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The Energy Distribution for a Spherically Symmetric Isolated System in General Relativity

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ABSTRACT

The problems of the total energy and *quasilocal* energy density for an isolated spherically symmetric static system in general relativity (GR) are considered with examples of some exact solutions. The field formulation of GR developed earlier by L. P. Grishchuk, et al (1984), in the framework of which all the dynamical fields, including the gravitational field, are considered in a fixed background spacetime, is used intensively. The exact Schwarzschild and Reissner-Nordstrom solutions are investigated in detail, and the results are compared with those in the recent work by J. D. Brown and J. W. York, Jr. (1993) as well as discussed with respect to the principle of nonlocalization of the gravitational energy in GR. Those examples are illustrative and simple because the background is selected as Minkowski spacetime and in fact the field configurations are studied in the framework of special relativity. It is shown that some problems of the Schwarzschild solution which are difficult to resolve in the standard geometrical framework of GR get resolved in the framework of the field formulation.

I. INTRODUCTION

The problem of interpretation of the energy of the gravitational field in general relativity (GR) has attracted considerable attention from many researchers over many years (see Ref. 1 for a review and references therein). It is recognized that the gravitational field in GR is different from all other physical fields, the reason being that the gravitational field is described by an intrinsically geometrical theory. As a result, for the gravitational field, unlike the other fields, there is no unique expression for the energy density. The action of the gravitational force can be recognized only through its global effects[1]. However, one can obtain a localized energy or localized energy density if some reasonable restrictions or conditions are used (see discussion in Sec. V). In spite of the considerable efforts, questions still remain, relating both to the total energy of a gravitating system and to the localization of energy density (which should more appropriately be called as a *quasilocal* energy density). Works related to these fundamental issues continue to appear (see, for example, Refs. 2 - 4) and we continue this discussion.

The numerous references in Ref. 3 give an extensive and representive set of approaches to the problem. As a rule, for each of these approaches the concerned authors have used examples of the simplest solutions of GR to demonstrate their effectiveness. Indeed, simple but explicit solutions explain the problem more clearly. Moreover, the simplest solutions are very close to real systems, especially the Schwarzschild one which is used as the basis in several investigations (see, for example, the recent works in Refs. 3 - 6).

It is well known that analogies in physical theory can be very useful. Often the comparison of a general problem in a more complete theory with a specific but fully worked out problem in a more simple theory helps to resolve the former. For example, consider the problem of spectral shifts in GR[7] and the problem of motion of massless particles in the Schwarzschild field[8] which were discussed recently. In the first case, in his book Synge[9] has emphasized that the spectal shift in GR "is not a gravitational effect, because the Riemann tensor appears nowhere in our formulae" and has described the gravitational redshifts within the framework of the more familar Doppler effect. This approach was developed in more detail and in a more simplified form by Narlikar[7] where the generalized expression for the gravitational spectral shifts in GR is given in the form of the expression for the Doppler effect generalized from special relativity. In Ref. 8, the well known ideas and techniques of classical mechanics (the optical-mechanical anology) were sucessfully applied to the problem of classical optics in GR.

In this paper we shall use the field formulation of GR which has already been developed in detail[10−12]. On the one hand, the field formulation of GR is equivalent to the ordinary geometrical formulation of GR. On the other, in this framework all the dynamical fields, including the gravitational field, are considered in a fixed background spacetime (curved or flat). The field formulation is a four-covariant formulation and contains the stress-energy tensor for the gravitational field and its matter sources. Thus, the field formulation of GR is very similar to any field theory in a fixed spacetime and analogies with the latter can be used to resolve problems in GR which are difficult to deal with in the geometrical framework.

The existence of the stress-energy tensor (not pseudotensor!) for the gravitational field is one of the advantages of the field formulation. This is the main reason why we use this formulation to consider the energy problem in GR with the help of examples of some exact solutions of GR. We consider these solutions as field configurations in a given spacetime, namely the Minkowski one. Thus, our description is very close to that in special relativity and can be explained and undestood easily. Then, for these configurations we construct the stress-energy tensor and obtain the energy distribution and the total energy with respect to the selected background. Thereafter that we focus attention on the specific aspects of the problems mentioned above.

The paper is organized as follows. In section II, we outline some problems of the Schwarzschild solution in the geometrical formulation of GR. In section III, the necessary notions of the field formulation of GR are given. In section IV, we consider the case of the ordinary spherically symmetric static body with a normal equation of state and two exact solutions, namely Schwarzschild and Reissner-Nordstrom solutions. It is shown that some outstanding problems problems of the Schwarzschild solution indeed get resolved in the field formulation of GR. In section V, these results are discussed and compared with those in the recent works, in particular in Refs. 2 - 4.

II. THE PROBLEMS OF THE SCHWARZSCHILD SOLUTION

We begin with the discussion of the distribution of masses and energy for a static spherically symmetric isolated system in GR as given by Narlikar[13]. There it was noted that the corresponding problem in Newtonian gravity is resolved very simply when Poisson equation for the gravitational potential is considered within the matter and outside the matter. The constants of integration for the potential outside the matter distribution are fixed by the assumption that the potential vanishes at infinity. One can use both the integration over the surface of the spherical source and over the physical volume and the same formulas can be applied to a point mass with mass distribution described by a δ-function.

In GR the situation is not so simple. Let us repeat the discussion in Ref. 13 in some detail. For the spherically symmetric case the most general line element

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.1}$$

when substituted into the Einstein equations

$$G_{\mu\nu}(g_{\alpha\beta}) = \kappa T_{\mu\nu} \tag{2.2}$$

leads to the equations

$$\kappa T_0{}^0 = -\, e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}, \tag{2.3a}$$

$$\kappa T_1{}^1 = -\, e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2}, \tag{2.3b}$$

$$\kappa T_2{}^2 = \kappa T_3{}^3 = -\frac{1}{2}\, e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right). \tag{2.3c}$$

Here, $\kappa \equiv 8\pi G/c^4$. Greek indices take the values 0, 1, 2, 3 and describe the spacetime coordinates, $g_{\alpha\beta}$ is the metric of spacetime in GR, $G_{\mu\nu}$ are the components of the Einstein tensor and $T_{\mu\nu}$ are the components of the energy momentum tensor. Recall that we consider the static case only: $\lambda = \lambda(r)$ and $\nu = \nu(r)$, and here $(') \equiv d/dr$. In empty space these equations have the solution

$$\nu + \lambda = 0, \qquad e^{-\lambda} = 1 - \frac{B}{r}, \qquad B = const. \tag{2.4}$$

In order to determine the constant B it is assumed that at spatial infinity the gravitational effect of the massive body becomes weak and after comparison with the Newtonian theory one gets

$$B = \frac{2mG}{c^2} \equiv r_g. \tag{2.5}$$

In the limit of the weak field GR tends to the Newtonian theory. Therefore, the definition of the constant B is correct. Only one has to remember that B in (2.5) is related to spatial infinity! Recently new interesting results about constants in Schwarzschild-like solutions were obtained by Dadhich[14]. A generalized definition of empty space in GR, for which $R_{\lambda\mu}u^\lambda u^\mu = 0$ instead of $R_{\lambda\mu} = 0$, is proposed. It can be argued that so long as there exists energy distribution outside the empty space region, the generalized definition seems to be more appropriate for its description. The spherical Schwarzshild-like solution in this empty spaces has a nonvanishing constant.

Now, let us repeat, like in Ref 13, the execise by Landau and Lifshitz[15] in order to define the mass m. The Equation (2.3a) is rewritten in the form

$$\frac{d}{dr}\left[r\left(1 - e^{-\lambda}\right)\right] = \kappa r^2 c^2 T_0{}^0 = 8\pi G r^2 \rho, \tag{2.6}$$

where ρ is the mass density for the body. Then, after integration of (2.6) up to the surface of the body with $r = r_s$ and comparison with (2.4) and (2.5) one obtains for the mass of the body

$$m \equiv m(r_s) = 4\pi \int_0^{r_s} r^2 \rho(r) dr. \tag{2.7}$$

This apparently innocent definition of gravitational mass is not as natural as it looks. Note that for the line element (2.1) the physical volume element on a spacelike hypersurface $t = const$ is not $4\pi r^2 dr$ but $4\pi r^2 e^{\lambda/2} dr$. In the one case this fact is explained as a defect of masses[15]. In the other case to make it appear more natural the mass m in (2.7) is rewritten as in Ref. 1:

$$\begin{aligned} m = 4\pi \int_0^{r_*} r^2 e^{\lambda/2} \rho_N dr + 4\pi \int_0^{r_*} r^2 e^{\lambda/2} (\rho - \rho_N) dr \\ + 4\pi \int_0^{r_*} r^2 e^{\lambda/2} \rho (e^{-\lambda/2} - 1) dr \equiv m_N + \frac{U}{c^2} + \frac{\Omega}{c^2}. \end{aligned} \tag{2.8}$$

Here m_N is the nucleonic mass of the body being made of rest mass densities ρ_N of all particles in it. The quantity U is the internal energy accounting for the density difference $\rho - \rho_N$, while Ω is the gravitational potential energy. In the weak field approximation one has

$$\Omega = -4\pi \int_0^{r_*} r^2 \rho \frac{Gm(r)}{r} dr$$

in agreement with the Newtonoian potential energy.

It is well known that the problem of the point mass in the Newtonian gravity is resolved very simply. One has to assume that the mass distribution has the form $m\delta(r)$ where δ-function satisfies the ordinary Poisson equation

$$\nabla^2 \left(\frac{1}{r}\right) = 4\pi\delta(r)$$

where in spherical coordinates

$$\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}.$$

Then, the Newtonian potential will apply to the whole space including the point $r = 0$. If we try to use the Schwarzschild solution in order to describe a point mass in GR a conceptual difficulty arises. If we assume that solution for the gravitational potentials (2.4), (2.5) is fulfilled in the whole spacetime, including the worldline $r = 0$, the matter distribution will have the form:

$$T_0{}^0 = T_1{}^1 = 0, \qquad T_2{}^2 = T_3{}^3 = \frac{mc^2}{2}\,\delta(r). \tag{2.9}$$

It will not be possible to obtain the correct total mass for this distribution if the ordinary volume integration, like in (2.7), is used. Indeed, for (2.9) the mass density is equal to zero. The situation cannot be saved even if one remembers that the time coordinate and the radial coordinate change their sense inside the horizon.

In Ref. 13, in order to save the situation the trace of the field equations (2.2) was used. Under this consideration the solution (2.4), (2.5) can be defined by the trace of the stress-energy tensor $T_{\mu\nu}$:

$$g^{\alpha\beta}T_{\alpha\beta} \equiv T = mc^2\ \delta(r) \tag{2.10}$$

where δ-function describes a point mass in a natural way. However, on closer examination dificulties are noticed[13] even for the trace version of the energy momentum tensor (2.10). In the Schwarzschild coordinates a particle at rest is to have only the timelike component of the stress-energy tensor nonzero. However, from (2.4) and (2.3a,b), it follows ${T_0}^0 = {T_1}^1$, i.e., it cannot be that one of these components is equal to zero and the other is not.

Let us return to the formulas (2.4) - (2.8). For the formula (2.8), Bondi[16] has pointed out the pitfalls in the definition of m_N and has shown how it is not an invariant. Now, recall that the definition of the parameter m in (2.5) was obtained by reference to the distant observer. It is clear that the Newtonian force, which the distant observer experiences, can be caused by any spherically symmetric distribution of energy under the spherical surface surrounding the source. Only the total mass has to be equal to m. In the framework of GR one has to calculate the total mass related to the observer at infinity, i.e., the mass of the source itself and the mass of the gravitational field created by the source. From this point of view the positive-energy theorem[17,18] was proved. The mass m in the formula (2.7) cannot be interpreted as the mass of the body. The formula (2.7) can be interpreted only as the solution of Eq. (2.6) (or (2.3a)) which determines the parameter m as a constant of integration.

In the general case the gravitational energy is not localized[1]. The infinitely removed observer is in fact placed in a flat spacetime. From his point of view the total mass is just localized within the spherical surface surrounding the source. Thus, we note again the important role of the distant observer. (The problem of the localization of the gravitational energy will be discussed below in Sec. V.)

Bearing the above consideration in mind we now wish to expand the description of the energy and the mass distribution for an isolated spherically symmetric static system from the point of view of the infinitely removed observer. His frame of reference at spatial infinity is taken to be the Minkowski spacetime. The field formulation allows to us incorporate this idea naturally. We will therefore consider an isolated system in the entire physical spacetime of GR as a field configuration in an auxiliary Minkowski spacetime.

III. THE FIELD FORMULATION OF GENERAL RELATIVITY

For our goal we use the field formulation of GR developed earlier by Grishchuk, Petrov and Popova[10−12]. The field formulation of GR has the properties which are very similar to those of any field theory in a fixed backround spacetime (curved or flat). All the physical fields, including the gravitational field and its matter sources, are considered against this fixed background. The field formulation is a Lagrangian based formulation and all the expressions and the equations are coordinate independent. The field formulation of GR also has the gauge invariance properties similar to those in ordinary gauge theories of the Yang-Mills type.

The equations for the gravitational field $h^{\mu\nu}$ in the field formulation of GR have the form:

$$G^L_{\mu\nu}(h^{\alpha\beta}) = \kappa t^{tot}_{\mu\nu} \tag{3.1}$$

where

$$G^L_{\mu\nu}(h^{\alpha\beta}) \equiv \frac{1}{2}(h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + \gamma_{\mu\nu}h^{\alpha\beta}{}_{;\alpha\beta} - h^{\alpha}{}_{\mu;\nu\alpha} - h^{\alpha}{}_{\nu;\mu\alpha}),$$

$$t^{tot}_{\mu\nu} \equiv t^g_{\mu\nu} + t^m_{\mu\nu}.$$

Here, $h^{\mu\nu}$ is a symmetric tensor; $\gamma_{\mu\nu}$ is the background metric; $(;\alpha)$ means the covariant derivative with respect to $\gamma_{\mu\nu}$ and $\gamma \equiv \det \gamma_{\mu\nu}$. Equations (3.1) were obtained after varying the action in the field formulation of GR with respect to $h^{\mu\nu}$ along with some algebraic operations. The stress-energy tensors $t^g_{\mu\nu}$ and $t^m_{\mu\nu}$ can be obtained after varying the part of the action for the free gravitational field and the part of the action for the matter interacting with the gravitational field, respectively with respect to $\gamma_{\mu\nu}$. The first of them has the form:

$$\kappa t^g_{\mu\nu} = -\,(KK)_{\mu\nu} + \frac{1}{2}\,\gamma_{\mu\nu}(KK)_{\alpha}^{\ \alpha} + Q^{\sigma}{}_{\mu\nu;\sigma} \tag{3.2}$$

where

$$(KK)_{\mu\nu} \equiv K^{\alpha}{}_{\mu\nu}K^{\beta}{}_{\beta\alpha} - K^{\alpha}{}_{\mu\beta}K^{\beta}{}_{\nu\alpha},$$

$$\begin{aligned}2Q^{\sigma}{}_{\mu\nu} \equiv &-\gamma_{\mu\nu}h^{\alpha\beta}K^{\sigma}{}_{\alpha\beta} + h_{\mu\nu}K^{\alpha}{}_{\alpha}{}^{\sigma} - h_{\mu}{}^{\sigma}K^{\alpha}{}_{\alpha\nu} - h_{\nu}{}^{\sigma}K^{\alpha}{}_{\alpha\mu} + h^{\beta\sigma}(K^{\alpha}{}_{\mu\beta}\gamma_{\alpha\nu} + K^{\alpha}{}_{\nu\beta}\gamma_{\alpha\mu})\\ &+ h_{\mu}{}^{\beta}(K^{\sigma}{}_{\nu\beta} - K^{\alpha}{}_{\beta\rho}\gamma^{\rho\sigma}\gamma_{\alpha\nu}) + h_{\nu}{}^{\beta}(K^{\sigma}{}_{\mu\beta} - K^{\alpha}{}_{\rho\beta}\gamma^{\rho\sigma}\gamma_{\alpha\mu}).\end{aligned}$$

The tensor $K^{\alpha}{}_{\beta\gamma}$ is symmetric with respect to the lower indices and satisfies the equation

$$h^{\alpha\beta}{}_{;\gamma} - (\gamma^{\alpha\beta} + h^{\alpha\beta})K^{\pi}{}_{\pi\gamma} + (\gamma^{\alpha\pi} + h^{\alpha\pi})K^{\beta}{}_{\pi\gamma} + (\gamma^{\pi\beta} + h^{\pi\beta})K^{\alpha}{}_{\pi\gamma} = 0. \tag{3.3}$$

Thus, in the field formulation of GR the total energy distribution is described by the stress-energy tensor (not by pseudotensor!) $t^{tot}_{\mu\nu}$ in the background spacetime. This stres-energy tensor consists of two parts which are also tensors. Note that if Eq. (3.1) is satisfied then the total stress-energy tensor can be obtained with the use of the left-hand side of Eq. (3.1). Thus, the total energy distribution both inside the matter sources and outside them can be described by the gravitational potentials only.

We stress especially that the field formulation of GR and the ordinary geometrical formulation of GR are two different formulations of the same Einstein theory and can be applied to the same physical problems[10,19]. The equivalence between the field and the geometrical formulations of GR can be stated after the simple identification

$$\sqrt{-g}g^{\mu\nu} = \sqrt{-\gamma}(\gamma^{\mu\nu} + h^{\mu\nu}) \tag{3.4}$$

where $g \equiv \det g_{\mu\nu}$. Then, the equations (3.1) change over to the Einstein equations (2.2) exactly, without any approximations. The relation (3.3) relates to the ordinary connection between the Christoffel symbols and the dynamic metric in GR

$$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma}(g_{\mu\nu}). \tag{3.5}$$

Besides, one obtains the relation

$$\Gamma^{\alpha}{}_{\beta\gamma} = K^{\alpha}{}_{\beta\gamma} + C^{\alpha}{}_{\beta\gamma} \tag{3.6}$$

where $C^{\alpha}{}_{\beta\gamma}$ are the Christoffel symbols constructed with the use of the background metric. Finally, the source stress-energy tensor in (3.1) is connected with the matter stress-energy tensor in (2.2) by the relation

$$t^{m}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T_{\alpha\beta}\, g^{\alpha\beta} - \frac{1}{2}\, \gamma_{\mu\nu}\, \gamma^{\alpha\beta}\, (T_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, T_{\pi\rho}\, g^{\pi\rho}). \tag{3.7}$$

In order to transfer from the geometrical formulation to the field one of GR one can take the following steps[11]. Equations (3.1) - (3.3), (3.6), (3.7) can be constructed after using the decompositions (3.4) and (3.6) in Eqs. (2.2) and (3.5). In this case on the right hand side of (3.7) all the components $g_{\mu\nu}$ are the functions of $h^{\mu\nu}$ and $\gamma_{\mu\nu}$. In order to obtain the concrete the gravitational field configuration one has to select the concrete background spacetime (the background metric) and use also the decompositions (3.4) and (3.6). This method will be used below for specific solutions of (2.2).

IV. THE FIELD CONFIGURATION AND THE ENERGY DISTRIBUTION FOR A STATIC SPHERICALLY SYMMETRIC SOLUTION IN GR

In this section we consider the solution (2.1) and construct the field configuration for it in the framework of the field formulation. We select the flat spacetime which coincides with a flat physical spacetime at infinity as a background spacetime for our model. In the spherical coordinates the metric of the background spacetime has the form:

$$\gamma_{00} = 1, \quad \gamma_{11} = -1, \quad \gamma_{22} = -\, r^2, \quad \gamma_{33} = -\, r^2 \sin^2\theta. \tag{4.1}$$

Then, after using the relations (3.4) and (3.6) and after changing λ and ν to

$$\phi \equiv e^{\frac{\nu-\lambda}{2}}, \quad \psi \equiv e^{\frac{\nu+\lambda}{2}} \tag{4.2}$$

it is not difficult to obtain for the solution (2.1) the gravitational field configuration:

$$\begin{aligned} &h^{00} = \phi^{-1} - 1, \quad h^{11} = 1 - \phi, \\ &h^{22} = r^{-2}(1-\psi), \quad h^{33} = r^{-2}\sin^{-2}\theta(1-\psi) \end{aligned} \tag{4.3a}$$

and

$$\begin{aligned} &K^1{}_{11} = \frac{1}{2}\left[\ln\left(\frac{\psi}{\phi}\right)\right]', \quad K^0{}_{10} = \frac{1}{2}\,[\ln(\psi\phi)]', \quad K^1{}_{00} = \frac{1}{2}\,[\ln(\psi\phi)]'\,\phi^2, \\ &K^1{}_{22} = r\,\frac{\psi-\phi}{\psi}, \quad K^1{}_{33} = r\sin^2\theta\,\frac{\psi-\phi}{\psi}. \end{aligned} \tag{4.3b}$$

The other components of $h^{\mu\nu}$ and $K^\alpha{}_{\beta\gamma}$ are equal to zero.

Let us assume that Eq. (3.1) is fulfilled at every point of Minkowski spacetime. Then, after using the left-hand side of Eq. (3.1) in the general form it can be easily seen that the total energy can be calculated over the surface integral at infinity. Indeed, for the background metric (4.1) one has for any static configuration:

$$\begin{aligned} E^{tot} &= \lim_{r\to\infty}\int_{S_t} d^3x\sqrt{-\gamma^{(3)}}\; t^{00}_{tot} \\ &= \frac{1}{2\kappa}\lim_{r\to\infty}\int_{S_t} d^3x\sqrt{-\gamma^{(3)}}\left(h^{00}\gamma^{\alpha\beta} + h^{\alpha\beta}\gamma^{00} - h^{0\beta}\gamma^{0\alpha} - h^{0\beta}\gamma^{0\alpha}\right)_{;\alpha\beta} \\ &= \frac{1}{2\kappa}\lim_{r\to\infty}\int_{S_t} d^3x\sqrt{-\gamma^{(3)}}\left(h^{00}\gamma^{ij} + h^{ij}\gamma^{00}\right)_{|ij} \\ &= \frac{1}{2\kappa}\lim_{r\to\infty}\int_{S_t} d^3x\left[\sqrt{-\gamma^{(3)}}\left(h^{00}\gamma^{ij} + h^{ij}\gamma^{00}\right)_{|i}\right]_{,j} \\ &= \frac{1}{2\kappa}\lim_{r\to\infty}\oint_{\partial S_t} d\sigma_j\sqrt{-\gamma^{(3)}}\left(h^{00}\gamma^{ij} + h^{ij}\gamma^{00}\right)_{|i}. \end{aligned} \tag{4.4}$$

Here, S_t is the spacelike section $t = const$ in Minkowski spacetime with the metric (4.1); Latin indices number the spatial coordinates on S_t; $\gamma^{(3)} \equiv \det\gamma_{ij}$; the vertical line implies covariant derivatives with respect to γ_{ij}; $d\sigma_j$ is the two-dimentional coordinate volume.

Note that according to (4.4) the value of the total energy can be obtained without knowledge about the structure of the source. It can thus be an ordinary body with a normal equation of state or the pure Schwarzschild black hole with the physical singularity. However for us it is more interesting to consider the distribution of energy over space for several sources. After using the left-hand side of Eq. (3.1) we write out for the field (4.3b) the 00-component of the total stress-energy tensor:

$$t^{tot}_{00} = \frac{1}{2\kappa}\{(\nabla^2\phi)\,(-1+\frac{1}{\phi^2}) - \frac{2\phi'^2}{\phi^3} + \frac{2}{r^2}\,(1-\phi) - \frac{2\phi'}{r} - \left[\frac{2}{r}\,(1-\psi)\right]' - \frac{4}{r}\,(1-\psi)\}. \tag{4.5}$$

It will be very useful to evaluate the 00-component of the stress-energy tensor (3.2) for the free gravitational field (4.3) under the condition $\psi = 1$ which applies to 'empty' exterior solution:

$$t^g_{00} = \frac{1}{4\kappa}\left[(\nabla^2\phi)\,\frac{(1-\phi)(1+\phi)(2+\phi)}{\phi^2} - \frac{4\phi'^2}{\phi^3}\right]. \tag{4.6}$$

These expressions give the distribution of the total energy and the free gravitational field energy with respect to an auxiliary Minkowski space.

a) The ordinary isolated body.

We do not consider an explicit interior solution for λ and ν. However, we assume that the body has a normal equation of state. Then the functions λ and ν (consequently, ϕ and ψ in (4.2)) have to be smooth and restricted enough[15], and the energy distribution within the matter sources is given by the general expressions (4.5). Thus, although we do not use the explicit volume integration within matter, we know that this will not meet difficulties. In order to obtain the total energy E^s in the space restricted by the surface of the body with $r = r_s$ we will use the surface integral (4.4) at $r = r_s$ and match the values of $h^{\mu\nu}$ inside the surface with the values of $h^{\mu\nu}$ outside the surface of the body. The field configuration outside the matter is obtained after substitution (2.4), (2.5) into (4.2) and (4.3), i.e., for $\phi = 1 - r_g/r$ and $\psi = 1$, and acquires the form:

$$h^{00} = \frac{r_g}{r}\,\frac{1}{1-\frac{r_g}{r}}, \qquad h^{11} = \frac{r_g}{r} \tag{4.7a}$$

and

$$K^1{}_{11} = -\,\frac{1}{2}\,\frac{r_g}{r^2}\,\frac{1}{1-\frac{r_g}{r}}, \quad K^0{}_{10} = \frac{1}{2}\,\frac{r_g}{r^2}\,\frac{1}{1-\frac{r_g}{r}}, \quad K^1{}_{00} = \frac{1}{2}\,\frac{r_g}{r^2}\left(1-\frac{r_g}{r}\right),$$
$$K^1{}_{22} = r_g, \quad K^1{}_{33} = r_g\,\sin^2\theta. \tag{4.7b}$$

Thus, for the total energy of the interior of the body

$$E^s = mc^2\left[\frac{1}{2}\,\frac{r_g\,(2r_s - r_g)}{(r_s - r_g)^2} + 1\right]. \tag{4.8}$$

After substitution (4.7), namely $\phi = 1 - r_g/r$ and $\psi = 1$, into (4.5) and (4.6) one obtains the explicit expression for the density of energy outside the body:

$$t^{tot}_{00} = t^g_{00} = -\,\frac{r_g^2}{\kappa r^4}\,\frac{1}{\left(1-\frac{r_g}{r}\right)^3}. \tag{4.9}$$

As is seen, this density is negative and after volume integration from $r = r_s$ to $r = \infty$ one obtains

$$E^{out} = -\,\frac{mc^2}{2}\,\frac{r_g\,(2r_s - r_g)}{(r_s - r_g)^2}. \tag{4.10a}$$

If $R > r_s$, then the energy of the gravitational field outside the surface $r = R$ is

$$E^{outR} = -\frac{mc^2}{2}\frac{r_g(2R - r_g)}{(R - r_g)^2}. \qquad (4.10b)$$

From (4.8) and (4.10a) it is seen that the total energy of the system is mc^2. It is natural that the total energy can be also calculated with the use of surface integration. After substitution (4.7a) into (4.4) it is easily seen that

$$E^{tot} = mc^2 \qquad (4.11)$$

again. This is a correct result and it agrees with the conclusion of many others (see Refs. 1 and 3 and references therein).

Let us discuss the result. The energy of the gravitational field outside the body is negative. This coincides with a naive understanding of the nature of the gravitational field. For example, in order to remove each star in a binary system from another one has to inject into the system an additional positive energy. Therefore, the energy of the gravitational binding has to be negative[1]. Next, it is assumed that the total energy of the closed universe is equal to zero[1,15]. This means that the positive energy of the matter sources is compensated exactly by the negative energy of the gravitational field. Note also that the potential energy of the gravitational field in (2.8) is supposed as negative. From (4.9) it is seen also that the density is stronger near the body which is also natural. These conclusions about the negativeness of the gravitational energy coincide with those of others[1,3,4].

One question arises. We note that the total energy within the body (4.8) is more than mc^2. This result appears strange in the light of the mass-defect property. The reason is that in (4.8) we cosider the energy of the body without the energy of the external gravitational field. In fact the mass-defect exists and may be interpreted in the following way. Let the body consist of two closed parts which are connected by gravitational forces. In order to remove each of the parts from the other to infinity and each of them from the observer to infinity we have to add positive energy. This means that the finished state has more energy than the initial one. That is the sum of the total energies of these parts (*together with the energies of their own gravitational fields*) is more than mc^2. So, under the concept of the distant observer the mass-defect exists.

Thus, instead of the formulas (2.6) - (2.8) we suggest the use of formulas of the present subsection in the framework of the field formulation of GR.

b) The Schwarzschild solution.

We also consider in the framework of the field formulation of GR the energy distribution for a black hole. For this, the field configuration (4.7) has to be considered over the whole Minkowski spacetime. Then the expression (4.5) gives the energy density for any point of the Minkowski spacetime in the form:

$$t_{00}^{tot} = \frac{mc^2}{2}\delta(r)\left[1 - \frac{1}{\left(1-\frac{r_g}{r}\right)^2}\right] - \frac{{r_g}^2}{\kappa r^4}\frac{1}{\left(1-\frac{r_g}{r}\right)^3}. \tag{4.12}$$

It is natural that the total energy E^{tot} obtained with the use of (4.12) over the volume integral is also equal to mc^2, like (4.11). If one calculates the energy outside the horizon only one will obtain $-\infty$; the energy inside the horizon is equal to $+\infty$ (see Fig. 1). However the infinite contributions near horizon are compensated. From (4.12) one can see that the contribution to E^{tot} from the δ-function is equal to $mc^2/2$, while the contribution from the free gravitational field outside $r = 0$ is also equal to $mc^2/2$.

It is interesting to examine the contribution from the matter source and from the free gravitational field in the different parts of (4.12). After using (2.1), (2.4), (2.5) and (2.9) in (3.7) one obtains

$$t_{00}^{m} = -\frac{mc^2}{4}\,\delta(r)\left[1 - \frac{r_g}{r} - \frac{1}{1-\frac{r_g}{r}}\right]. \tag{4.13}$$

From (4.6) over the whole Minkowski spacetime one has for the Schwarzschild solution:

$$t_{00}^{g} = -\frac{mc^2}{4}\,\delta(r)\frac{r_g}{r}\left[1 + \frac{3}{1-\frac{r_g}{r}} + \frac{2}{\left(1-\frac{r_g}{r}\right)^2}\right] - \frac{{r_g}^2}{\kappa r^4}\frac{1}{\left(1-\frac{r_g}{r}\right)^3}. \tag{4.14}$$

It is not surprising that the sum of (4.13) and (4.14) gives (4.12). One can see that separately the δ-functions in Eq. (4.13) and Eq. (4.14) make $(-\infty)$-contribution and $(+\infty)$-contribution to the total energy. However, it is in the spirit of GR that $t_{\mu\nu}^m$ can not be considered separately from $t_{\mu\nu}^g$. Thus, the infinite contributions cancel each other.

Let us also write the other components $t_{\mu\nu}^m$ and $t_{\mu\nu}^{tot}$ (from them it is easy to obtain $t_{\mu\nu}^g$). In addition to (4.13) we obtain

$$\begin{aligned} t_{11}^{m} &= \gamma_{11}\frac{mc^2}{4}\,\delta(r)\left(1 - \frac{r_g}{r} - \frac{1}{1-\frac{r_g}{r}}\right),\\ t_{22}^{m} &= \gamma_{22}\frac{mc^2}{4}\,\delta(r)\left(2 - \frac{r_g}{r} + \frac{1}{1-\frac{r_g}{r}}\right),\\ t_{33}^{m} &= \gamma_{33}\frac{mc^2}{4}\,\delta(r)\left(2 - \frac{r_g}{r} + \frac{1}{1-\frac{r_g}{r}}\right). \end{aligned} \tag{4.15}$$

and likewise, besides (4.14), we obtain

$$\begin{aligned} t_{22}^{tot} &= \gamma_{22}\left(\frac{mc^2}{2}\,\delta(r) - \frac{2r_g}{\kappa r^3}\right),\\ t_{33}^{tot} &= \gamma_{33}\left(\frac{mc^2}{2}\,\delta(r) - \frac{2r_g}{\kappa r^3}\right). \end{aligned}$$

All the other components of $t^m_{\mu\nu}$ and $t^{tot}_{\mu\nu}$ are equal to zero.

In the above we have assumed that Eq. (3.1) is satisfied at all points of Minkowski spacetime (including $r = 0$), an assumption that now finds a confirmation. Indeed, we have the solution (4.7) at all of the points of the Minkowski spacetime (including $r = 0$) if the matter tensor (4.13), (4.15) is used. As is seen, the situation is more comprehensive than for the point mass in the Newtonian gravity where the δ-function enters the matter energy density only. Nevertheless, we can use the volume integration over the whole Minkowski spacetime and obtain the energy values in the natural way after using the expressions (4.12) - (4.14). Thus, the problem of the point mass is resoved with the use of the field approach, unlike the formulas (2.7), (2.9) and (2.10) in the ordinary approach. It is naturally expected that in the framework of GR one has to take into account the stress-energy tensor of the gravitational field which also contains the δ-function at $r = 0$ (see (4.14)).

In this regard, we note also the work in Ref. 20. These authors noted that for the Schwarzschild vacuum solution the stress-energy tensor is concentrated on the region $r = 0$ usually excluded from spacetime, resulting in the physically unsatisfactory situation that curvature is generated by zero stress-energy tensor. Using distributional technique they made dimensional regularization of the Schwarzschild metric and curvature. It turns out to be a well defined tensor distribution with the $\delta(r)$-function. Thus, indeed the Eistein equations can be treated at $r = 0$ and the curvature of this geometry obtains natural physical interpretation.

Let us discuss Eq. (4.12) (see also (4.14)) with the help of the Fig. 1. In fact we extend the concept of Minkowski spacetime from spatial infinity up to the horizon $r = r_g$, and even under the horizon including the worldline $r = 0$. However, in reality the distant observer cannot see the space within the horizon. Therefore, it is more useful to consider the situation outside the horizon. So, if we approach the horizon from outside, then we have the infinite negative density for the graviational energy. Naively this picture can be explained as follows. From the point of view of the distant observer (and absolutely) if the test particle moves closer from outside to the horizon then it finds it more difficult to escape from the black hole. Indeed, the negative density of the gravitational energy (and. consequently, the attraction) is stronger near the horizon. At the horizon (the density $t^g_{00} = -\infty$) it is impossible to escape the balck hole.

c) The Reissner-Nordstrom solution

In our framework it would be interesting also to consider the charged black hole. Indeed, in this case we have the electrovac solution. Therefore, we have to consider the gravitational field together with the electromagnetic field at all the points of the Minkowski spacetime, unlike the subsection 4b) where we considered the pure and free gravitational field everywhere exluding the worldline $r = 0$.

The metric of the Reissner-Nordstrom solution corresponds to the metric (2.1) where (see (4.2)) one has to choose

$$\phi = 1 - \frac{r_g}{r} + \frac{Q^2}{r^2}, \qquad \psi = 1, \tag{4.16}$$

Q being the charge of the black hole. Then, after substitution of (4.16) into (4.3a) one obtains the field configuration

$$h^{00} = \frac{\frac{r_g}{r} - \frac{Q^2}{r^2}}{1 - \frac{r_g}{r} + \frac{Q^2}{r^2}}, \qquad h^{11} = \frac{r_g}{r} - \frac{Q^2}{r^2}. \tag{4.17}$$

Because the configuration (4.17) is static, one can use the expression (4.4) in order to calculate the total energy of the system. We obtain the result (4.11) again. It is not surprising because the additional terms in (4.17) with respect to (4.7a) have the behavior r^{-2} at spatial infinity. Such terms can not give a contribution to the total energy[19,21]. Thus, the parameter m in (4.17) determinies the total energy of an isolated system, as in (4.7). The result (4.11) covers all the possibilities of the charged solution, namely $Q < r_g/2$, $Q = r_g/2$ and $Q > r_g/2$.

As earlier, it is more interesting to consider the distribution of the energy. Let us substitute (4.16) into (4.5) and (4.6). Then one obtains

$$\begin{aligned} t^{tot}_{00} &= \frac{mc^2}{2}\delta(r)\frac{A^4 - r^4}{A^4} + \frac{1}{\kappa A^6}\left[Q^2A^2 - \left(r^2 - A^2 - Q^2\right)^2\right] \\ &= \frac{mc^2}{2}\delta(r)\frac{A^4 - r^4}{A^4} + \frac{1}{\kappa A^6}\left[r^2\left(Q^2 - {r_g}^2\right) + 3Q^2 r_g r - 3Q^4\right] \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} t^{g}_{00} =& \frac{mc^2}{4}\delta(r)\frac{\left(A^4 - r^4\right)\left(2r^2 + A^2\right)}{r^2A^4} \\ &+ \frac{1}{2\kappa r^6 A^6}\left[Q^2A^2\left(r^2 - A^2\right)\left(r^2 + A^2\right)\left(2r^2 + A^2\right) - 2r^6\left(r^2 - A^2 - Q^2\right)^2\right] \end{aligned} \tag{4.19}$$

where

$$A^2 \equiv r^2 - r_g r + Q^2. \tag{4.20}$$

For simplisity we obtain the matter stress-energy tensor after substruction of (4.19) from (4.18):

$$t^{m}_{00} = -\frac{mc^2}{4}\delta(r)\frac{A^4 - r^4}{r^2A^2} + \frac{Q^2}{2\kappa r^6 A^2}\left(-r^4 + 2r^2A^2 + A^4\right) \tag{4.21}$$

Naturally, this expression can be obtained in the direct way, like (4.13). Below we give a qualitative assessment of (4.18) - (4.21) in the three cases $Q < r_g/2$. $Q = r_g/2$ and $Q > r_g/2$.

1) The case $Q < r_g/2$. In this case the black hole has horizons at $r = r_+$ and $r = r_-$. The quantity A^2 from (4.20) can be rewritten as $A^2 = (r - r_+)(r - r_-)$ and can acquire positive, negative and zero values. Because the external observer can see the space up to the outer horizon $r = r_+$, we show in more detail the energy distribution outside $r = r_+$ where $A^2 \geq 0$. From (4.19) one concludes that the behavior for the pure gravitational energy density is the same, as in the Fig. 1 outside the horizon $r = r_g$. Everywhere at $r_+ < r < \infty$ this density is negative, it falls off to zero at infinity and approaches $-\infty$ at $r = r_+$. From (4.21) the behaviour outside $r = r_+$ for the energy density of the electromagnetic field interacting with the gravitational field, is as follows. At infinity this density falls off to zero from the positive value. But it approaches $-\infty$ at $r = r_+$. The latter is not surprising because we consider the matter interacting with the gravitational field. If one 'excludes' the gravitational field from (4.21) one will obtain a positive value everywhere. Fig. 2 ullustrates the behaviour of the total energy density (4.18).

2) The case $Q = r_g/2$. In this case $r_+ = r_-$. The behavior for (4.19) and (4.21) outside the horizon is the same, as in the previous case. The behavior for the total energy density is described in Fig. 3.

3) The case $Q > r_g/2$. In this case there are no horizons and $A^2 > 0$ everywhere in the Minkowski spacetime.

The behavior of the energy density of the free gravitational field (4.19) is as follows. At infinity the density falls off to zero from the negative value; it approaches $-\infty$ at $r = 0$ (exluding the term proportional to $\delta(r)$). However, there is the positive hump near $r = 2Q^2/r_g$. The latter fact, it appears, is not satisfactory. However, in this regard it is interesting to recall that for $Q > r_g/2$ the Reissner-Nordstrom solution is not physical[1] near $r = Q^2/r_g$.

The matter density (4.21) has positive and negative parts. At large distances this density is positive and falls off to zero at infinity. For a large enough value of Q this density is positive at all points of the Minkowski spacetime (excluding the term proportional to $\delta(r)$).

It is interesting to see how the behavior of the total density (4.18) changes with Q. The curves in the Figs. 4 (a, b, c) can be interpreted as follows. For $r_g/2 < Q < r_g$ at a large enough distance from the source the negative gravitational energy dominates over the positive matter energy. For $Q \geq r_g$ the situation is changed, namely the positive matter energy dominates over the gravitational energy. Only the negative infinity of the gravitational energy prevails near the world line $r = 0$.

Finally, we have also resolved the problem of the point mass for the Reissner- Nordstrom solution. Thus a reasonable value for the total mass can be obtained by performing the valume integration of (4.18). Further, the Einstein equations for the solution (4.17) and for the stress-energy tensors (4.18) - (4.21) are satisfied at all points of the Minkowsi spacetime including the worldline $r = 0$.

V. DISCUSSION

At first sight, the use in Sec. IV of the field formulation of GR may seem to be of doubtful validity in the interior of a black hole. Indeed, for a physical spacetime the coordinates t and r (which have the ordinary sense of time and radial coordinates outside the horizon) change their sense within the horizon. In contrast, in our consideration these coordinates have the same sense at each point of Minkowski space including $r \leq r_g$ (or $r \leq r_+$). Moreover, the geometry of the auxiliary background spacetime in the field formulation of GR is not affected by the motion of test particles. Therefore, the use of the flat background, espessially near and inside the horizon may seem unphysical.

In spite of the above remarks we claim the following in favour of the present approach. GR in the field formulation is a theory in which all the dynamic fields, including the gravitational field, are considered in a fixed background spacetime. Moreover, the field formulation of GR can be constructed in the ways where the background spacetime is used as a fundamental concept[10,12]. Thus, the field configurations (4.[illegible]) - (4.7) and (4.17) are simply the solutions of a selfconsistent field theory which is not different from Einstein's theory. Hence, studying the energy distribution of these configurations with respect to a Minkowski spacetime at all of its points is quite justified. Note that, as discussed earlier in Ref. 22, the trajectories near the horizon can be also studied successfully with the use of the Minkowski spacetime. In general, the concept of a fixed spacetime (fixed metric) in the energy density problem is not new (see Refs. 1 and 3 and references therein). Only in the field formulation of GR do we find the possibility of using the concept of the Minkowski spacetime in full measure and over the entire spacetime.

Now, we want to give the curious example in favour of the idea that inside the horizon r could be selected as a spatial coordinate. It is known that using the Møller superpotential gives the correct answer mc^2 for the total energy of the Schwarzschild black hole. Recently[23] it was shown that Tolmen's and Møller's formulae for the total energy are equivalent. The naive definition of sources in the geometrical formulation of GR for the Schwarzschild geometry has the form (2.9). Then, if one tries to use this stress-energy tensor in the Tolman formula, it will give the correct result mc^2. Only r has to be interpreted as a spatial coordinate.

Let us give another argument. Consider the movement of a test particle near the horizon in the Schwarzschild coordinates, where one finds that in the coordinates of the distant obserever, the horizon is approached by the particle with increasing slowness taking infinite time to approach it[15]. On the other hand, the particle in its own frame passes through the horizon without any obstacle and finite proper time. To distance observer, however, his own coordinates are more real. Indeed, as a distant observer we never can observe the transition from an ordinary star to the true black hole. This was highlighted by Narlikar[13]. Here we describe the energy density as well as the motion of the falling particle in the framework of the distant observer.

In the recent work by Brown and York[3] a Hamilton-Jacobi-type analysis was carried out very carefully and an interesting and useful method for defining the quasilocal energy, momentum and spatial stress was suggested. Here we discuss the quasilocal energy only. In Ref. 3 a surface stress-energy tensor is defined by the functional derivative of the action with respect to three-metric on ${}^3B$, the timelike three-dimensional boundary surrounding a system. Surface energy density is defined by projecting the surface stress-energy tensor normally to a family of spacelike two-surfaces B that that foliate ${}^3B$. The integral of the surface energy density over B is the quasilocal energy associated with spacelike three-surface Σ whose ortogonal intersections with ${}^3B$ is the boundary B.

The formula for the quasilocal energy[3] has the form:

$$E = \frac{1}{\kappa}\int_B d^2x\sqrt{\sigma}\left(k - k^{(0)}\right) \tag{5.1}$$

where the energy E is defined in a three-dimentional domain Σ restricted by the two-surface B. Here, σ_{ik} is the metric on B; $\sigma \equiv \det \sigma_{ik}$; k_{ik} is the external curvature of B as embedded in Σ; $k^{(0)}_{ik}$ is the external curvature of B as embedded in a reference three-space; $k \equiv k_{ik}\sigma^{ik}$; $k^{(0)} \equiv k^{(0)}_{ik}\sigma^{ik}$. The use of the reference space and $k^{(0)}$ defines a "zero" for the local energy.

Let us note some properties of the quasilocal energy as given in (5.1). (i) The quasilocal energy E is defined as minus the variation in the action with respect to a unit increase in proper time separation between B and its neighboring two-surface. Thus, the quasilocal energy equals the value of the Hamiltonian that generates unit time traslations orthogonal to Σ at the boundary two-surface B. One can see that it is a very natural definition. (ii) In their approach Brown and York used a minimun number of assumptions. Therefore, their rules have no ambiguities, and they are simple and practical. Indeed, in (5.1) one only has to select a two-surface B in the physical spacetime and then to select a reference three-space in such a manner that B can be embedded isometrically into it. (iii) The formula (5.1) is covariant. Thus after B and the reference three-space are selected, one obtains no difference in the value of E for different coordinates used. For example, for the Schwarzschild geometry whether one uses the Schwarzschild coordinates or the isotropic coordinates the answer is the same. The important point is that B has to be the same. (iv) One simple property that the quasilocal energy possesses is additivity.

Now let us compare our results with those of Brown and York. Formula (5.1) was used to examine the the energy distribution for a compact star or black hole with both the Schwarzschild and the Reissner-Nordstrom extiriors[3]. For the physical spacetime the standard Schwarzschild-like coordinates were used, and the flat reference three-space with metric in spherical polar coordinates was selected. The two-surface B was selected as a two-sphere of the radius $r = R$ surrounding a source and which is outside the horizon. For the Schwarzschild and the Reissner-Nordstrom cases the energy at infinity was obtained respectively as

$$E(\infty) = E(R) - \frac{E^2(R)}{2R} \tag{5.2}$$

and

$$E(\infty) = E(R) - \frac{E^2(R)}{2R} + \frac{Q^2}{2R} \tag{5.3}$$

where $E(R)$ is the energy within a domain bounded by the sphere $r = R$. Only note that in Ref. 3 the signature $(- \; + \; + \; +)$ is used.

One can see that qualitatively the results (5.2) and (5.3) agree with ours in Sec. 4. Indeed, in (5.2) and (5.3) the total energy of the system is $E(\infty) = mc^2$. We have also obtained the same result (4.11) for both the Schwarzschild and the Reissner-Nordstrom solutions. Then, as for the formula (5.2), it is easily seen that the energy within the surface $r = R$ is

$$E(R) > mc^2, \tag{5.4}$$

i.e, the energy of the gravitational field outside the surface $r = R$ is negative. Qualitively this agrees with our conclusions (4.10a,b).

Next, the formula (5.3) has to be considered under the three possibilites, as in subsection 4 c). Note that since $E(R)_{|R\to\infty} = E(\infty)$, we have to select the smaller solution $E(R)$ to the equation (5.3). For $Q < r_g/2$ it follows from (5.3) that $E(R) > mc^2$ and the negative gravitational field energy dominates over the positive energy of the electromagnrtic field outside the surface $r = R$. For $Q = r_g/2$ it likewise follows that $E(R) = mc^2$. This means that outside the sources the gravitational energy exactly cancels the electromagnetic energy. For $Q > r_g/2$ one has $E(R) < mc^2$, i.e., outside the surface $r = R$ the electromagnetic energy prevails. Note that for $R \leq Q^2/r_g$ one has $E(R) \leq 0$. Thus, the result (5.3) (with some differences) qualitatively agrees with ours in subsection 4 c).

Thus, as compared to Ref. 3 (and also to Ref.4) we have obtained the same answer mc^2 for the total energy but different answers for the local values of the energy densities. This relates also to the others works (see the references in Refs. 1 and 3). This is not surprising because, in principle, the energy of the gravitational field is not localized[1] and a unique definition of the energy density of the gravitational field does not exist.

If reasonable and clear rules of localization are violated then immediately the general properties of nonlocalization in GR show themselves, and one can obtain less reasonable answers. Let us demonstrate this by the following example. Let the metric for the Schwarzschild solution be in the isotropic coordinates, instead of the coordinates in (2.1):

$$ds^2 = \frac{\left(1 - \frac{r_g}{4r}\right)^2}{\left(1 + \frac{r_g}{4r}\right)^2} c^2 dt^2 - \left(1 + \frac{r_g}{4r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{5.5}$$

Note that r here is different from r in (2.1). Now, let us examine the formula (5.1) with respect to (5.5). We select the spacelike sections Σ as $t = const$. The metric σ_{ik} is

$$\sigma_{22} = \left(1 + \frac{r_g}{4r}\right)^4 r^2, \quad \sigma_{33} = \left(1 + \frac{r_g}{4r}\right)^4 r^2 \sin^2\theta, \quad \sqrt{\sigma} = \left(1 + \frac{r_g}{4r}\right)^4 r^2 \sin\theta. \tag{5.6}$$

Next, we use the the ordinary formulas for the external curvature[1]:

$$k_{ik} = -\frac{1}{2} L_{\vec{n}} \sigma_{ik}, \qquad k_{ik}^{(0)} = -\frac{1}{2} L_{\vec{n}_0} \sigma_{ik} \tag{5.7}$$

where $L_{\vec{l}}$ is the Lie-derivative with respect to the three-vector $\vec{l}$; $\vec{n}$ and $\vec{n}_0$ are unit normals to B in Σ and in the reference flat space with the metric $\gamma_{11} = 1$, $\gamma_{22} = r^2$, $\gamma_{33} = r^2 \sin^2\theta$, respectively. (The opposite sign is because of the signature $(- \; + \; + +)$ in Ref. 3.) Note that the metric (5.6) descibes a Riemannian manifold with the topology of a two sphere and everywhere positive curvature, therefore it can be embedded in flat space in a unique way[3]. Then, one can use (5.1) and after taking into account (5.6) and (5.7) one obtains

$$E(R) = mc^2 \left(1 + \frac{r_g}{8R}\right) \left[1 - \left(\frac{r_g}{4R}\right)^2\right]. \tag{5.8}$$

From here it is seen that $E(\infty) = mc^2$ again. However, (5.8) gives the energy distribution different from (5.2). Indeed, outside the horizon $r = r_g/4$ for $r_g/4 < R \leq r_g/4(\sqrt{2} - 1)$ one has $E(R) \leq mc^2$, unlike (5.4). Only for $R > r_g/4(\sqrt{2} - 1)$ the inequality (5.4) is fulfilled which means that the energy density of the gravitational field outside the surface $r = R$ is negative. Moreover, the formula (5.2) gives the correct Newtonian limit[3], whereas the formula (5.8) doesn't. Thus, we have obtained different and less reasonable results. The reason is that we have violated the Brown-York procedure (see point (iii) after (5.1)), i.e., we have selected the two surface B described by (5.6) and (5.7) which differs from B used by Brown and York[3] that gives (5.2).

Many approaches, like ours, use the concept of auxiliary reference flat spacetime. For resolving some problems it can be permissible that a choice of the flat spacetime can be made in different ways. For example, in the cases where at spatial infinity the fall-off of the gravitational potentials with respect to this flat spacetime is fast enough and only then, will one have the correct answer for the total energy of an isolated system[19,24,25]. At the same time the energy distribution itself can be arbitrary with respect to different flat spacetimes. We demonstrate this fact for our approach.

Let us examine the metric (5.5) again. Let the background metric be in the form (4.1):

$$\gamma_{00} = 1, \quad \gamma_{11} = -1, \quad \gamma_{22} = -\, r^2, \quad \gamma_{33} = -\, r^2 \sin^2\theta. \tag{4.1'}$$

After a coordinate transformation which transforms (5.5) to (2.1) one finds that in the coordinates of (2.1) the metric of the flat spacetime (4.1′) has another form different from the metric (4.1) in the same coordinates. Thus, (4.1) and (4.1′) describe different flat background spacetimes.

Taking (4.1′) as the flat background metric we reduce (5.5) to the field form. We then change the spherical coordinates in (4.1′) to the Cartesian ones, and this metric acquires the Lorentz form:

$$ds^2 = dt^2 - {dx^1}^2 - {dx^2}^2 - {dx^3}^2.$$

Then the field configuration of GR for the solution (5.5) acquires the form:

$$h^{00} = \frac{\left(1 + \frac{r_g}{4r}\right)^7}{1 - \frac{r_g}{4r}} - 1, \quad h^{11} = h^{22} = h^{33} = \left(\frac{r_g}{4r}\right)^2. \tag{5.9}$$

The total energy for this configuration can be obtained with the use of (4.4) and is equal to mc^2 again (see Ref. 10 also). However, for (5.9) the energy density differs from (4.12). At the same time, the character of the energy distribution is reasonable and the same outside the horizon $r = r_g/4$. That is t_{00}^{tot} is negative for $r > r_g/4$, near the horizon $t_{00}^{tot} \to -\infty$. Why we have the difference in the energy density is because we use the very weak restrictions for localization. In fact the fall-off conditions in (5.6) are the same as in (4.7).

Now we give another example with the use of the background spacetime. The idea of mapping the points of spacetime on a fixed background spacetime with the purpose of localizing energy was used by Katz[26], and the covariant superpotential defining the energy density for the gravitational field and its sources was suggested. The problem of a resonable and unique mapping was investigated by Katz and Ori[2] and some mapping rules for the background spacetime to obtain conserved densities were given. First, mapping should have well defined equations with a unique solution; second, a field energy density has to be positive in physical spacetime; third, the topology of physical spacetime has to have the topology of the flat spacetime. Then, it was found that the solution to the problem of embedding thin shells gives a unique mapping, i.e., a unique localization. Thus for this particular problem of GR one has a resolution of the localization problem, and one cannot find another flat background to change the energy density.

The indefinite character of the energy density for the gravitational field appears also within the other approaches although not explicitly. In Ref. 4, for example, for an asymptptically flat spacetime the three-covariant energy density for the gravitational field $(1/\kappa)\ \partial_j \left(eT^j\right)$ (which gives the ADM energy) was assumed. The author notes that the ADM energy for an asymptotically flat spacetime can be also obtained by the use of the three-covariant expression

$$\frac{1}{\kappa}\ \partial_k \left(NeT^k\right) + \partial_k \left(\Pi^{jk} N_j\right) \tag{5.10}$$

in a coordinate system such that for $r \to \infty$, one has $N \to 1$, $N_j \to 0$. Here, the notation of the Hamiltonian teleparallel description of GR developed by Maluf[27] is used. We note that the density (5.10) is more suitable because after substituting the constraints the Hamiltonian of the system[4] is exactly equal to (5.10).

Thus, in the examples of Refs. 2 - 4 and in the present study one can see some peculiarities of the localization of the gravitational energy in GR. The importance of the problem of localization was better realized after the proof of the positive-energy theorem[17,18] (see references in Ref. 3 and, for example, some recent works in Ref. 28). Studies of the problem were being developed in the three broad directions: (i) The total 4-momentum is considered inside a closed two-surface. One of the main restrictions is that the 4-momentum has to be timelike. (ii) Constructing positively defined Hamiltonians. (iii) Mapping a physical spacetime onto a fixed background spacetime. The problem of uniqueness (unique localization) was being resolved in different ways in several approaches.

We think that solution of the problem of a unique localization is necessary. However, in the general case a unique localozation may be restrictive and not always useful. In our approach we are restricted by the concept of the distant observer only and have enough freedom in the selection of a flat background spacetime. We admit negative energy density, for example. On the other hand, each of a concrete selection of a background is a concrete clear localization. In relation to this freedom of localization in our approach we make two remarks.

First, let us consider the pure Schwarzschild solution. At $r = r_g$ as seen from (4.12) (see Fig. 1) the energy density has a discontinuity. This highlights the fact that in the standard formulation of GR one has a coordinate singularity at $r = r_g$. This is not a real singularity. Indeed, in the field formulation of GR this break in the energy density can be countered with the use of an appropriate choice of a flat background[22]. Thus, the local energy distribution is changed but the total energy of the system is left unchanged at mc^2.

Second, the fact that we discuss the *quasilocal* energy density of the gravitational field in the field formulation of GR is expressed as follows. Recall that the gauge transformations in that formulation are directly connected with another choice of the background under the transition from the standard to the field formulation of GR[10,11,19]. The equations of motion in the latter are invariant under the gauge transformations. However, the total stress-energy tensor is gauge invariant up to a covariant divergence only: $t^{tot}_{\mu\nu} \rightarrow t^{tot}_{\mu\nu} + (\ldots_{\mu\nu})^{\alpha}_{;\alpha}$. This means that the energy $\Delta E = t^{tot}_{00}\Delta V$ in any three-dementional domain ΔV can be changed, i.e., it can not be localized in the general case[1].

Finally, let us note the role of the distant observer. If an observer is in a state of free fall he will not feel the gravitational forces; locally he is in a flat spacetime. Thus, if the distant observer is free falling, he will also not feel the Newtonian force and, consequaently, he can not obtain the total energy of the system. It is noted[1] that all the effects in which the gravitational energy participates are global effects, not local effects. In our consideration the frame of the distant observer is in fact the global frame of all the distant observers fixed at spatial infinity. For a real isolated system in this global frame the fall-off of the gravitational potentials can not be made (with the help of gauge transformations) faster than the fall-off of the Newtonian potential at spatial infinity[19,25]. (In the standard framework of GR the fall-off of the metric at spatial infinity can not be made with the help of coordinate transformations faster than the fall-off of the Schwarzschild type metric[21].) At the same time for our global frame we have the weakest fall-off conditions for the gravitational potentials[19,25]. We are thus able to obtain correct and unambiguous answer for the total energy of an isolated system, and resolve some problems of interpretation of the energy in the Schwarzschild and Reissner-Nordstrom solutions.

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FIGURE CAPTION

Fig. 1

The total energy density for the Schwarzschild black hole is shown. The horizon is denoted by the line $r = r_g$. Excluding the worldline $r = 0$ one has the energy density for the free gravitational field only.

Fig. 2

The total energy density for the Reissner-Nordstrom solution under the condition $Q < r_g/2$. The horizons are denoted by the lines $r = r_-$ and $r = r_+$.

Fig. 3

The total energy density for the Reissner-Nordstrom solution under the condition $Q = r_g/2$. The horizon is denoted by the line $r = r_- = r_+ = r_g/2$.

Fig. 4

The total energy density for the Reissner-Nordstrom solution under the condition $Q > r_g/2$. Horizons are absent. (a) $r_g/2 < Q < r_g$; (b) $Q = r_g$; (c) $Q > r_g$.

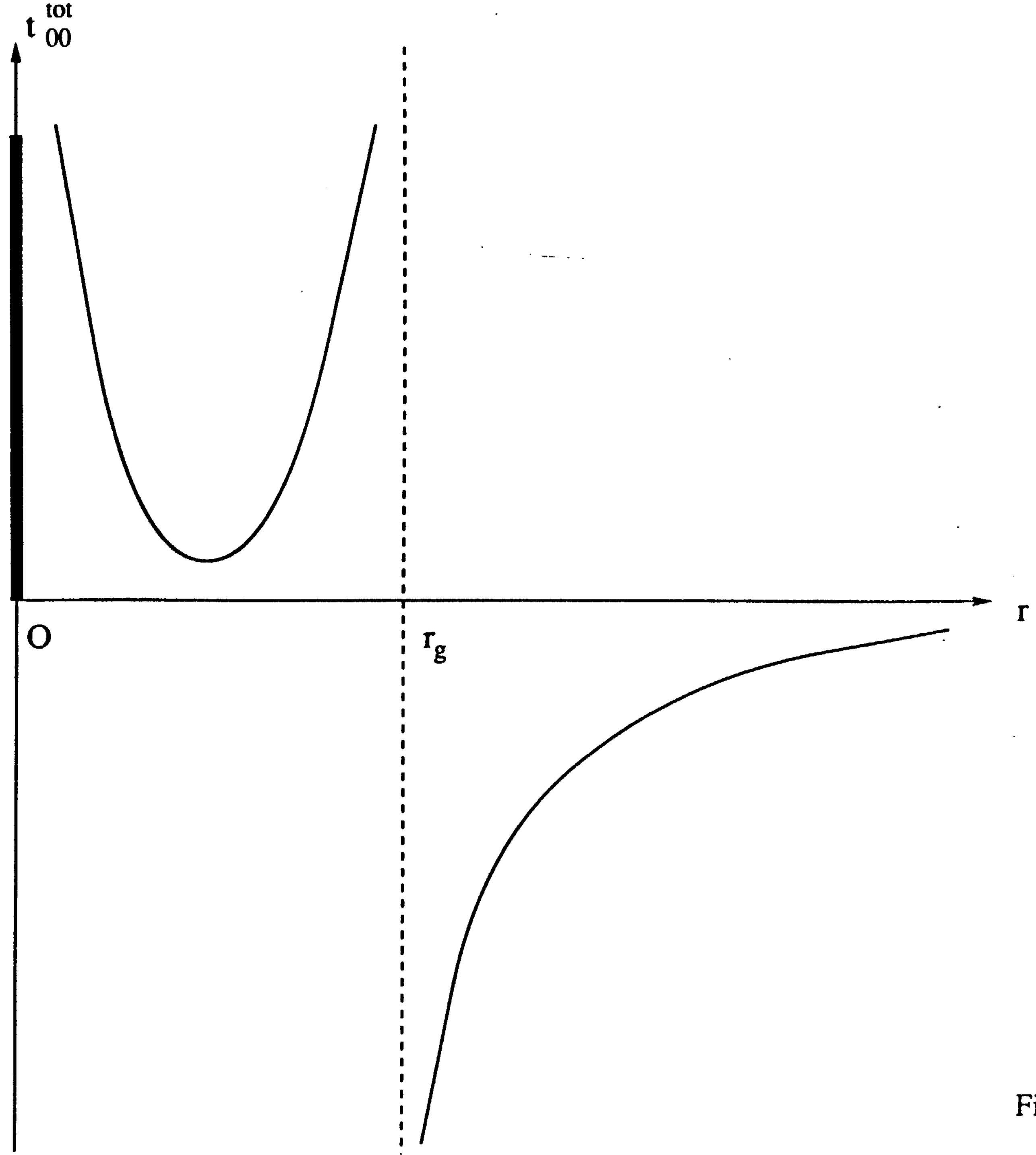

Figure 1

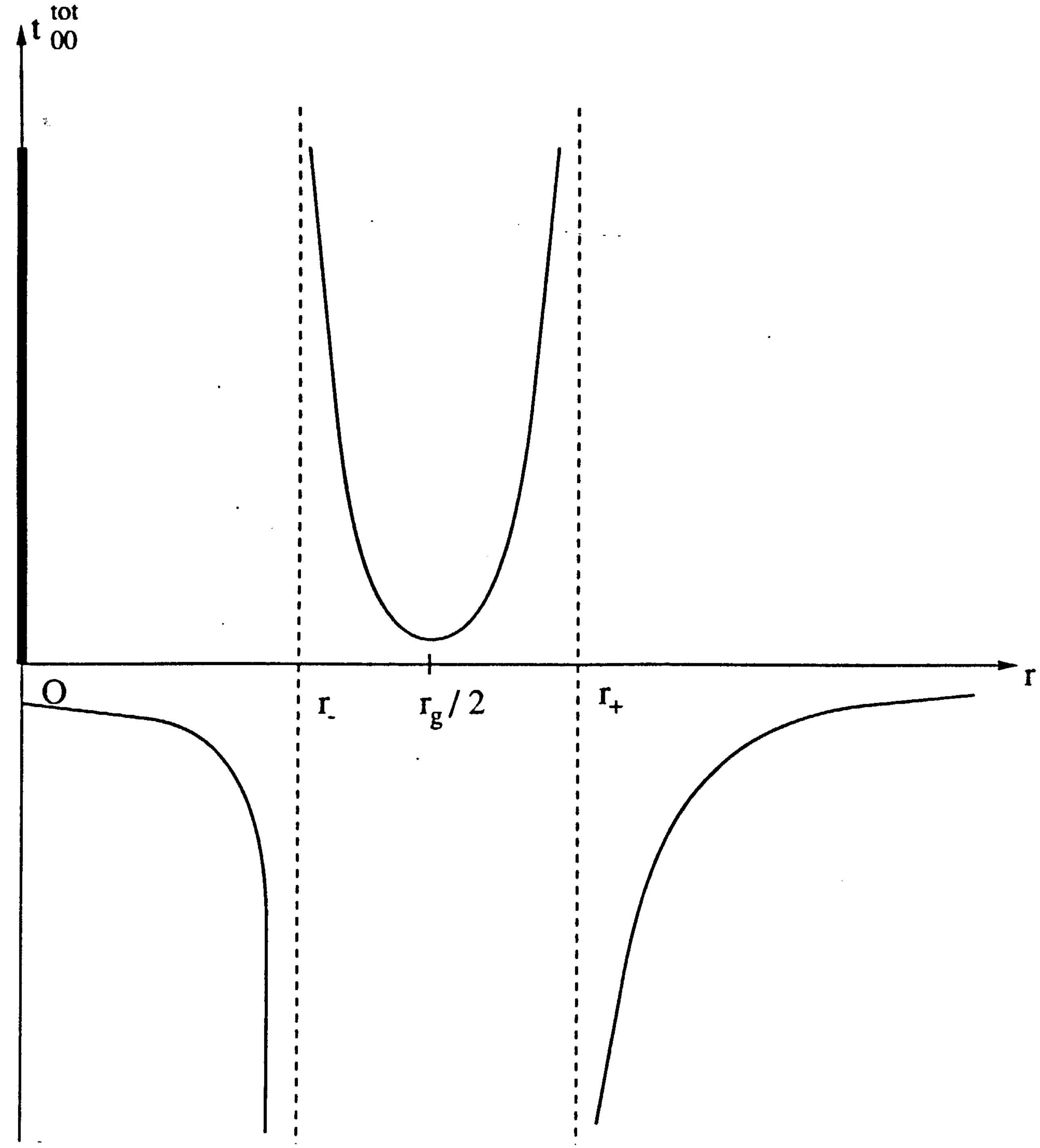

Figure 2

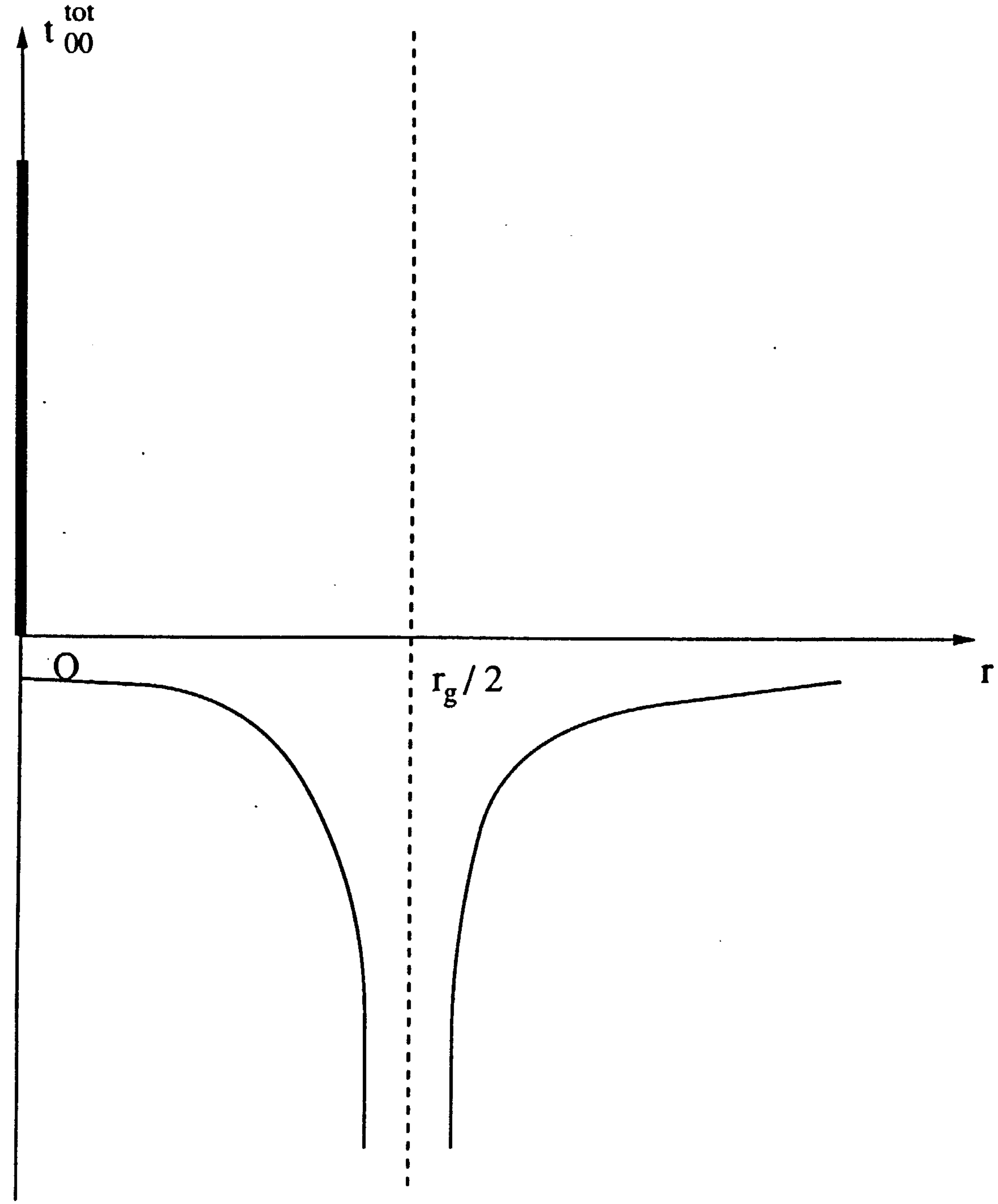

Figure 3

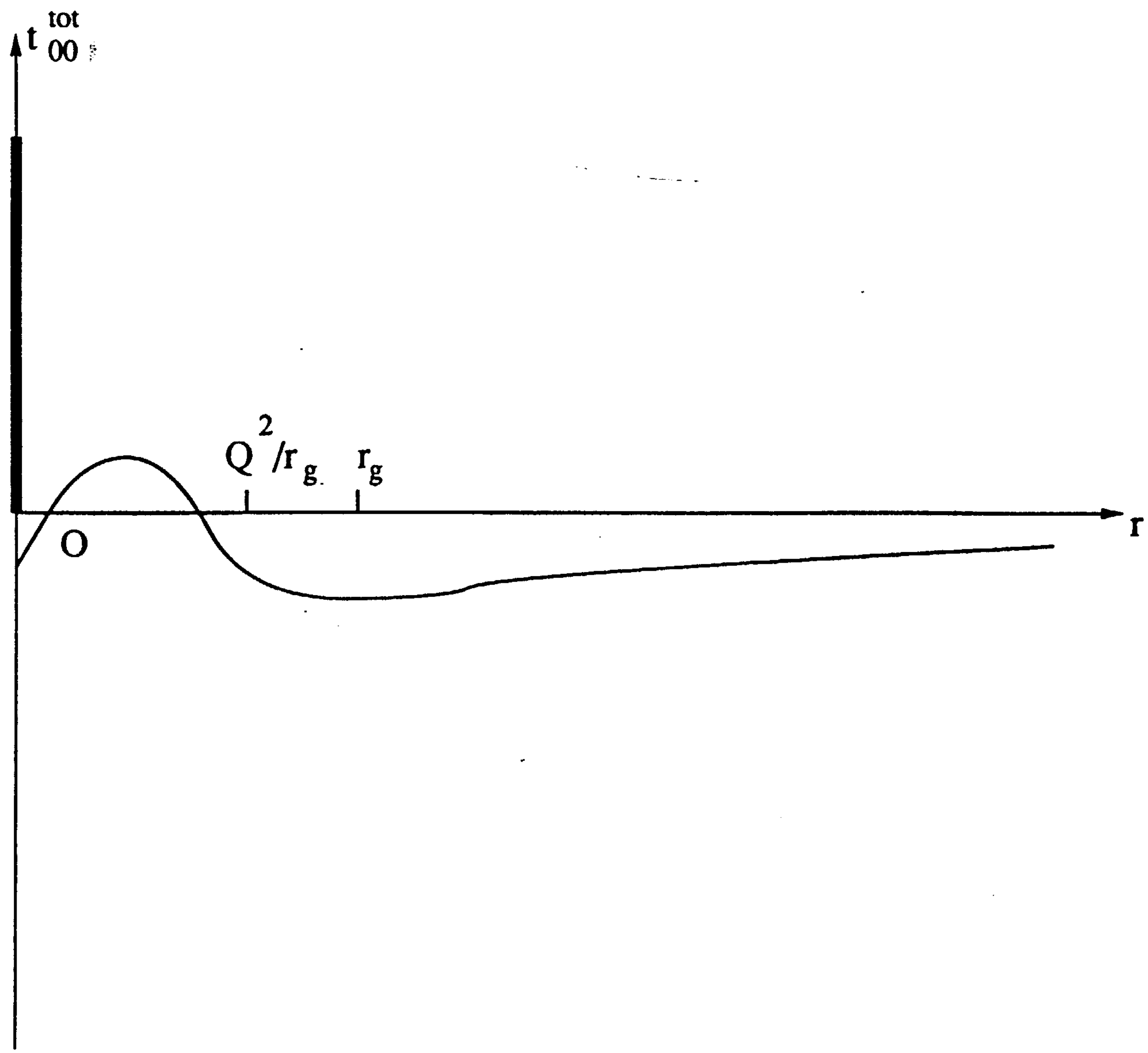

Figure 4(a)

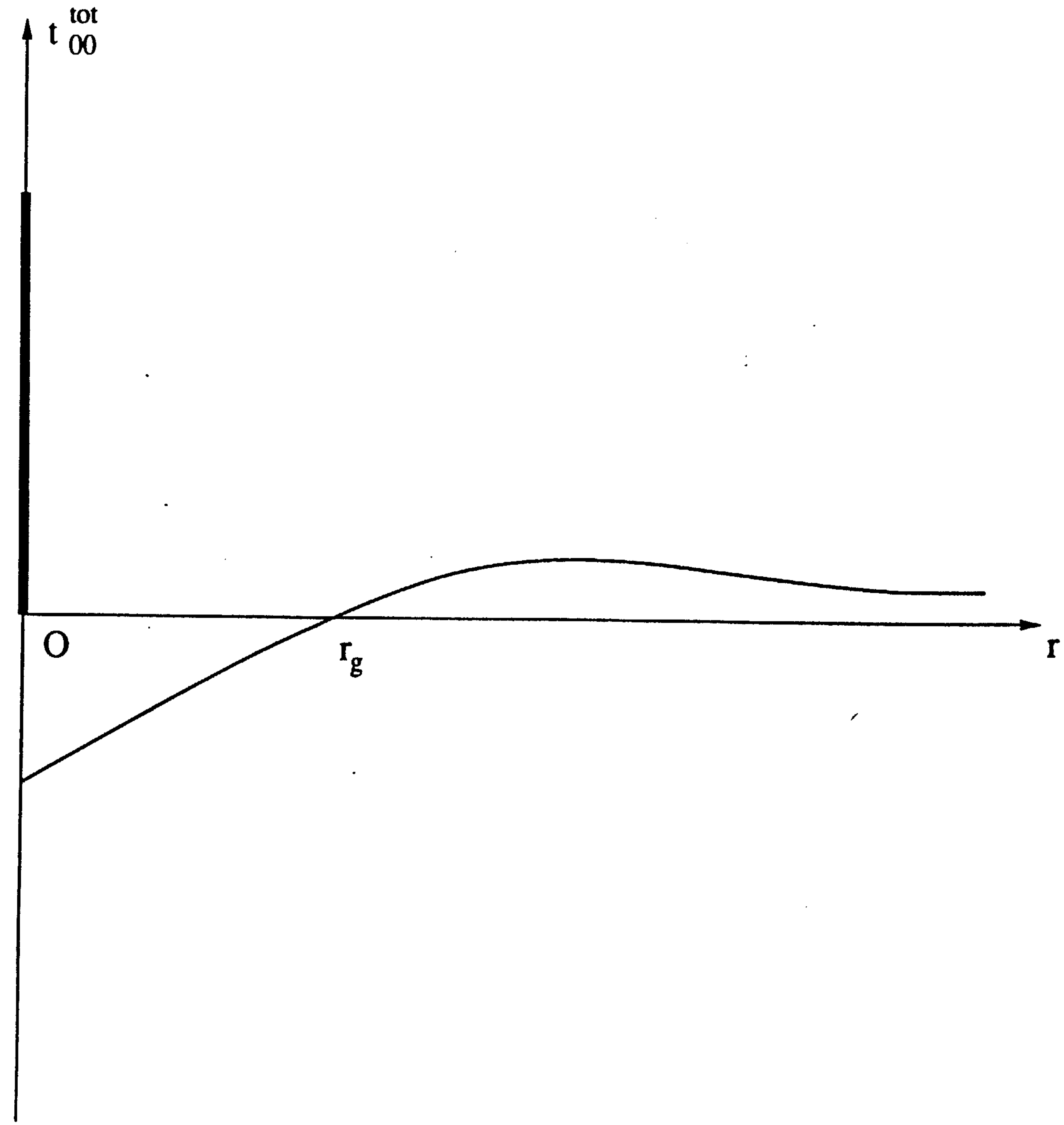

Figure 4(b)

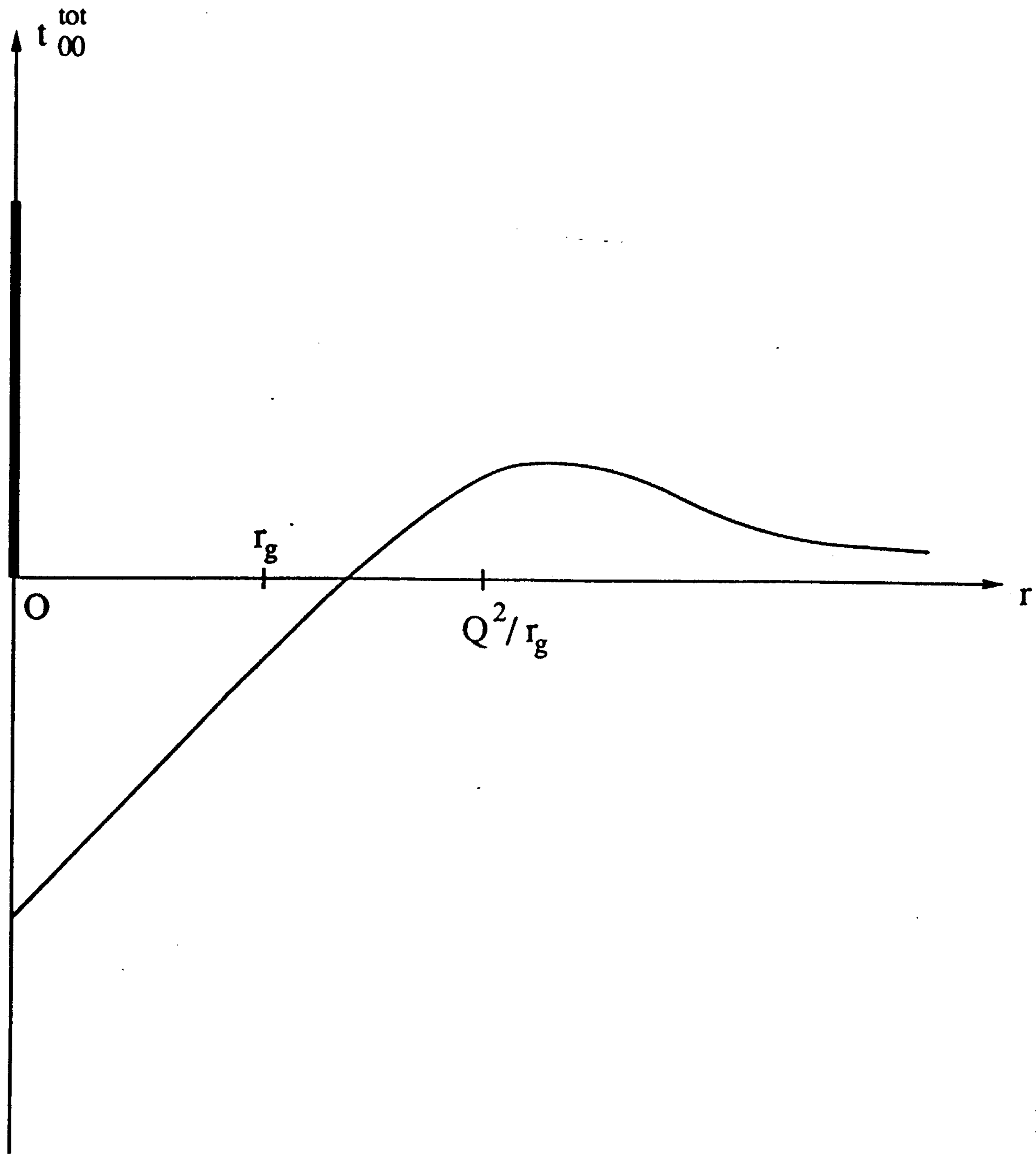

Figure 4(c)