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## The language user as an arithmetician

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### Abstract

Dutch, like other languages, has approximative expressions with two numerals, for example: “twee, drie boeken” (lit. two, three books; two or three books). This construction is analysed. It turns out that the choice of number words is not arbitrary. Various kinds of factor are involved, as is shown using language materials from large corpora of Dutch texts. The interval between the two numbers has to be 1, 2,  $2\frac{1}{2}$  or 5, multiplied by  $10^n$ , at least in the decimal number system. It is argued that in daily life this set of so-called “favourite numbers” has a special role. Coins and banknotes, prices of special offers, bidding conventions in auctions are based on, or make use of, this set of numbers. An explanation for this favouritism is offered in the framework of the triple-code model of human number processing proposed by Dehaene. The explanation substantiates Dehaene’s claim of the existence of an analogue magnitude code used in estimating and comparing. Human cognition seems to be able to perform simple calculations with quantities (e.g., halving and doubling), independently of any counting or number system.

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### 1. Introduction

In his stimulating article *Varieties of numerical abilities* Dehaene presents a triple-code model of the architecture of human number processing (Dehaene, 1992). The model postulates the existence of three cardinal

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representations, one of which, the so-called magnitude representation, is supposed to be responsible for comparing and estimating capacities. Dehaene also claims this magnitude representation to be analogue instead of digital, as opposed to the other representational modes of the model: the auditory verbal word frame and the visual arabic number form.

In the [...] *auditory verbal word frame*, which is created and manipulated using general-purpose language modules, an analogue of a word sequence (e.g., /six/hundred/) is mentally manipulated. In the [...] *visual arabic number form*, numbers are manipulated in arabic format on a spatially extended representational medium [...]. In the *analogue magnitude code*, numerical quantities are represented as inherently variable distributions of activation over an oriented analogical number line [...]. (1992, p. 30)

In this article, we will present some observations which bear upon questions raised by the triple-code model, especially upon Dehaene's description of the magnitude representation. Our observations concern situations in which one or more numbers, numerals and number words are used. We will try to characterize the hidden regularities in the estimating and calculating activities of language and number users.

The second section deals with the way people use their language when estimating quantities. We will show that some aspects of the use of numerals and number words in approximative expressions can be explained by a few arithmetical principles, which seem to be of an analogue rather than a digital nature. In the third section, we will present observations from other domains. We will discuss some findings on currency systems of different countries, and we will present data related to the use of numbers in prices of "special offers" in ordinary Dutch shops. These phenomena, approximations with number words, denominations of bank notes and coins, and prices of "special offers", prove to be related to each other in a non-trivial way. It seems to be possible to explain the facts in these domains with a small set of arithmetical principles, the cognitive status of which will be discussed in the fourth section of this article.

## 2. Two-number approximative expressions

In Dutch, as in other languages, expressions like "twee, drie boeken" and "twee à drie boeken" (two or three books)<sup>1</sup> are normal utterances. We call expressions with two number words, with or without the preposition "à", approximative expressions. The use of two numerals indicates an uncertainty as to the quantity of the items that the speaker is referring to. The same

<sup>1</sup> In English, this expression has a disjunctive and an estimative reading. This ambiguity does not exist in Dutch if the expression only contains two number words partitioned by a comma. Whenever we use the translation with *or*, we restrict ourselves to the estimative reading.

applies to expressions like “four or five articles” and “ten or twelve journals”. In such expressions not all combinations of numerals are possible. This is in agreement with observations by Channell (1980) and Sigurd (1988), the only researchers who have paid some attention to estimation pairs. In accordance with their judgments, we claim that cases like “two or seven books” and “twelve or ten journals” are not acceptable as expressions in Dutch or any other language.

In this section we will present the results of an analysis of two Dutch language corpora, and we will try to formalize our findings. We have consulted the so-called “Eindhoven corpus” and the “Vijf-miljoen corpus” (Five million corpus) of the Instituut voor Nederlandse Lexicologie (Institute for Dutch Lexicology). The Eindhoven corpus contains large quantities of spoken and written Dutch. The other corpus (henceforth: “INL corpus”) is restricted to written Dutch, although some of the texts were meant to be spoken. Both collections consist of randomly chosen fragments and are available for all kinds of linguistic research. The materials in these collections of utterances have been coded, so that specific parts of speech and linguistic constructions can be selected.<sup>2</sup>

For the sake of simplicity, we shall present the cases we found using numerals rather than number words, even when the expression was written with number words. In accordance with linguistic practice we will use an asterisk (\*) when the expression is not acceptable as an expression of the Dutch language. Although we have used Dutch language materials only, we do claim that the rules and regularities that come to light through these sentences also exist in English and in other languages.

In the Eindhoven corpus and the INL corpus, we found respectively 46 and 44 two-number approximative expressions without “à”. Examples are “Drie, vier dagen geleden werd de pijn heviger” (Three or four days ago the pain became more intense), “Vier, vijf nationaliteiten op een schip, dat is tegenwoordig doodnormaal” (Four or five nationalities on a ship, that is absolutely normal nowadays), and “[. . .]n stuk of tien, twaalf van die hondjes had ze bij d'r” (She had some ten or twelve doggies with her).<sup>3</sup> We also found respectively 32 and 65 approximative expressions with two

<sup>2</sup> The Eindhoven corpus was named after the university where it was collected and prepared for analysis. It contains approximately 720 000 words, and is divided into six equally large sub-corpora, one of which is “spoken language”. The approximately 5 000 000 words in the INL corpus are all taken from written texts. Of the Eindhoven corpus we used the version that was made available by the Free University of Amsterdam. We also used a third corpus, the “27mln95 corpus”, to which we will refer below. That corpus consists of 27 billion words of newspaper texts, made available by the INL for linguistic research. We wish to thank Eindhoven University, and the Institute for Dutch Lexicology at Leiden, for making the corpora available for our research.

<sup>3</sup> In the Eindhoven corpus, we also found four expressions with three numerals: 2, 3, 4; 3, 2, 3; 4, 5, 6; and 5, 10, 20. We will not include these cases in our analysis, but it may be noted that they are in line with the regularities for two-number approximative expressions that we will describe below.

cardinal numbers separated by the preposition “à”, for example “tien à twaalf docenten” (ten or twelve teachers). We did not find important differences between the sets of cases with or without “à”.

Table 1 shows what kinds of expression we found, with their frequencies. For reasons that will become obvious below, the cases are arranged according to the differences between the numbers involved.

The expressions we found show some striking regularities. To start with: there is an *ordering rule* which states that the smallest number comes first. There are only three exceptions to this rule:  $[2, 1\frac{1}{2}]$ ,  $[5, 4]$ , and  $[12, 1]$ . They come from the sub-corpus “spoken language” in the Eindhoven corpus. In view of their contexts, the first two exceptions should probably be regarded as corrections of an error by the speaker. The third exception  $[12, 1]$  is a time expression; we will discuss this case in section 4.

Secondly, the cases we found seem to obey a *roundness rule*. As was observed by Channell (1980), for a two-number approximative expression to be acceptable, at least one of the numbers has to be “round”. Channell holds that the following numbers are “round”:<sup>4</sup> the integers 1–20, the even numbers 6–20 and the multiples of 5, 10, 20, 50 or 100. And, indeed, all cases in the corpora contain at least one number that is “round” in her sense. The same applies to all 45 examples presented in Sigurd (1988). Channell’s concept of “roundness” and her generalization, however, are not correct. The second set she mentions (the even numbers between 6 and 20) is superfluous and the last five sets should be narrowed: not all multiples of 5, for instance, can be considered as round (compare, for example, 165395). Channell’s generalization that in a two-number approximative expression one of the numbers has to be round, cannot be correct either. “Roundness” is not an absolute, but a relative property of numbers. Numbers are more or less round. In Jansen and Pollmann (in press) empirical evidence is presented that (relative) roundness is a function of accumulated number properties. What is more, the roundness of the numbers in estimation pairs is not an independent feature. As will become clear below, this roundness is related to another regular feature of these expressions.

The third regularity has to do with the *difference* between the numbers. We call it the *difference rule*. As can be seen from Table 1, in the vast majority of cases (149 times) the difference is 1, 10, 100, 1000, etc.:  $10^n$  ( $n \geq 0$ ). Twenty-five times the difference equals  $1/2$ , 5, 50, etc.:  $1/2 * 10^n$ , and 11 times the difference is 2 or 20. Two times the difference equals 3:  $[1, 4]$  and  $[12, 15]$ . We consider the first of these cases to be an anomaly, for which we have no explanation to offer. To the other one will return in section 4 below.

Although we did not find specific examples in the two corpora, it was our

<sup>4</sup> To support her claim, Channell refers to her own intuitions and to empirical research done by Rosch (1975). In Rosch’s study, however, only the numbers 10, 50, 100, and 1000 were hypothesized and shown to function as “reference points” in a decimal system.

Table 1  
Two-number approximative expressions and their frequencies in Dutch language corpora,  
ordered by the differences between the numbers

Difference	Examples	Frequency
$\frac{1}{2}$	1, $1\frac{1}{2}$ 8, $8\frac{1}{2}$	6
1	5, 6 12, 13	119
2	10, 12 12, 14	10
3	1, 4 12, 15	2
5	10, 15 90, 95	14
10	10, 20 70, 80	17
20	120, 140	1
50	50, 100	1
100	200, 300 700, 800	3
1000	2000, 3000 6000, 7000	2
5000	5.000, 10.000 40.000, 45.000	4
10.000	30.000, 40.000 60.000, 70.000	2
100.000	300.000, 400.000	1
1000.000	5.000.000, 6.000.000 7.000.000, 8.000.000	2
10.000.000	30.000.000, 40.000.000 80.000.000, 90.000.000	2
1000.000.000	3.000.000.000, 4.000.000.000	1
Total		187

intuition that expressions like [50, 75], [100, 125] or [1000, 1250] are perfectly acceptable estimation pairs. Here the difference between the members equals 25 or 250. To check this intuition we looked for such cases in the so-called “27mln95 corpus” of the Leiden Instituut voor Nederlandse Lexicologie, which is much larger than the “5 miljoen corpus”, but which was – at the time when we collected the data – not as accessible as the smaller INL corpus that we used. We found the following examples of estimation pairs in which the difference between the numbers is  $2.5 * 10^n$ : [0.25, 0.5], [25, 50], [75, 100], [100, 125], [750.000, 1.000.000] and [500.000.000, 750.000.000]. We consider this as evidence that pairs where the difference is  $[2.5 * 10^n]$  are normal estimatory expressions in the Dutch language.

To sum up, the differences between the two numbers of an approximative expression seem to be restricted to the set of numbers  $[10^n * (1 \text{ or } 2 \text{ or } 2.5 \text{ or } 5)]$ .

Clearly, the three rules we mentioned, the ordering rule, the roundness rule and the difference rule, are related. They seem to be covered by a “rule of sequence”. This rule states that the numbers in a two number approximative expression have to be part of the same arithmetic sequence, and have to follow each other immediately in this sequence.

There are three restrictions:

(1) The starting point of the sequence has to equal the difference between each member. When the difference is 1, for instance, the starting point is 1, and the resulting arithmetic sequence is 1, 2, 3, 4, 5, etc.:  $a_n = n$ . When difference and starting point equal 2, the sequence is 2, 4, 6, 8, 10, 12, etc. ( $a_n = 2 * n$ ) and when 10 is the difference and the starting point, the sequence is 10, 20, 30, 40, 50, etc. ( $a_n = 10 * n$ ). Formulated like this, our “rule of sequence” correctly predicts that expressions like [3, 4], [4, 6] and [20, 30] are acceptable, while expressions like [\*1, 3] or [\*2, 5] are not.

(2) Only sequences where starting point and difference equal  $10^n$ ,  $5 * 10^n$ ,  $2 * 10^n$ , or  $2.5 * 10^n$  can be used as sources for approximative expressions. Combinations like [\*3, 6], [\*40, 80], or [\*600, 1200], which would form part of sequences with differences equalling ( $[3 * 10^n]$ ,  $[4 * 10^n]$  or  $[6 * 10^n]$ ), are not acceptable as estimation pairs. With the exception of the pair [12, 15], such examples do not occur in our material.

(3) The sequence involved can only be used up to its twentieth number at most: [12, 13] and [90, 95] are in our collection and [\*32, 33] or [\*150, 155] are not. If the difference and starting point is 2 or 20, etc., or 2.5 or 25, etc., the sequence might even be shorter than 20. (Relatively) large numbers with a (relatively) small difference are not possible members of an estimation pair. This property of the sequences expresses the dependency of the roundness of the numbers on the difference between them. The roundness required by the roundness rule is to be seen as a consequence of this last restriction. The unacceptability of a pair like [147, 148] is not due to the fact

that [147] and [148] are relatively “unround”, but to the fact that the sequence of which this pair is a part can only be used up to its twentieth number. The relation between magnitude and difference of the numbers seems to reflect Weber’s law in one way or another. Rabelais’ joke that someone had “between 2435768 and 2435769 sheep” (*Gargantua et Pantagruel*, III, 2) is apparently partly based on a violation of the same principle.<sup>5</sup>

It might be possible to derive from the language material we presented a pragmatic theory of an estimating language user. However, in the rest of this paper we do not want to explore such a theory and we shall not try to offer a complete explanation for all the possible combinations we found. Rather we want to elaborate on the difference rule which describes the restrictions on the differences between the members of the pairs.

As we explained above, the difference between the numbers in an approximative expression equals a member of a special set of numbers. We call this set “the set of favourite numbers”. We claim that the following principle holds:

In any numeration system in base  $n$ , there is a set of favorite numbers comprising

- any integer power of the base
- half, double, and half of half of any integer power of the base.

We call this *the principle of favourite numbers*. In our numerical system, where the base is 10, this principle leads to the set  $F(10)$ .

$$F(10) = \{f \mid f = 10^n * (1 \text{ or } 2 \text{ or } \frac{1}{2} \text{ or } \frac{1}{4}) \text{ for any value of } n; n \text{ in } \mathbb{Z}\}$$

This set of favourite numbers contains the numbers (0.01, 0.1, 1, 10, 100, 1000, etc., i.e., the  $n$ -powers of 10), and the doubles (0.2, 2, 20, 200, 2000, etc.), the halves (0.5, 5, 50, 500, etc.) and the halves of the halves (0.25, 2.5, 25, 250, etc.) of these numbers. In a duodecimal number system the base numbers are 1, 12, 144, etc., leading to the set  $F(12)$  which consists of the base numbers, and the doubles (2, 24, 288, etc.), the halves ( $\frac{1}{2}$ , 6, 72, etc.) and halves of these halves of the base numbers ( $\frac{1}{4}$ , 3, 36, etc.). In section 4, the sets  $F(10)$  and  $F(12)$ , and the status of *the principle of favourite numbers* will be discussed in further detail.

There is a striking lack of symmetry in the principle of favourite numbers.

<sup>5</sup> Our collection of estimation pairs contains the case [1918, 1920], an apparent violation of the rule that the largest number in an estimation pair is never larger than 20 times the difference between the two numbers. The numbers in this case denote dates. In spoken Dutch, as in spoken English, the numbers have a special form, which is different from the form these numerals have, if they denote quantities. Presumably this case is to be explained by the special character of the number line of dates. Each century might have its own line of date numbers, starting at the beginning of that century.

The principle states that a half of a given number might be halved again, but not that a double might be doubled in its turn. We have no explanation for this unbalanced situation. In our data we did not find any grounds for the supposition that 4, 40, 400 and so on might be members of  $F(10)$ , whereas there is evidence for 2.5 and its derivations to be in  $F(10)$ .

In section 3 we will argue that the concepts “double”, “half”, and “half of half” play a hidden, but important, role in some other arithmetical facts of everyday life as well.

### 3. Favourite numbers in everyday life

In this section we will examine other cases in which the set  $F(10)$  of favourite numbers in the decimal system seems to be used. The examples we have are examples of a rather trivial usage of numbers in everyday life. They are non-linguistic in nature.

#### 3.1. Coins and banknotes

In the Netherlands, the currency contains coins with the denominations of 0.05, 0.10, 0.25, 1, 2.50, and 5 guilders. The Dutch banknotes have the values 10, 25, 50, 100, 250, and 1000 guilders. It is easy to see that all these denominations are denoted by numerals that are elements of the set  $F(10)$  of favourite numbers. This phenomenon is not restricted to Dutch currency. On the contrary, it seems that all currencies all over the world use coins and banknotes with favourite number denominations only. We assume that all denominations used in a specific currency system are a subset of  $F(10)$ , and we claim that this is not a coincidence.

To test our assumption, we checked as many travel guides as we could find in the city library of our home town Utrecht. We collected data from 84 countries from all parts of the world.<sup>6</sup> This way we collected approximately a thousand different denominations. With only 13 exceptions all these denomi-

<sup>6</sup> Albania, Argentina, Australia, Austria, Bahrain, Baltic States, Belgium, Bolivia, Brazil, Bulgaria, Canada, Chile, China, Colombia, Cook Islands, Cuba, Cyprus, Czechoslovakia, Denmark, Dominican Republic, East Germany, Ecuador, Egypt, Fiji Islands, Finland, France, French Polynesia, Great Britain, Greece, Guyana, Hong Kong, Hungary, Iceland, Indonesia, Iran, Ireland, Italy, Japan, Jordan, Kuwait, Luxembourg, Malaysia, Malta, Mexico, Mongolia, Nepal, New Caledonia, New Zealand, Nigeria, North Korea, North Yemen, Norway, Oman, Paraguay, Peru, Phillipines, Poland, Portugal, Qatar, Romania, Russia, Saudi Arabia, Singapore, Solomon Islands, South Yemen, South Africa, South Korea, Spain, Surinam, Sweden, Switzerland, Syria, Taiwan, Thailand, Tonga, Tunisia, Turkey, United Arab Emirates, Uruguay, USA, Vanuata, Venezuela, West Germany, Yugoslavia. Not all, but most, of these countries were still existing in the summer of 1995; some of these countries do not exist any more. Of course, this does not change our main point.



nations were in  $F(10)$ . Every system uses its own subset of  $F(10)$ .<sup>7</sup> The 13 exceptions are the following. Since the currency reform of 1534 Russia has had a 3-ruble coin or banknote. Albania has a 3-lek banknote, and the Baltic States used to have 3-ruble banknotes. Rumania has a 3-lei coin. There exists a 3-dollar banknote in the Cook Islands. Mongolia has a 3-mongo banknote and Cuba has a 3-pesos banknote. The Cubans have also a 40-centavos coin, the United Arab Emirates have a 1.50-fils coin, Mongolia a 15-mongo coin and in Russia and the Baltic States there exist/existed 15-kopek pieces. In Romania people have a 15-bani coin.

The history of money systems shows some other exceptions. The Greek “tetradrachme”, for example, must have been a 4-drachme piece. And surely, the Dutch “vierduitstuk” was a piece of 4 “duiten”. Many people will remember the traditional denominations of the currency in Great Britain before the reform of 1971; the *threepence* and the *sixpence* are clearly exceptions to our rule as it is stated. However, in a duodecimal system these particular values are perfectly regular, 6 being half and 3 being half of half of 12.

Notwithstanding the exceptions which we encountered, we conclude that there is overwhelming evidence supporting the claim that currency denominations are generally denoted by elements of the set of favourite numbers.

### 3.2. *Other direct usage of favourite numbers*

There are other cultural conventions which seem to be based on the existence of  $F(10)$ .

We think that our observations about the denominations of currencies can easily be extended to all nominal values of all kinds of “securities”: bonds, shares, lottery tickets, certificates, vouchers, travellers cheques, gift coupons and so on. This prediction, however, we did not test.<sup>8</sup>

We also found a description of bidding conventions at Dutch book auctions. These conventions state that

Bidders can take the line that the auctioneer will advance the price of the lots/parcels as follows: up to Dfl 100 [by] Dfl 5, from Dfl 100–Dfl 250 [by] Dfl 10, from Dfl 250–Dfl 1000

<sup>7</sup> In the Dutch system there are no coins or banknotes denominating 2, 20 or 200 guilders. But in a large number of other countries this type of coin and/or banknote does exist. In Germany, for instance, there are pieces of 2 Pfennig and of 2 Mark, and there are banknotes of 20 and of 200 Mark. But the Germans do without coins or banknotes of 0.25, 2.5 or 25 DM. There are currency systems in which the set of coins and banknotes comprises values derived from both 2 and 2.5.

<sup>8</sup> Traditional jubilee years also seem to be denoted by favourite numbers (100, 50, 25). However, we could not find any systematic cross-cultural survey of jubilee customs to test this supposition. We are aware, though, that in Western culture there are some jubilee numerals (especially for wedding anniversaries) that seem to be counterexamples:  $12\frac{1}{2}$ , 40, 60.

[by] Dfl 25, from Dfl 1000–Dfl 2000 [by] Dfl 50, from Dfl 2000–Dfl 10.000 [by] Dfl 100, from Dfl 10.000 upwards [by] Dfl 500. Other bidding commissions will be changed to that effect. (From *Auction sale of Books and Prints*, Bubb Kuyper, Haarlem, 1994)

This description of the bidding system makes it sufficiently clear that in one way or another the system must be based on the existence of  $F(10)$ .

To name an example in a somewhat different field, we point at conventions in determining levels of statistical significance. Kerlinger states: “A level of significance is to some extent chosen arbitrarily. But it is certainly not completely arbitrary. [. . .] The .05 and .01 levels correspond fairly well to two and three standard deviations from the mean of a normal probability distribution” (1973: 169). Both values Kerlinger mentions, as well as other values frequently used as levels of significance (.1, .025, .001), are in  $F(10)$ .

### 3.3. Bargain price numerals

The example that we will use in this sub-section to demonstrate a more indirect usage of the set  $F(10)$  is even more trivial than the currency system discussed above. It concerns the numbers which are used to denote prices of “special offers” in shops. In the Netherlands, as elsewhere, commodities of all kinds are sometimes offered as bargains. These bargains are priced in an allegedly attractive way. One finds prices like *not fl. 750.-- but fl. 698.--*, *not fl. 22.000.-- but fl. 19.500.--* and so on. We will refer to the lower numbers in these offers, like 698 and 195, as “bargain numbers” (in discussing them we will disregard zeros at the end and commas or dots in a middle position).

We assume that bargain numbers turn out to form a special set of the set of natural numbers, a set that we claim to be related to the set of favourite numbers. More specifically, we assume that most bargain numbers are numbers which become a multiple of 10 when 1 or 2 is added, and a multiple of 100 when 25 or 5 is added. In other words, we assume that the last digits of most bargain numbers are 8, 9, 75 or 95. In the examples presented below, the numbers in the second column are the favourite numbers 1, 2, 5 and 25.

<i>Bargain number</i>	<i>Favourite number</i>	<i>Multiple of 10</i>
89.--	+1	=90
198.--	+2	=200
6.49	+1	=650
319.--	+1	=320
468.--	+2	=470

<i>Bargain number</i>	<i>Favourite number</i>	<i>Multiple of 100</i>
5.95	+5	=600

37.50	+25	=400
16.75	+25	=1700
375.--	+25	=400

We tested our assumption in the leaflets, newspaper advertisements, brochures and so on that we found in our postbox during the third week of June 1994. We collected the prices of all special offers made to us in this period by 26 different firms. In this way, we collected 1618 items: offers with an “old” price and a lower “special price”. The last ones were the numbers that we put on our list. Of the numbers collected, 55 proved to be themselves members of the set  $F(10)$  as defined in section 2. Examples are 1, 2, 10, 25, 1000.

Table 2 shows the distribution of the last digits of the other 1563 numbers (again, we disregard commas and zeros at the end of the numbers). The results in Table 2 show a division as predicted: in the upper part of the table we find the digits “5”, “9” and “8”; the other digits can be found in the lower part of the table; “5”, “9” “8” appear far more frequently as last digits in bargain numbers than the other digits do.

The distribution of the last two digits in numbers ending with “5” is shown in Table 3. The results in Table 3 again show a division as predicted: in the upper part of the table we find “95” and “75”; the other combinations can be found in the lower part of the table.

All in all, the data support our assumption that special offers preferably have prices consisting of multiples of powers of 10, minus a favourite number as defined in  $F(10)$ . One may suppose that the facts of the bargain numbers are directly related to the facts on coins and banknotes discussed earlier in this section. At first sight, bargain prices seem to be prices to which only one coin or banknote has to be added to become a “round” price. However, this generalization is apparently wrong for the Dutch

Table 2  
Distribution of last digits in bargain numbers, arranged according to frequency

Last digit	Frequency
5	740
9	730
8	63
7	16
4	4
2	4
6	3
3	2
1	1
Total	1563

Table 3

Distribution of the last two digits in bargain numbers ending with 5, arranged according to frequency

Last two digits	Frequency
95	569
75	58
25	30
15	23
45	17
85	13
35	12
55	8
65	7
05	3
Total	740

situation, there being bargain price numbers ending in . . .8 in the absence of coins and banknotes with denominations of 2, 20, 200, etc.

#### 4. The cognitive status of sets of favourite numbers

In sections 2 and 3 we argued for the special position of  $F(10)$  in everyday language and in everyday life. Now we come to the question of how the facts can be explained.

It is easy to see that the arithmetic involved in the use of elements of  $F(10)$  is very simple. Manipulation of numbers which are elements of  $F(10)$ , the deduction of “bargain numbers” and the use of numbers in approximative expressions must belong to the most easy arithmetic operations there are. No special training is needed to recognize “bargain numbers” and to connect them to numbers that are more round. People simply seem to *know* what differences are allowed between members of approximative expressions. And it would be peculiar to presume that denominations of currencies all over the world do not reflect the need to handle money and to calculate with amounts of money in the easiest possible way. We do not know of any case where monetary authorities made an appeal to principles to justify their choice of monetary denominations, other than intuitions about “ease of use”. It is therefore an important question what kind of knowledge it is that facilitates this position of  $F(10)$  in ordinary language use and in everyday life.

Looking for explanations for the *principle of favourite numbers* we turned to the psychology of arithmetic learning, to the history of numbers and arithmetic, to linguistics and to cognitive science.

Our exploration of the literature on learning principles of arithmetical

knowledge was rather disappointing. We consulted Piaget (1965), Gelman and Gallistel (1978), Gallistel and Gelman (1992), Nesher and Kilpatrick (1990), the collection of articles in Campbell (1992) and references given there. In this literature we did not find any awareness that the learning of the concepts of “half” and “double” is important for the manipulation of magnitudes on an elementary level. Even in the well-known studies of Piaget and his followers on the child’s conception of number, we could not find any awareness of the special role of the concepts “half” and “double”. There are no traces of a supposition that  $F(10)$  is ever to be learned as such. Occasionally, however, we did find a rather implicit use of the concepts “half” and “double” in descriptions of educational materials, for example Goffree (1992) and Flexer (1986). Of course, sometimes one may find traces of an awareness that half and double are “easy”, but we did not find theoretical explanations (see, for example, Hart, 1981, pp. 69, 76, 89). Calculating with the elements “half” and “double” within a given magnitude seems not to be learned in the explicit sense of the word. Of course, the words for these concepts have to be learned, but they do not seem to be part of any regular arithmetic education. In the literature on the learning process of mathematical abilities, full attention is paid to the learning of numbers, counting and computations like addition, subtraction, multiplication and division. The learning of “magnitudes”, or “orders of magnitude”, however, does not yet seem to have received much attention from researchers.

A look into the results of historical research was more rewarding. Up until the Renaissance period, people used to be able to make multiplications with 2 (i.e., they were able to double a given quantity), before they could make multiplications in the way people compute them nowadays.

Multiplication was a succession of *duplications*. In the same way division was reduced to *mediation*, i.e., halving a number. (Dantzig, 1962, p. 26)

Menninger states the following in his discussion of doing arithmetic on a counting board:

The counting board directly promotes doubling and halving, because these procedures can be carried out without any computation at all, simply by adding or removing counters according to a fixed rule [. . .] Doubling and halving are primitive operations, early forms of multiplication and division. They occur even in ancient Egyptian computations [. . .] and are thought to be still in use among Russian peasants. (1969, p. 359)

People have evidently been accustomed for a very long time to make use of many different magnitudes which are not related to each other in a decimal or otherwise regular way. Weights and surfaces, distances and time-spans have been measured and still are measured in all kinds of magnitude, which are not decimally related (see Verhoeff, 1983). Of course, the relation between the magnitudes which are in use is mathematically expressible but not on a regular decimal basis. All these different mag-

nitudes answer the need to measure different things in different ways (see Ifrah, 1987, esp. ch. 2). This suggests that the principle of calculating with magnitudes is independent of the number of digital units within this magnitude. The magnitude does not even need to be enumerable. It might be: the length of this particular stick, the content of this particular bucket, the value of this particular piece of gold. Despite the absence of enumerability, a special kind of computation-like operation with these magnitudes is possible. One can always take something, divide it into two pieces, bring those pieces together again, etc., without having to think of these actions in terms of a numerical system. The principle of the favourite numbers may therefore turn out to be a special form of a *principle of favourite quantities*.

Before we investigate this more basic principle, let us first turn to linguistics. In her analysis of approximative expressions Channell (1980) opposes a truth-semantic approach. In the footsteps of Lakoff, Sadock and Wachtel (Channell 1980, p. 461), she is looking for a pragmatic and/or sociolinguistic framework to describe the regularities in the use of numbers in approximative expressions. In our opinion, however, it does not make sense to assume that the difference rule or the rule of sequence is of a linguistic nature. The *principle of favourite numbers* does not look like a rule of grammar or like any pragmatic or sociolinguistic principle of any known kind. And even if it did function as some kind of semantic or pragmatic filter in constructions with two numerals, it would be absurd to think that the same linguistic principle defines the denominations of bank notes. The values of a given currency are not determined by a rule of the language. The knowledge of the *principle of favourite numbers* cannot be lexical either. The rule generalizes over a set of numbers and thus over a set of number words. All this makes it improbable that the principle is an expression of a rule of language.

What comes to mind is that the *principle of favourite numbers*, in one way or another, might be an aspect of (non-linguistic) human cognition, and that in this respect it would resemble the kind of knowledge that linguists investigate, even though this specific knowledge is not of a linguistic nature. The *principle of favourite numbers* out of which  $F(10)$  may be deduced might make up a part of the basic arithmetical competence of the human mind. This seems to be the only way to explain the occurrence of the same numerical phenomena in the different languages and domains in which we saw them operate.

The *principle of favourite numbers* resembles the modules of Universal Grammar in the Chomskian sense. It may be a cognitive module comparable with Government theory, Case theory and the like (Chomsky, 1981, p. 135). Each of these theories consists of a set of principles that function as a kind of filter that discriminates between “possible” and “impossible” expressions. Note that the *principle of favourite numbers* is independent of a specific choice of the base of numeration. The existence of the digits 1 to 9

makes the system decimal. The set of these digits can therefore be regarded as a parameter determined by the culture in which the principle is used. In linguistic theory it is held that parameters are the *learned* part of the cognitive elements that make up the Universal Grammar. It is clear from historical research that the set of the digits 1 to 9 is not universal. Our claim, however, is that “double” and “half” are universal across number systems, and that at least the concept of “magnitude” as the basis for some computational operations is also universal.

It is a prevailing view in the field of cognitive numerical research that computing abilities are organized in separate modules.<sup>9</sup> We refer to the survey article of Dehaene and the other articles in the special issue of *Cognition* 44 (1992). Dehaene and Mehler (1992: 2) state that a “a minimal set of principles of elementary arithmetic might be universal, possibly regardless of culture, language or level of education”. The authors refer to the preverbal system of counting and arithmetical reasoning that humans share with a broad range of animals, and to the distance effect in numerical comparison.<sup>10</sup>

We think that our findings support the supposition of Dehaene (1992, pp. 30–32) that in a model for “numerical cognition” there should be a so-called *analogue magnitude code*. We are now in a position to give some substance to this claim. Dehaene’s analogue magnitude code apparently comprises the knowledge of quantities and magnitudes (numerical and non-numerical alike) and some procedures for working with these quantities. Dehaene mentions the procedures “comparing” and “estimating”, and we add to these “doubling” and “halving”. They are ways of manipulating quantities that may be characterized as analogical. There are no numerical abilities presupposed; instead, these procedures can be carried out by performing only one action – if desired an endless number of times – with only one object (or abstraction), or by “measuring” it on a continuous scale. On the other hand, counting, adding and subtracting, and “higher” forms of multiplication and division, are made possible by the concept of “digits”.

According to Dehaene the analogue magnitude code can operate independently of other subsystems of arithmetical cognition. One has to conceive of a cognitive computational system without digits, operating on all kinds of quantities. Such a subsystem does not presuppose the knowledge of

<sup>9</sup> Another view is taken by Hurford (1987, p. 3). He assumes that “the number faculty largely emerges through the interaction of central features of the language faculty with other cognitive capacities” [...] and that it “is therefore not necessary to postulate an autonomous ‘faculty of number’ as a separate module of mind”. It seems likely, however, that these “other cognitive capacities” could turn up as principles like the principle of favourite numbers which are candidates for a separate “module of mind” in the sense of Chomsky (1981) or Fodor (1982) see also Chomsky (1980, p. 249).

<sup>10</sup> This refers to the phenomenon that comparison times decrease with increasing distance between operands. This phenomenon has been found with subjects from different linguistic communities; it resists extensive training, and it is already present in 6-year-old children.

numbers. It will not be able to count, to add, to subtract or to perform other “higher” arithmetical procedures. These arithmetical abilities are directly related to the existence of numbers. But the subsystem can manipulate all kinds of quantities: it can halve and double them. This is why we can generalize the *principle of favourite numbers* to a *principle of favourite quantities*, extending it to quantities other than numerical. The principle might be stated as follows.

Every quantity comprises a small set of favourite quantities. This set consists of

- the quantity itself
- the half, the double, and half of half of this quantity.

We think that this principle, in connection with the existence of the integer powers of base 10, the digits 1 to 9 which are presupposed by this base and the magnitude code, explains the existence of  $F(10)$ . It explains the “ease” of its use in everyday life, and its universality. And the principle also explains, at least partly, the approximative expressions we found in the data.

This explanation is in line with an assumption made in the literature on the psychology of arithmetic learning that there is

A distinction between *logico-physical understanding*, which results from thinking about either procedures applied to physical objects or spatio-physical transformations of these objects, and *logico-mathematical understanding*, which results from thinking applied to either procedures or transformations dealing with mathematical objects. (Bergeron and Herscovics, 1990, p. 36)

The set of favourite numbers would reflect a common ground between the realm of the cognitive representation of physical objects and the realm of the cognitive representation of mathematical objects; magnitude numbers seem to function and be able to be manipulated as physical objects.

It is difficult to find direct empirical data to substantiate or falsify the claim about the universality of the *principle of favourite quantities*. As far as we know, duodecimal or sexagesimal number systems exist only a very rudimentary form these days. We do not know of any cultures in which other number systems are alive, let alone whether relevant data related to such cultures would be available. This makes it impossible to systematically search for numerical phenomena of approximative expressions, denominations of coins and bargain prices in cultures with non-decimal numerical systems. But what we have seems to be in favour of our claim. There are duodecimal traces in contemporary time measuring. We would like to point to the partitioning of the year into 12 months, and into periods of 3 or 6 months (and the absence of “natural” periods of 4, or 5 months and the like); and to the partitioning of watch faces into periods of 15 minutes (or 3 hours) as indications of the presence of the predicted favourite numbers in a



duodecimal or hexagesimal number system. Perhaps the pair [12, 15], already mentioned in section 2, is an acceptable approximate expression after all. Possibly [12] functions as the magnitude here. If that is the case, the interval would be correct: [15] deviates by 1/4 times the magnitude from [12], the other number in the pair. The existence of the threepence and the sixpence in the former British currency, which was partly duodecimal, fits in the pattern too, 3 and 6 being favourite numbers in a duodecimal number system. Although this evidence is rather incidental, we consider the available data to be in favour of the parameter status of the base of numeration in the *principle of favourite quantities*.

The evidence we have presented thus far has to do with the applicability of the *principle of favourite quantities* to numerical cases and not with non-numerical ones. Do we, therefore, have to limit ourselves to the weaker *principle of favourite numbers*, being the numerical specification of it? Are there examples of the usage of *favourite quantities* (as opposed to *favourite numbers*)? Until now, we have not discussed expressions with estimation pairs of time indications. It is in this field that we will find cases in which a quantity of time seems to be partitioned into two and four parts in a non-numerical way. The reader will find this argument below.

In Dutch it is possible to express time estimations more or less in the same way as was demonstrated in section 2 for quantity estimations in general. The examples (1) we found in our corpora:

- (1) een uur of 2, 3  
approximately 2h00, 3h00
- elf uur, half twaalf  
approximately 11h00, 11h30
- een uur of vijf, kwart over vijf  
approximately 5h00, 5h15
- een uur of zes, kwart over zes  
approximately 6h00, 6h15
- kwart voor tien, tien uur  
approximately 9h45, 10h00

[. . .] en die was zondags was die *twaalf*, *een* uur waren die open  
(lit. [. . .] and on sundays, it was at twelve, one o'clock they were open)

The last case we already mentioned in section 2. Although it shows two restarts of the speaker, not uncommon in spoken utterances, it offers a clear and convincing example of a time-measuring approximative expression.

The examples (1) seem to be governed by the same rules that govern all two-number approximative expressions:

- The ordering rule:  $x$  has to be smaller than  $y$ .
- The roundness rule:  $x$  or  $y$  has to be “round”.
- The difference rule: the interval between  $x$  and  $y$  has to be in  $F$ .

However, in time measuring  $x$  is not *smaller* than  $y$ , but *earlier*. In the expressions (1) the first time indication indicates a point in time that is earlier than the second. That the “number line” of time measuring has a directionality of its own is shown by the last example in (1). There being only one magnitude “hour”, the interval between the two points in time must equal “one hour”, “half an hour” or “a quarter of an hour”. Our corpora do not contain a case where the interval equals “two hours”. A sentence like “She will come at 4, 6 o’clock” is apparently inadmissible. Otherwise the set of rules explains the facts in the cases (1). The facts in (1) give some independent evidence to the claim that a non-numerical system of time estimation makes use of a *principle of favourite quantities*.

Note that the Dutch language has special expressions for the half hours and the quarters of an hour. These points in time are not indicated with the normal numerical expressions for  $\frac{1}{2}$  and  $\frac{1}{4}$ , although the expressions clearly are related. For example, the point in time “5h15” is called *kwart over vijf* (lit. quarter over five), not *vijf-en-eeen-kwart uur* (lit. five and a quarter hour). In the same way, the expression for 5h30 is not *vijf-en-eeen-half uur* (lit. five and a half hour), but *half zes* (lit. half six).

To conclude, the competence of using different non-numerical magnitudes and of doubling and halving these magnitudes makes it possible to do some simple arithmetic of quantities without a set of digits.

Does this prove that a computing ability as reflected in the *principle of favourite quantities* is prior to the invention of counting? And does it prove that this knowledge is innate? The answer to the first question is that calculating with “magnitudes”, “halves” and “doubles” is independent of any counting system. The evidence does not decide on the issue of the priority of the two systems, although multiplication, division and estimation in their simple analogue form seem to be older than this kind of number manipulation in the digital form. As far as the issue of innateness is concerned, on the basis of the evidence we presented, it seems hard to deny that the knowledge of favourite quantities is universal and easy to apply. But without further research it is impossible to decide whether notions like “half” or “double” are innate or “just” easy to learn.

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