that determines the elements of the matrix F_{jk}. Thus, as this argument goes, in this very particular sense, any equation of Wave Mechanics can be consistently translated into an equation of Matrix Mechanics.

Part 2 addresses the pressing issue of whether it is possible to establish the "inner connection" between Matrix Mechanics and Wave Mechanics and, hence, the agreement of both with Bohr's model. This part of the text is the key to the proof, as Schrödinger and others saw it, as it provides the *unidirectional argument for the ontological equivalence* as far Bohr's atom goes- by constructing suitable matrices from eigenfunctions.

Relying on the insights of Part 1, Schrödinger replaces the u_i of the $u_k = u_k(q)$ with the eigenfunctions of his wave equation. Thus, he obtains an operator function: $[H, \psi] = E\psi$. The operator's eigenvalues E_k satisfy the equation $[H, \psi_k] = E_k\, \psi_k$. As it turns out, solving this equation is equivalent to diagonalizing the matrix H.[18]

In *the final and decisive step* of Part 2, Schrödinger demonstrates that the matrices constructed in accordance with the elements of matrix F_{jk} given by the above-stated

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[18] In other words, the H turned out to be diagonal with respect to the specified basis (diagonalization of a matrix is a particular orthogonal transformation of the so-called quadratic form, i.e., its rotation).

equation, with the help of some auxiliary theorems, satisfy the Born-Jordan-Heisenberg laws of motion. More precisely, the Heisenberg-Born-Jordan laws of motion (Born, Heisenberg and Jordan, 1926) – the laws initially derived purely from Matrix Mechanics point of view (Jammer, 1989, 221) – are satisfied by (as Schrödinger characterizes the decisive step in the Introduction) "assigning the auxiliary role to a definite orthogonal system, namely to the system of *proper functions* [Schrödinger's italics] of that partial differential equation which forms the basis of my wave mechanics" (1926a, 46).

The first indication that Schrödinger believes that the main goal was already achieved in Part 2 with the construction of matrices from eigenfunctions, is his claim at the beginning of Part 3 that he "might reasonably have used the singular" when speaking of Matrix Mechanics and Wave Mechanics. *Yet if we believe that providing a logical proof of the isomorphism between Matrix Mechanics and Wave Mechanics was the central goal of the proof, Part 3 of the text must be at least as essential as Part 2*, as it tries (and ultimately fails) to establish the reciprocal equivalence required by such a goal.

Unlike the pressing issue dealt with in Part 2, the issue addressed in Part 3 is an 'academic' (in a pejorative sense of the word) one of logical isomorphism requiring the proof of reciprocal equivalence. Schrödinger states that "the equivalence actually exists, and it also exists conversely." But he never fully demonstrates this, nor does he make an outstanding effort to do so. Instead, he provides a vague idea of how one might proceed in proving this sort of logical equivalence.[19] More precisely, as Muller (1997a, 56)

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[19] Muller's view of what he calls "Schrödinger equivalence" is misleading – Schrödinger ended the proof *vis à vis* eigenvalues in Part 2, contrary to what Muller believes. The "moments problem" of a function issue referred to in Part 3, has to do with the preliminary discussion of the full-fledged logical proof and an attempt to argue for epistemological advantage of Wave Mechanics. Thus, Schrödinger promises "[t]he functions can be constructed from the numerically given matrices." (p. 58) If so, "the functions do not form, as it were, an *arbitrary* and *special* "fleshly clothing " for the bare matrix skeleton, provided to

correctly pointed out, Schrödinger does not prove the bijectivity of the Schrödinger-Eckart mapping, necessary for isomorphism.[20]

However, Muller (1997a, 55) misses the bigger picture when he reduces the proof to the narrow model of mathematical equivalence that could be implicit in Part 3 (as well as in his brief discussion of the possibility of Schrödinger's proof being a proof of ontological equivalence in (Muller 1997b)). He leaves out the agreement with Bohr's model of the atom, not realizing that the failure of Part 3 concerning the reciprocal equivalence is not alarming, as it is irrelevant to the central goal. (This is why Muller puzzles over Schrödinger commenting on the subject of bijectivity and reciprocal equivalence in a footnote (Muller, 52).)[21]

In general terms, the constructing of matrices from eigenfunctions in Part 2 becomes meaningful in itself, independently of the reciprocal connection, in light of the final ontological goal of providing a plausible big picture (i.e., Bohr's model). There might be an alternative explanation of the proof's goal[22], but such an explanation would have to take into account that Schrödinger (and others in their proofs) insisted on the derivation in Part 2 as central. Also, the insistence on the derivation in this direction

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pander to the need of the intuitiveness." In order to show this, he invokes the totality of the "moments" of a function.

[20] Moreover, Part 3 seems to have a further, arguably more important, ontological rather than logical, goal of demonstrating that Wave Mechanics was more than merely an *ad hoc* convenient tool, a sort of a shorthand for the superior Matrix Mechanics, as it was, in Schrödinger's view, perceived by Heisenberg and others (Bitbol, 1996, 68).

[21] It should be noted that despite warning his readers about the danger of reading in Von Neumann's terminology into Schrödinger's text that was developed much later, he does not seem to avoid it entirely himself (Muller, 1997a, 57; Muller 1997b, Section IIIa).

[22] E.g., the equivalence could have be seen as a precursor to the relativistic version of Schrödinger's account, especially because, otherwise, his brief discussion of this issue at the very end of last paragraph seems inserted. Even so, this might not be a competing but rather supplementary goal of the proof.

made sense especially because Matrix Mechanics was not suitable to account for single states.[23]

Moreover, the isomorphism of Matrix Mechanics and Wave Mechanics would have made sense as the *explanans* and as the key, and perhaps, the only goal of the proof, only if a full-blown empirical equivalence was established. Otherwise, given that the ontological and methodological status of Wave Mechanics and Matrix Mechanics was tentative, the more pressing issue of the relation between Matrix Mechanics, Wave Mechanics, and Bohr's model could have been resolved with a "softer" derivation (or, rather, the "construction" of matrices from eigenfunctions) – the kind of derivation devised in Part 2.

The key to the proof, then, is its purported demonstration of the formalisms as essential only through their coherence with Bohr's model. It is not clear why Schrödinger might have insisted on a more demanding and what, at the time, seemed a rather academic and esoteric issue, namely, the logical equivalence of possibly dispensable formalisms. Taken in historical context, the more tangible demonstration was more desirable, especially because establishing Bohr's model as an acceptable "big picture" did not require the logical equivalence (i.e., bi-directional derivation to prove isomorphism). Although ambiguous in his statement of the central goal of the proof, then, Schrödinger likely gave priority to the ontological goal.

A debate with Bohr that immediately resulted in doubts and later led to even more devastating doubts concerning the applicability of Wave Mechanics, took place around the time of writing the proof paper. As expressed in a letter to Wien shortly before the debate, Schrödinger's optimism was diminishing (also reflected in his second

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[23] See quote of Bohr's account on p. 17.

Communication – 1926c). This ultimately led him to refocus and to use the soft derivation in Part 2.

Were this not the case, it would be hard to explain the closing passage in the introductory section of the proof, where Schrödinger apparently gets his priorities straight. Although he explicitly emphasizes the construction of matrices from eigenfunctions at the beginning of the paragraph, he only vaguely hints at offering only "a short preliminary sketch" (1926a, 47) of a derivation in the opposite direction as well as the relativistic context of the wave-equation in the last section of the paper. Does this mean that Schrödinger was not keen on the (supposed) main goal of his proof? Or could the passage indicate that he perceived the issue as rather academic?

His characterization of the reversed "construction"[24] would be even more surprising if one believed that Part 3 was the key to the proof. Schrödinger tentatively says, "The following *supplement* [Schrödinger's italics] to the proof of equivalence given above is interesting" (Schrödinger 1926a, 58), before going on to discuss the possibility of the construction of Wave Mechanics from Matrix Mechanics and its implications for the epistemological status of Wave Mechanics.[25]

The assertive tone and the insistence on the exclusiveness and superiority of Wave Mechanics over both old quantum theory and Heisenberg's approach, very explicit in his first Communication (Schrödinger 1926b), and somewhat toned down in the second (Schrödinger 1926c), does not characterize the proof paper. In fact, quite the contrary: the tone of the proof paper is defensive. In Part 3, Schrödinger rather cautiously argues that Wave Mechanics may have the same epistemological significance as Matrix

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[24] "Construction" is a better word choice than "derivation" in this case, given that the latter might indicate the purely logical nature of the proof.

[25] See footnote 18 for the continuation of Schrödinger's discussion.

Mechanics does, and, judging by the above-cited passage, treats this portion of the paper as secondary. That Schrödinger set out to, first and foremost, demonstrate the significance of Wave Mechanics (motivated by its agreement with Bohr's model), a significance which was doubted because of its failure to account for the spectral line intensities, is in keeping with such a tone.

Even if, despite the above-presented indications to the contrary, Schrödinger were at first undecided as to the main goal of the proof, soon after publishing it and his four Communications, he and the quantum physics community embraced it and its limited ontological goal.

Thus, two years after the publication of his seminal work, in his correspondence with others, he continued to discuss the application of the Wave Mechanics and its meaning. Moreover, judging from the following excerpt from Bohr's 1928 letter to Schrödinger, the key issue was still the nature of the agreement of Wave Mechanics and Matrix Mechanics with Bohr's (revised) model. Bohr is still concerned with an (implicit) assumption of Matrix Mechanics regarding stationary states as a limitation on the applicability of Wave Mechanics:

> In the interpretation of experiments by means of the concept of stationary states, we are indeed always dealing with such properties of an atomic system as dependent on phase relations over a large number of consecutive periods. *The definition and applicability of the eigensolutions of the wave equation are of course based on this very circumstance*. (emphasis added; Bohr's letter to Schrödinger (May 23, 1928), in Bohr, 1985*,* V. 6, 49)

It is also important to compare Schrödinger's effort with similar efforts by others. For instance, in his letter to Jordan (12 April 1926), Pauli talks about "a rather deep connection between the Göttingen mechanics and the Einstein-de Broglie radiation field." (Mehra and rechenberg, 1982, 656) He thinks he has found "a quite simple and general way [to] construct matrices satisfying the equations of the Göttingen mechanics," a description of the proof's goal which is analogous to the moderate goal of Schrödinger's proof. It is also striking to what extent the use of Bohr's model was critical in constructing the proofs.[26]

Section 5: The moral of the story

Thus, the 1920s agreement on equivalence appears to be an agreement on a 'myth' only if we leave out the ontological goal of providing a coherent overall model of the atom, and focus solely on the purely formal goal. However, only at a later stage of development was the proof worked out in the terms which Muller's historical and conceptual analysis takes to be central to the 1920s agreement. And although the equivalence of the 1920s was perhaps more provisional than that of the 1930s, it was justified by virtue of its ontological aim.

It is not at all clear, however, that the proof of the equivalence provided by Von Neumann in the 1930s could have settled the issue at the time of the appearance of Schrödinger's proof, given the tentative standing of the formalisms. As Hanson notes,

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[26] See (Mehra and Rechenberg, 1982, 657) on Pauli's proof, (D'Abro, 1951, 874) on Dirac's, (Mehra and Rechenberg, 1982, 150) on Heisenberg's and (Scott, 1967, 57) on the proof of Eckart's.

“Von Neumann’s theory was a splendid achievement. But it was also a precisely defined mathematical model, based on certain arbitrary, but very clearly stated assumptions concerning quantum theory and it physical interpretation” (Hanson, 1963, 124). In particular, the well-known scattering phenomena could not be formulated in a satisfying way within the limitations of his approach at the time.[27]

In the stage of the development of Quantum Mechanics at which the first set of equivalence proofs was provided, the community of quantum physicists was keen on severe experimental testing of the corpuscular and wave mechanical hypothesis concerning the microphysical processes and their implications. Only after the experiments were judged to have provided satisfying results with respect to the available theoretical accounts (Bohr’s model, Matrix Mechanics and Wave Mechanics) did the development of the theory enter the next stage, where an answer to the question of logical equivalence of the two formalisms became significant.

Later commentators understood Schrödinger’s proof in the same spirit as Von Neumann (and Muller is right in claiming this) because of the changing tide in quantum physics. The second stage of the quantum revolution had already begun, and physicists concentrated their efforts on the formal aspects of research, grounded on firmly established experimental results. But we should not confuse the subsequent equivocation with the actual understanding of the goals in the 1920s quantum physics community. It is a mistake to judge these two stages of the development of quantum theory by a criterion that applies only to the second stage.

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[27] Jammer (1989, 335) also points out some of the difficulties with Von Neumann’s approach.

Assuming that the proof was justifiably perceived as a breakthrough, what is the moral of the story? How did this affect the understanding of quantum phenomena at the time? What was the importance of the proof, given its domain-specific ontological goal?

Here, Pauli's attitude regarding the results of his own proof are informative. After presenting the relation between Matrix Mechanics and Wave Mechanics in his letter to Jordan, he concluded that, "from the point of view of Quantum Mechanics the contradistinction between 'point' and 'set of waves' fades away in favor of something more general" (Mehra and Rechenberg, 1982, 657). This is strikingly similar to the complementarity view devised by Bohr in response to the same developments.

Also, although at the time there was still a lack of the agreement on the full-fledged empirical equivalence, the proofs demonstrated that the two approaches added up to a coherent account of the atom – at least as far as the known facts went.

In order to appreciate the relevance of this point, it is important to understand that interpretations, formalisms, and the relevant experiments were closely related aspects of the same endeavor. Disentangling them by introducing rigid distinctions might misguide us in our attempts to reconstruct the relevant views and arguments. Both the development of quantum mechanics and its interpretation were closely dependent on the experimental results: the view of the interpretation(s) arising from the theory, and the theory arising from the experiments, is misleading. It is more accurate to say that all three components informed each other.

Thus, the roots of what has become the Copenhagen Interpretation might be found, to a great extent, in the domain-specific ontological equivalence of Matrix Mechanics and Wave Mechanics, not in the manufacturing of consent among physicists and philosophers. If we leave out Bohr’s model as the background to the proof(s) and concentrate on the equivalence as a purely mathematico-logical issue, the loose agreement represented by the Copenhagen Interpretation seems to have been enforced. In other words, if we take the background into account, the agreement seems to be a reasonable step forward.

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