Modal Truths from an Analytic-Synthetic Kantian Distinction

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Abstract

In the article of 1965 *Are logical truths analytic?* Hintikka deals with the problem of establishing whether the logical truths of first-order logic are analytic or synthetic. In order to provide an answer to this issue, Hintikka firstly distinguishes two different notions of analyticity, and then he shows that the sentences of first-order logic are analytic in one sense, but synthetic in another. This interesting result clearly illustrates the non-triviality of the question. In this paper we aim at answering the question *Are modal truths analytic?* In order to elaborate a satisfactory answer to this question, we will follow the strategy of Hintikka and we will exploit some recent results on the proof theory for modal logic. Finally, our conclusions will shed new lights on the links between first-order logic and modal logic.

1 Introduction

The main aim of this paper is to answer the following question: are modal truths analytic? The very same type of question was posed by Hintikka [3] with respect to first-order logical truths (while, as far as we know, nobody has ever wonder about the analyticity of modal logic). Hintikka showed that the answer, contrary to what one might think at the first glance, is far from being trivial. In order to develop our answer, we will closely tread the same path as Hintikka.

Hintikka distinguishes three notions of analytic:

(I) analytic truths are sentences that are true by sole virtue of the meaning of the terms they contain,

(II) analytic truths do not convey any factual information,

(III) analytic truths can be shown to be true by strictly analytic methods.

Let us get a closer look to these different notions of analyticity. The first notion of analytictiy has to be completely disregarded. Indeed, the attacks raised by Quine ([11]) towards such notion have clearly shown that it is, as it stands, unsatisfactory. Moreover it makes all the logical truths trivially analytic, and it is therefore irrelevant to our purposes. So, let us turn our attention to definitions II and III of analyticity. Following Hintikka, (a certain part of) first-order logic is analytic in the sense II of analyticity, but synthetic in the sense III of analyticity. Our goal is to show that modal logic is also synthetic in the sense III of analiticity. We will not deal with the question of whether modal logic is analytic in the sense II of analyticity. Such a question can be the subject of future research.

Let us start our task by analysing the notion III of analyticity.

2 Analytic by means of analytic methods

Let us ask what can be said of the sense of analyticity defined by III. Here it is advisable to first precisely define the concept of analytical argument-step and then to extend the definition to the larger concept of argument. The basic idea of sense III seems to be expressible as follows:

> III(a) All that is said by the conclusion of an analytic argument step is already said in the premise(s)

A conclusion of an analytic argument-step merely repeats what has already been explicitly stated or simply mentioned in the premise(s). This seems to be the more standard and common sense of the word *analysis*. Therefore our definition III(a) seems to be correct. Although correct, definition III(a) is admittedly very vague. Hence it is our purpose to make it somewhat clearer. One way of making it clearer, it is to formalise it. Despite the precision and transparency that a formalisation gives us, we will not go in this direction. Indeed we aim at being clear and general at the same time; unluckily a formalisation would oblige us to choose a precise formalism, and therefore to lose the broader view on the notion of analyticity. So we have to find another solution. We propose the following one:

> III(b) In the conclusion of an analytic argument step no more objects are considered together at one and the same time then were considered together in the premise(s)

In order to clarify the meaning of analytic argument step, we use the notion of objects and their interrelations. For the conclusion to merely repeat what is said in the premise(s), the number of objects and their interrelations occurring in the conclusion must be the same as the number of objects and their interrelations occurring in the premise(s). We underline that in [3, p. 180], Hintikka

proposes a very similar notion of analytic-argument step: the only difference being that wherever we use the term "object", he uses the term "individual". We think that the term "object" is more general, and thus make definition III(b) applicable not only to first-order logic, but also to modal logic. In the next section we will specify what we exactly mean by number of objects and their interrelations.

We have thus elucidated the notion of analytic-argument step. But our goal was to define the broader notion of argument. For this, it suffices to generalize what we have said up to this point. We can claim that a proof of q from p is analytic in sense III(b) if no more objects are considered at any of the intermediate stages than are already considered either in p or in q. A modal true sentence p will be said analytic if it can be proved to be true by strictly analytic means, i.e. by an argument where no more objects are considered together than in p.

Let us conclude this section by relating definition III(b) with Kant's distinction between analytic and synthetic (see [4, 5, 6]). In order to explain such a bound, let us firstly relate the sense III(b) of the term "analytic" to the sense that this word has traditionally had in geometry. A geometrical argument can be called analytic in so far as no new construction is carried out in it, i.e. in so far as no new lines, points circles and their like are introduced during the argument. On the other hand, a geometrical argument is said to be synthetic if these new entities are introduced. Let us compare this notion of analytic with the notion introduced in III(b). It seems enough clear that the former notion is just a more specific version of the latter notion. Indeed in the notion III(b) we generally say that in an analytic argument step, no new object should be introduced passing from the premise to the conclusion; in the geometrical notion, we specify what kind of object (i.e. point, lines, circles) should not be introduced passing from the premise to the conclusion.

According to Hintikka [2], Kant's usage of the term "analytic" and "synthetic" largely follows the geometrical paradigm. Therefore Kant's usage of these terms comes pretty close to the sense that these terms have in definition III(b).

3 First-Order Logic

As we have already said in the introduction, Hintikka deals with the question of whether the sentences of first-order logic are analytic in the sense III of analyticity that we have just explained. Let us dedicate this section to a presentation of Hintikka's solution.

First of all, let us introduce the key notion of *degree* of a sentence. In order to define the degree of a sentence, we have to introduce the notion of *depth* of a sentence. The depth of a sentence A, in symbols $d(A)$, can inductively be defined in the following way:

$d(A) = 0$, if A is an atomic sentence or an identity

$d(A_1 \wedge A_2) = d(A_1 \vee A_2) = \max\left((d(A_1), d(A_2)\right)$

$d(\exists xA) = d(\forall xA) = d(A) + 1$

In intuitive terms, the depth of a sentence is nothing but the maximum number of quantifiers whose scopes all overlap in it. For instance, we have: $d(P(a,b)) = 0$; $d(\exists xP(x,a)) = d(\exists xP(a,x)) = 1$; $d(\exists xP(a,x) \wedge \exists xP(x,a)) = 1$; $d(\forall x(\exists yP(x,y) \vee \exists y\exists z(P(y,z) \wedge P(z,x)))) = 3$.

Once the notion of depth of a sentence clarified, we can introduce the *degree* of a sentence. The degree of a sentence corresponds to the sum of its depth plus the number of free individual symbols occurring in it (constants or free variables).

The notion of degree of a sentence will be important in what follows, so it is worth explaining it carefully. The degree of a sentence serves to identify the number of objects (in case of first-order logic, the number of individuals) whose properties and interrelations one considers (or might consider) in that sentence. Of course this number includes the individuals referred to by the free individual symbols of the sentence. It also includes all the indefinite individuals introduced by the quantifiers within the scope of which we are moving. It does not include any other individual. The maximum number of all these individuals is just the degree of the sentence in question, which is therefore the maximum number of individuals we are considering together in the sentence.

Recall now definition III(b). In that definition, we used the expression "objects considered together at one and the same time". We promised to clarify this expression. The notion of degree serves this goal. Indeed the degree of a sentence is nothing but the number of objects considered together at one and the same time in that sentence.

As we have just said, Hintikka claims that certain sentences of first-order logic are not analytic in the sense III(b) of analyticity. Now we have all the means to explain how Hintikka justifies his claim. Let us then consider the rule of existential instantiation of the natural deduction calculus for first order logic. Such a rule has the following form:

$$\frac{\begin{array}{c}d\\ \vdots\\ \exists xPx\end{array}}{P[a/x]}\ EI$$

where a is a free individual symbol that should not have been used elsewhere in the derivation d and $P(a/x)$ is the result of replacing x by a in P (wherever it is bound to the initial quantifier $\exists xPx$).

The rule of existential instantiation is a synthetic rule: if we take objects to be individuals, it increases the number of objects considered together at one and the same time by adding a new one. Indeed in the rule EI we pass from the formula $\exists xPx$ to the formula $P[a/x]$, hence, thanks to the rule EI, we might pass from the formula $\exists xPx$ to the formula $\exists xPx \rightarrow P[a/x]$: while the formula $\exists xPx$ has degree 1, the formula $\exists xPx \rightarrow P[a/x]$ has degree 2. Therefore

the rule of existential instantiation is synthetic because it allows to infer a conclusion which has a degree higher than that of the premise(s). Therefore the first-order logic sentences that are provable by using this rule are synthetic because they cannot be proved by strictly analytic methods. An example of this type of sentences are the laws of exchanging adjacent quantifiers.

Given this situation, one could naturally ask whether the conclusion that has just been drawn is not an accidental peculiarity of natural deduction. One might think that another formalism could allow to prove all the first-order sentences in a purely analytic way. Unluckily it is not so. Let us consider the rule that eliminates the existential quantifier in tableaux calculi, i.e. the exposition rule:

$$\exists x P x$$

$$\Big|$$

$$P(x/a)$$

where a is a constant that has not been used elsewhere in the derivation. Even in this case, the degree of the conclusion is higher in the degree of the premise. The same happens in the sequent calculus with the rule that introduces the existential connective on the left side of the sequent[1]

$$\frac{P(x/a), M \Rightarrow N}{\exists x P x, M \Rightarrow N}$$

All these rules are synthetic. This is a feature of every complete proof procedure in quantification theory. Indeed every such proof procedure makes use of sentences of higher degree than that of the sentences to be proved. This way Hintikka concludes that certain sentences of first order logic are synthetic in the sense III(b).

4 Modal Logic

Let us now focus on modal logic. As we have already said our aim is to show that certain modal logical truths are synthetic in the sense III(b) of analyticity. In order to draw this conclusion, let us proceed as follows. Let us first of all remind the reader of an important result that links modal logic to first-order logic.

Definition 4.1. Let x be a first-order variable. The standard translation ST_x taking modal formulas to first-order formulas is defined as follows:

[1]We take for granted that the reader knows that this rule corresponds to the exposition and the instantiation rules in tableaux calculi and natural deduction calculi, respectively (for further detail see, for example, [12]).

- $ST_x(p) = Px$
- $ST_x(\bot) = x \neq x$
- $ST_x(\neg A) = \neg ST_x(A)$
- $ST_x(A \vee B) = ST_x(A) \vee ST_x(B)$
- $ST_x(\diamond A) = \exists y(Rxy \wedge ST_y(A))$

where y is a fresh variable (that is, a variable that has not been used so far in the translation).

It should be clear that this definition makes good sense: it is essentially a first-order reformulation of the modal satisfaction definition. For any modal formula A, $ST_x(A)$ will contain exactly one free variable (namely x); the role of this free variable is to mark the current state; this use of a free variable makes it possible for the global notion of first-order satisfaction to mimic the local notion of modal satisfaction. Furthermore, observe that modalities are translated as bounded quantifiers, and in particular, quantifiers bounded to act only on related states; this is the obvious way of mimicking the local action of the modalities in first-order logic. Finally it is clear from this translation that there is a strict correspondence between the existential quantifier of first-order logic and the modal operator $\diamond$ (as there is a strict correspondence between the universal quantifier and the $\Box$).

Given this correspondence, and taking into account what we have previously shown for first-order logic through the rule that eliminates the existential quantifier in natural deduction, let us consider the rule that eliminates the diamond in the natural deduction calculus for modal logic. Unluckily, as far as we know, it does not exist a sort of standard natural deduction calculus for modal logic. Therefore we do not have a standard rule for the elimination of the diamond operator that we can refer to. The problem lies in the fact that proof theory for modal logic is still a pretty young enterprise and that the research has obtained better results in the sequent calculus framework. Let us then have a look to the rule that introduces the diamond on the left side of the sequent in a sequent calculus. In this case, we have different rules at our disposal. Each rule belongs to a different calculus, and each calculus for modal logic is a different generalisation of the standard sequent calculus. The latest extension of the Gentzen sequent calculus is the tree-hypersequent method (see [10]). In what follows, we will mainly deal with this formalism since it is the best suited for our purpose, but we will also make reference to the other formalisms.

Let us try to explain the tree-hypersequent method in the most general and least formal possible way (for a more accurate and formal description see [10]). First of all, in the tree-hypersequent method, instead of dealing with one sequent a time (as is the case in the classical sequent calculus), i.e. with objects of the form

$$M \Rightarrow N$$

where M and N are multisets of formulas, we deal with n sequents a time. These n sequents, that are standardly called *hypersequents*, are arranged in such a way that they can form a tree; hence they are named *tree*-hypersequents There exists a semantic way to look at tree-hypersequents: they can be seen as tree-frames of Kripke-semantics, where each sequent represents a world of the tree-frame. The rule that introduces the diamond on the left side of the sequent in the tree-hypersequent setting is the following one:

$$\frac{G[M \Rightarrow N/ \Rightarrow A]}{G[M \Rightarrow N, \diamond A]} \diamond L$$

The rule $\diamond L$ should be read (bottom-up) in the following way: in a tree-frame G, if there is a world x (denoted by the sequent $M \Rightarrow N, \diamond A$) that contains the formula $\diamond A$, then we can construct a new world y (denoted by the sequent $\Rightarrow A$) such that xRy (the relation xRy is denoted by the slash) and y contains the formula A. What can easily be noticed by looking at the rule $\diamond L$ is that in the passage from the premise to the conclusion (reading the rule bottom-up), we add some new structure, namely a slash and a new sequent. Let us further examine what this addition of structure involves.

From the standard translation of Definition 4.1., we know that there exists a strict link between the possible words of modal logic and the individuals of first order logic. Both are the objects that the first-order and modal sentences talk about. In first-order logic, in order to be able to identify these objects and their interrelations, we have introduced the notion of degree. We have to introduce a similar notion for modal logic. We do it by defining the concept of $mdegree$ in the following way. The mdegree of a sentence A, in symbols $md(A)$, is the number of sequents occurring in the longest branch of a tree-hypersequent. A brief reflection will allow the reader to realise that the notion of mdegree of a modal sentence tries to capture the same idea that is at the base of the notion of degree i.e. that of counting the number of objects and their interrelations (in the modal case, related worlds) that are represented by a sentence.

Thanks to this notion of mdegree, we are now able to check whether the rule $\diamond L$ is a synthetic rule, according to definition III(b). Indeed it is: the rule $\diamond L$ allows us to infer, reading it bottom-up, a conclusion with a mdegree higher than the premise. We can then draw the conclusion that the modal truths that are provable by means of this rule are synthetic in the sense III(b) of the distinction between analytic and synthetic sentences.

Let us then briefly sum up what has been said up to now. Between modal logic and first order logic there is an evident parallelism that can even be formalised by a rigorous definition. Sentences of modal logic talk about worlds and their interrelations, as sentences of first-order logic talks about individuals and their interrelations. In the case of first-order logic, the number of objects and their interrelations expressed by a sentence is calculated by summing up the free individuals occurring in the sentence plus the number of quantifiers whose scope all overlay in the sentence. In the case of modal logic, thanks to

the tree-hypersequent method, the number of objects and their interrelations expressed by a sentence is much more easily calculated by counting the number of sequents occurring in a the longest branch of a tree-hypersequent. In the rule of existential instantiation, as well as in the rule $\diamond L$, the number of objects considered together at one and the same time increases from the premise to the conclusion. This fact allows us to claim that both rules are synthetic, and that the sentences of first-order logic and modal logic, respectively proved by means of the rule of instantiation and the rule $\diamond L$, are synthetic too according to definition III(b).

Let us conclude this section by briefly seeing other formalisms where we have a rule that introduces the diamond on the left side of the sequent. For instance, let us consider the display method, as well as the multiple sequent method (for an accurate description of these two generalisations of the sequent calculus see [10] and [13]). In both these cases we have the following rules:

$$\frac{A \Rightarrow \bullet N}{\diamond A \Rightarrow N}\ \diamond L* \qquad \frac{A, M \diamond\!\!\Rightarrow N}{\diamond A \Rightarrow N}\ \diamond L**$$

where the symbol $\bullet$ is the characteristic structural connective of the display method, while $\diamond\!\!\Rightarrow$ is the characteristic structural connective of the multiple sequent method. Without dwelling on the interpretations of the two new structural connectives of the display method and the multiple sequent method, which would lead us too far away from our principal goal, it is easy to notice that in the two rules $\diamond L*$ and $\diamond L**$ we can observe the same phenomenon that we have underlined in the rule $\diamond L$, i.e. a new structural element appearing. This new structural element could be taken as the new introduced object that renders the rules concerning the diamond all synthetic. Hence these examples clearly show that the fact that certain modal sentences are synthetic do not depend on a particular formalism,[2] but it is, on the contrary, an unavoidable feature of modal logic, as it was the case for first-order logic.

5 First-Order Logic and Modal Logic

We have thus shown that certain modal sentences are synthetic because provable by synthetic means. Note that such a conclusion could also serve to shed new lights on the relations between first-order logic and modal logic. Let us first of all remind the reader the following theorem, which is based on Definition 4.1

Theorem 5.1. *Let A be a modal formula, then:*

for all models $\mathfrak{M}$, $\mathfrak{M} \models A$ if, and only if, $\mathfrak{M} \models \forall x ST_x(A)$

[2]Note that there also exist standard sequent calculi for modal logic, e.g. [8] and [9]. In these calculi, the rule that introduces the diamond on the left side of the sequent is analytic, i.e. it does not introduce any new structure. On the other hand, in these calculi the cut-rule, the non analytic rule *par excellence*, is not eliminable, therefore the lack of analyticity is there anyway.

In other more informal words, Theorem 5.1 states that modal formulas are equivalent to first order formulas in one free variable. On purely syntactic grounds it is obvious that the standard translation is not surjective (standard translations of modal formulas contain only bounded quantifiers). The question is then: could every first-order formula (in the appropriate correspondence language) be equivalent to the translation of a modal formula? No. This is very easy to see: whereas modal formulas are invariant under bisimulations, first-order formulas need not be; thus any first-order formula which is not invariant under bisimulations cannot be equivalent to the translation of a modal formula (for further details see [1]). This means that modal logic is a subset of first-order logic, under an adequate translation. Given this situation, it could have been that the first-order formulas that we have classified as synthetic were exactly the first-order formulas not translatable in modal logic. In that case modal logic would have been the analytic subset of first-order logic. In this paper we have on the contrary shown that certain modal formulas are synthetic and therefore that the modal subset of first-order logic contains (at least) some of the synthetic first-order formulas.

6 Kant and the analytic-synthetic distinction

In Section 2 we have explained the distinction between analytic and synthetic from a geometrical, and more generally, from a mathematical point of view. We have also briefly said that Kant was very close to this kind of distinction and therefore to the sense of analyticity explicitly stated in III(b). In this Section we will further examine these issues in the light of the conclusions that we have obtained in Section 4.

Let us go back to Euclid and Aristotle. A reason for uniting these two illustrious thinkers lies in the fact that they both used the word, *echtesis*, for denoting two procedures that seem, at the first sight, pretty different. On the one hand, we can call echtesis, or exposition, the part of an Euclid's theorem where a new figure is introduced or drawn for the first time. On the other hand, we can call echtesis, or exposition, a procedure, used in syllogistic theory, that pretty much resembles to the rule of existential instantiation (for further and more precise references see [7]). So we have (i) a geometrical exposition, (ii) a syllogistic exposition, and (iii) an inferential exposition represented by the rule that eliminates the existential quantifier in first-order logic. The crucial common point of these steps is that they all introduce a new element. As we have seen in the last section, even in modal logic we have a similar situation. When it come to the diamond operator, we introduce in the derivation some structure that was not present in the premise(s). What Kant would have thought of these four different procedures? As it has been suggested in [2], Kant would have probably indicated them as synthetic procedures. Indeed Kant seems to claim that mathematical truths are synthetic because they are based on the use of constructions. But construction is exactly what characterise (i)-(iii) plus the modal rule $\diamond L$. In the rule $\diamond L$, when we pass from the conclusion to the premise, we

literally construct a new world-sequent linked to the others by a relation-slash. In (ii) and (iii) the construction is in terms of introduction or "exhibition" of an individual idea to represent a general concept. In (i) the construction is done by drawing a new figure. So for Kant these fours procedures would have been perfect examples of synthetic modes of reasoning in mathematics. In particular (iii) and $\diamond L$ would have led him to agree with us that certain first-order logic sentences, as well as certain modal sentences are synthetic because provable by synthetic means.

Francesca Poggiolesi got her Ph.D thesis in 2008 at the University of Florence and at the University of Paris 1 Sorbonne. The thesis received the special mention for the best Ph.D thesis discussed in 2008-2010. Moreover, the thesis has been improved in such a way that it has become the monograph "Gentzen Calculi for Modal Propositional Logic" published by Springer for the prestigious collection *Trends in Logic*. From 2008 until 2011 Francesca Poggiolesi has been a post-doc researcher at the Vrije University of Brussels; since March 2011 she is a post-doc researcher at the IHPST in Paris. She is the author of several articles published in international reviews such as *Synthese*, *Studia Logica* or the *Review of Symbolic Logic*.

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