The Nature of Mathematics – an interview with Professor Karlis Podnieks

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Dr. Karlis Podnieks studied mathematics at the University of Latvia from 1966-71. In 1979, he received his Ph.D. in mathematics from the Computing Centre of the USSR Academy of Sciences in Moscow, where he did research on machine learning algorithms. Initially, his research interests included work on the foundations of mathematics. In 1980, he turned his attention to the theoretical and practical aspects of computation, including computer programming, database design, information systems, and graphical tool development for business process modeling. He has, however, continued to write about the philosophical foundations of mathematics, and his writings are notable for their insight, humor, and originality. He became Professor of Information Technologies at the University of Latvia in 2005. This interview took place during April of 2009.

J.T. > Professor Podnieks, some mathematicians talk about "discovering" mathematics. Others talk about "creating" mathematics. The difference is important. One discovers things that have an existence of their own. Most people would agree that one discovers planets, for example, or species of birds. By contrast, one creates things that did not exist prior to the act of creation – symphonies, for example, or automobiles. Do you think that mathematics is created or discovered? Or to put it another way: How much of mathematics has an objective existence, and how much do we simply imagine into existence?

K.P.> I would prefer the term "inventing" instead of "creating". Speaking strictly, one discovers neither planets nor species of birds. One is inventing models of the world – or of parts of it. Some time ago, planets were thought of as lights attached to crystal spheres. Were these spheres "discovered" or invented? Some time later, a new model was invented in which planets were thought of as massive bodies orbiting the Sun. This picture remains stable after essential refinements of the model due to Kepler, Newton, Einstein et.al., and after the new evidence obtained recently by Gagarin, Apollo crews, etc. In fact, this stable part of model evolution is what people are calling "discovered final truth". As to symphonies and automobiles – after creation, they can be discovered just as planets were. Staying with the usual naive notion of "discovering truth about reality", we will never be able to understand the nature of mathematics.

J.T.> I thought of using the word "invent" rather than "create," but I think that some decisions about mathematics – the choice of axioms for a particular discipline, for example, and the choice of problems to study – depend on aesthetics. Mathematicians sometimes make these choices because the results that they obtain appeal to their sense of beauty. In this sense, mathematics has a good deal in common with art. But I can see that some mathematics may better be described as "invented" because it is developed in response to specific, often predetermined problems – especially those arising in engineering and the sciences. Anyway, that was my thinking...

K.P.> On the above human “modeling panorama”, where should the place of mathematics be? Physicists, chemists, biologists, economists, psychologists et.al. are inventing models for their “part of the world”. Then, what are mathematicians doing?

For many years, I have been promoting the broadest possible notion of mathematical models. Many people think that mathematical models are built using well-known “mathematical things” such as numbers and geometry. But since the 19th century, mathematicians have investigated various nonnumerical and non-geometrical structures: groups, fields, sets, graphs, algorithms, categories etc. What could be the most general distinguishing feature that would separate mathematical models from non-mathematical ones?

I would describe this feature by using such terms as autonomous, isolated, stable, self-contained, and – as a summary – formal. Autonomous and isolated – because mathematical models can be investigated "on their own" in isolation from the modeled objects. And one can do this for many years without any external information flow. Stable – because any modification of a mathematical model is qualified explicitly as defining a new model. No implicit modifications are allowed. Self-contained – because all properties of a mathematical model must be formulated explicitly. The term "formal model" can be used to summarize all these features.

For example, toy automobiles are autonomous, isolated and stable models of "big" automobiles, but they are not self-contained because, as physical objects, toys possess a huge number of very complicated physical properties that a) are explained by complicated physical theories; b) do not play any role in modeling; c) are not separated explicitly from the properties really involved in modeling. Thus, to make our toy model self-contained, we should include (at least) quantum electrodynamics as part of it!

J.T.> I'm not sure I understand the analogy. First, do you mean that mathematical models should retain only essential features of the objects that they model? And second, by "self-contained," do you mean that these mathematical models should be complete within themselves in the same way, for example, that Euclidean geometry is complete? Euclidean geometry is whatever can be deduced from the axioms, and if a result cannot be deduced from Euclid's axioms then it is not part of Euclidean geometry. In this sense, it is self-contained. Is this what you mean when you say mathematical models should be self-contained?

K.P.> As with most models, formal models may include inessential and even "wrong" properties. For example, many good models of the Solar System represent planets not as massive bodies but "wrongly" as massive points. Thus, from the "goodness" point of view, mathematical models are as good or as bad as any other products of human intelligence.

Yes, indeed, the description of a self-contained model must include ALL assumptions that are allowed to derive new information (prove theorems) about the model. Thus, to make a self-contained model of a toy automobile, you must do one of two things: a) either separate explicitly, which properties of the toy are included in the model (for example, if you are interested only in the shape of the vehicle, then declare this explicitly, scan the shape into your computer, and allow the use of analytical geometry to derive information); or b) include in the model all physical, chemical etc. theories necessary to draw conclusions about physical properties of the toy (for example, how would it behave under very high temperatures, high gamma radiation, etc.). Following the first way, you will obtain a simple mathematical model containing only a few (but almost only essential) properties of the vehicle. Following the second way, you would obtain a very complicated mathematical model, containing a huge number of inessential properties.

Now, the move from mathematical models to mathematics is as follows: For me, the task of mathematics is developing methods for creating and exploring mathematical models as defined above. As put by Morris Kline: "More than anything else mathematics is a method."

J.T.> So with respect to modeling sets, would you say that there is a sort of world of sets, and mathematicians develop mathematical models of this world? (This is the mathematician as an explorer of the mathematical landscape.) To use a specific example, would you call the theory of sets that arises from the axioms of Ernst Zermelo a mathematical model? That gets to the heart of the question. To quote what you said about physicists and chemists, is Zermelo's set theory a mathematical model for the mathematician's "part of the world?"

K.P.> In the philosophy of science, models and theories are treated as different categories. Theories are a popular means of model-building. For example, by using the theory of Newtonian mechanics with the Gravitation Law included, one can build models of various systems of "particles": planet systems, galaxies etc.

From the axiom and theorem point of view, mathematical theories and models are very similar – any of both can be represented as a set of axioms and rules of inference allowing one to generate theorems.

Zermelo-Fraenkel set theory (ZFC) arose, indeed, as a model – the second attempt to describe the vision of "the world of sets" invented by Georg Cantor in the 1870s. The first attempt at axiomatic description failed. The simplest possible system of set axioms (in fact, a single axiom – the so-called unrestricted comprehension schema) leads quickly to contradictions (the famous Russell's Paradox and some others). Is ZFC a "correct description" of Cantor's intuitive vision of sets? Or was Russell's Paradox already "built" into Cantor's vision, and hence, ZFC represents a new "better" version of the world of sets that is not identical to Cantor's world? Anyway, in ZFC, one can re-build all of the common mathematics (all except some exotic highly theoretical results that need additional axioms, for example, the so-called large cardinal axioms).

Is ZFC a model of the mathematicians' "part of the world"? I would answer "no" - it is not a model, it IS the mathematicians' part of the world, they do not know any better one.

J.T.> This seems inconsistent with what you said before. Do you mean that you think the ZFC model for sets is the best model that is currently available, or do you mean that it is an example of discovered final truth, a concept that you mentioned earlier?

K.P> Your question, as well as my sudden turn "off the modeling" come close to the biggest controversy in the philosophy of mathematics. ZFC started, indeed, as an attempt to describe a vision of the world of sets. The unrestricted comprehension axiom schema led to paradoxes. So Zermelo introduced a restricted set of comprehension axioms that wouldn't allow reproduction of the known paradoxes, but should be sufficient for the reproduction of theorems already proved about sets. Zermelo's idea was extremely successful. Even now, one hundred years later, ZFC still dominates the market of set theories.

After this, should we still think of ZFC as a model of some more prominent structure that exists independently of the axioms of ZFC? Cantor's intuitive "world of sets with Russell's Paradox inside" is not a good candidate for such a prominent structure. Thus, have mathematicians invented another world of sets, one that is better than Cantor's, and that is described correctly in the axioms of ZFC but exists independently of these axioms? Or has this "better world of sets" existed since the Big Bang, and mathematicians (starting with Cantor) have been trying to build a correct model of it?

This fantastic chain of questions can be answered in two ways. The minimalist way: cut the chain at the very beginning. ZFC, after being formulated, and after one hundred years of continued success, does not need any more prominent structure behind. The axioms of ZFC themselves ARE the best world of sets known to mathematicians. This point of view is called the "formalist philosophy of mathematics".

But there is also the maximalist way: Let us believe that, indeed, the "best" world of sets has existed since the Big Bang, and mathematicians are simply trying to build a correct model of it. This point of view is called the "Platonist philosophy of mathematics". (Plato introduced the "world of ideas" as something separate from the "world of things" 2,400 years ago.) At least until now, the so-called theory of large cardinals seems to support this point of view.

J.T.> To make the discussion more concrete... Bertrand Russell wrote a short article called "Definition of Number" in which he defines what is meant – or at least what *he* meant – by a natural number. In it, he describes the number 3 as something that all "trios" have in common. (When he says "trio," he means a set with three objects. Three particular people, three particular stones, and the set consisting of the words "paper, rock, scissors" are examples of trios.) Each such set is an "instance" of the number 3. When I read the article, I enjoyed it, and it made sense to me. But then I began to think about very large integers – integers, for example, that are much larger than the number of all the atoms in the universe. What would be an instance of this size number? And if there is no instance of such a large integer, in what sense does the integer exist?

K.P.> Of course, the (now-called) natural numbers 1, 2, 3, ..., billion, etc. arose from the human practice of counting. In mathematics, this human process of "number creation" ended with the axioms (for example, the so-called Peano axioms) describing the infinite natural number sequence as a whole. There is no problem with the existence of the axioms – one can write them down on paper. But what about the existence of the very very large numbers predicted by the axioms? According to the axioms, the number 10^{1000} can be obtained by adding 1 to 0 many times. But physicists know that the universe, as a computer, could not perform this "computation," even working continually since the Big Bang. Thus, the mathematical "world of numbers" is, in part, a kind of Disneyland – most really big numbers are of the Tom and Jerry kind.

J.T.> What do you mean "the universe, as a computer,"? And if large numbers are a sort of fiction, is it because they are too large to obtain by counting or because to the best of our knowledge no instance of such a number exists? For large enough numbers, of course, both properties must be true.

K.P.> Of course, most probably, the universe is not a computer (at least not a usable one). But if you could imagine a computer as big as the universe, how many bits could it store, and how many operations could it have performed since the Big Bang? Physicists say no more than 10^{120} bits and no more than 10^{120} operations (Seth Lloyd)!

But if we represent numbers not as people of primitive times (as sequences of 1's), but as normal computers (i.e., in binary notation), then operating with numbers of size 10^{1000} is not a problem (just use 3,500 bits to represent a single number, and use the well-known simple algorithms to add and multiply such numbers).

J.T.> But no matter how large the largest numbers that can be stored within a computer, most numbers will be bigger still –

K.P.> Yes, for example, the "tower of four tens" – 10^(10^(10^10)), where ^ stands for the power operation – never will be represented either in the binary or in the decimal notation. We can operate with such "numbers" only in a very limited sense. Aren't we, in fact, operating with number expressions rather than with numbers?

J.T.> Another way of thinking about the natural numbers is that they are "closed under addition." Most people accept the reality of small natural numbers and they accept the requirement that it is always possible to add 1 to a natural number to obtain the natural number that is "the next one over," but then the larger natural numbers must exist, because they are logical consequences of this closure requirement. We only need to begin with 1 and then we just add 1 until we have generated large natural numbers. But this, it seems to me, is closer to Aristotle's idea of the infinite. In his book *Physics* he wrote about the infinite in geometry. He said, "In point of fact they *[mathematicians]* do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish." What do you think?

K.P.> Most mathematicians do not agree with Aristotle, and they use the Axiom of Infinity to obtain big and bigger actually infinite sets. But, of course, Aristotle was a brilliant thinker of his time, and his idea that, in fact, mathematicians do not need the actual infinite (only the potential one) is not completely wrong. Moreover, today, we know (as you say, "to the best of our knowledge") that the finite straight line CANNOT be produced as far as we wish - in the universe, because of gravity, there are no very long straight lines at all. Potentially infinite straight lines are idealizations, but they appear to be very good for building useful mathematical models and – to some extent – the same is true of actually infinite sets.

J.T.> How have your ideas about the reality of mathematics affected your own mathematical research?

K.P> Unfortunately, I left mathematics for computer science at the age of 35 (now I'm 60). My recent 25-year experience includes the theory and practice of computer programming, database design, graphical tool development for business system modeling etc. If I would be allowed to carry out mathematical research, I would try to build a new arithmetic that would use arithmetical expressions and not numbers as the fundamental notion. But it seems I won't, so I would invite younger people to try this idea that was inspired by my life-long philosophical development.

There is another philosophical idea that I would be happy to develop mathematically. Reading Henri Poincare, I realized that arithmetic "should be" inconsistent, i.e. there should be a way to derive a contradiction from the axioms of arithmetic. The idea is as follows. In trying to axiomatize the notion of natural numbers, we are building a vicious circle: The notion of proof from the axioms includes the so-called mathematical induction, but this induction also represents the main feature of the natural number system that we are trying to axiomatize.

The most serious partial results in this direction were obtained by Edward Nelson. But, if we try searching the web for possible contradictions in mathematics, then we can find a serious announcement by Nikolai Belyakin: If we add to Zermelo-Fraenkel set theory the second weakest large cardinal axiom, then we obtain a contradiction. However, the full proof of this result is not yet published.

Of course, an inconsistency proof of arithmetic will not put an end to the banking business. Nor will this mean that Intel processors are built on a "wrong theory". No harm will be done to applications of mathematics because it is only the "Tom and Jerry part" of arithmetic that "should be" inconsistent!

J.T.> Mathematics seems to be a sort of cross-cultural language. Of course, there are people who find math inaccessible, but mathematicians from around the world usually seem to agree on when a theorem has been proved. This is remarkable to me because mathematicians often share no common spoken language and have very different cultural backgrounds. Depending on their backgrounds, they may approach mathematics in different ways, but they still agree on the main points – at least that is how it seems to me. Do you agree, and if so, do you think that this reveals more about how the human brain works than about anything that mathematics purports to describe?

K.P.> As I have been trying to promote for many years, the task of mathematics is developing methods of creating and exploring mathematical models (in the broadest possible sense). Are the general features of the "world of all the possible mathematical models" determined by the features of how the human brain works or by the features of how the physical world is or both? Could an alien civilization design its world of mathematical models in a radically different way from our way? For example, would they use the same kind of natural numbers that we are using? I guess that the answer should be "yes". But could we try proving this as a mathematical theorem? Would it be a theorem of our mathematics or theirs?

J.T.> Thank you for sharing your considerable insight into the nature of mathematics. I've enjoyed our conversation.