On the archetypal origin of the concept of matrix.

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In this paper we study the mathematical object matrix and compare its historical and psychological origins.

The literature on history of mathematics tells that this science started with the study of numbers and shapes, giving rise to the fields of arithmetic and geometry. The historical approach has been leading to an interpretation of algebraic structures as secondary concepts, derived from systems of equations representing numerical or geometrical constrains.

Our epistemological view supports the argument that matrices originate from a especial form of tabulation, which had and still have a widespread use, appearing in different sciences and forms of expression. Originally, these tables are not numerical and became matrices when numbers were introduced in them. This view leads to the assumption that these algebraic structures are just as fundamental as numbers and figures. Therefore we can assume that algebra is one of the most fundamental domains of mathematics, forming a basic trilogy with the other two fields.

This review gives rise to new interpretations in the physical realm, where observational languages are built *a priori* on suitable mathematical domains. It can show contrasting views between the historical approach and the epistemological view. Observational realism is our guiding epistemology.

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I. INTRODUCTION.

Mathematics comprises the basic fields of number, geometry, algebra and their interrelations. The foundations of arithmetic and geometry had been, together or in separation, extensively considered in literature. But the foundations of algebra are still obscure. It is frequently considered as a sort of hybrid field, originating from the other two.

A reason for misunderstanding is in the somehow arbitrary usage of the word algebra which has been adopted in very different senses. This name is used in the arithmetic context, when a quantity is not explicitly shown in an equation. It is also used in the geometrical domain, when classes of figures are expressed by means of constrains between variables, this is the field of the algebraic geometry. The same holds for invariance in the form of equations. Numerical analysis of geometry also produced formulas named algorithms, which involve variables and for this reason are named algebraic forms. The representation of series also adopts these formulas. And the study of vectors as geometrical elements gave rise to the field of linear algebra. So, the name algebra became associated to any kind of mathematical representation involving variables. And this arbitrariness does not help to make a sharp distinction between algebraic concepts, numbers and geometrical elements, which will be shown here.

In this work, we argue that the reason why algebra is a field of its own is because its elements are matrices. In other words, matrix is the most fundamental concept of algebra. And the historical view on the origin of these concepts adds to this mentioned obscurity, by suggesting that algebraic structures, such as numerical matrices, originated in the analysis of systems of equations. These systems are sets of independent equations but connected by common variables in them. Literature on this topic localizes the appearance of matrices in early Chinese texts from around 300 BC, dealing with these systems in the context of financial and administrative applications of mathematics [1]. The historian C. Boyer comments that a diagrammatic aspect, which he thought to be a characteristics of matrices, had a natural appeal to the Chinese culture. Apparently, no questions about the origins of these systems of equations emerged for those mathematicians, neither this was discussed by the Greek philosophers, who were familiarized with them, in the same practical context. The usage of matrices seems to have been so driven for practical purposes that its intrinsic significance did not call any attention. Or, perhaps, texts on these topics did not survive.

After nearly two thousand years, in the eighteenth century, strong interest on these structures appeared in the context of systems of equations again. Soon later, analysis and geometrical formulation of complex numbers, vectors and other concepts, forced a reflection on the essences of both mathematics and logic, resulting in an intermingle with them. Then, the old Aristotelian logic, originally applied to usual languages and arithmetics, was reviewed and extended to treat relations between 'abstract elements'. In this way, mathematical concepts became elements of logic and mathematical operations were identified with logic rules. Mathematical structures acquired a proper operational field and the modern field of abstract logic emerged. Operational contexts started to be discovered and organized in subfields named algebras or 'sub-algebras'. The concept of algebra of logic appeared in the nineteenth century, meaning the study of operational domains and their interrelations. Slowly, as we can realize, there happened a logic shift, which focused on the study of operations instead of focusing directly on the properties of the operated elements [2, 3]. The rules got an objective status and started to define the elements, not the other way around, which was the old starting point for logic and other sciences [4, 5]. Why this shift happened, is an interesting question for epistemology and it involves physics, mathematics and their relation. Mathematicians involved with this shift were in close collaboration with physicists, if not both in one. A natural consequence of this program was the elimination of innate ideas or any kind of *a priori* characterization for the elements of the logic and also of science.

On one hand, a great portion of these developments happened around the concept of matrix, but not only. The resurgence of matrices in the modern period did not liberate them from systems of numbers, in spite of the strangeness of these beliefs. It was accepted that matrices should not have a fundamental status, being logical elements derived or defined by their operations. Otherwise, they could be built from operations, involving other algebraic forms such as vectors or so. And matrices naturally entered the domain of the abstract logic. In a sense, there was no other alternative to understand the formation of these structures. On the other hand, theoretical physics had plenty of results which could not be understood and interpreted in connection with the usual physical space. This involved, between others, situations which required representation in complex spaces. This all revealed the existence of strange domains of interconnection between physics and mathematics. Moreover, a shift in the realm of physics had also emerged. Interactions, which before were relations, became objects in itself, in spite of all the strangeness in them, such the absence of medium and other paradoxes connected with the electromagnetic world.

The motivation behind the logic shift was probably to safeguard physics. The idea was that, if mathematics could be liberated from ontological responsibilities, concerning these non-real and structural elements, physics could be safe from metaphysics too. Nowadays, there is a trend in the theoretical physical community to think that these structures could even be primary in physics. But this is not in the sense of an observational primacy, as we considered it here. The other interpretation is that physical reality derives from an abstract mathematical realm.

In this paper, we study matrices. We follow arguments from the mathematician Friedrich Frege (1848 – 1925), in his clarification of the conceptual nature of numbers, [6, 7] to compare the origins of numbers and matrices. We find that there is some similarity in these origins but they are very different mathematical objects. In our opinion, this difference makes from arithmetics and algebra two separated sources of physical expressions, which together with geometry can produce broader observational domains. This work represents a step in the program of the observational realism, an epistemology designed to tackle the evolution of the physical observer by focusing on the observational languages and separating them inside the body of physical theories.

II. A FEW ELEMENTS IN THE HISTORY OF THE SCIENCE OF NUMBERS.

The historical origin of numbers, arithmetics and geometry are known to be in the very early days, when other languages emerged. Information from the very old past is not acquired through written registers, but interpreted from pictures, diagrams and symbolic organizations in burial and other archaeological means. Registers in tablets were done later and more elaborated mathematical knowledge was already present around 5000 BC.

Around 3000 BC, Egyptians had a kind of arithmetics, although different from ours in terms of representation of numbers. And around 2000 BC, civilizations in Mesopotamia had devices for calculation, the abacus, and written registers of their knowledge. Sumerian, Akkadian and Babylonian peoples had developed geometry and numerical analysis. Around the same time, in India, practical geometry was in use and the science being developed. Between 800 till 200 BC, they wrote religious texts, where they have shown to be well familiarized with simple equations, quadric and cubic equations originating from geometry, rudiments of limits and ideas of infinity and eternity in connection with cosmological subjects. And around the same time, in central America, there were sophisticated sciences of geometry, numbers, also applied to astronomy and construction.

There are Chinese documents, localized around 300 BC or much older, showing knowledge on the subject of systems of equations. Methods for solution of these systems were clearly presented in form of diagrams, later named matrices [1]. In Greece, the use of logical methods were implicitly adopted much before Aristotle's grounds of a science of logic. Thales from Miletus $(624 - 548BC)$ gave great contributions to mathematics, especially to geometry. And Pythagoras of Samos $(570 - 495BC)$ was the grounder of a school in which numbers received great symbolic importance. The number one had a special status, being generator of others. The idea that numbers could comprise a special reality, emerged at that time. Socrates $(469 - 399BC)$ seems to have been the first who discussed the possibility that we humans had an innate tendency to grasp the meanings of certain ideas, considering numbers. This was told by Plato in his dialogs *Meno* and *Phaedo*.

During these early periods, the concept of number was in development. Different approaches and applications of the original integers, generated new kinds of numbers. There emerged the rational, irrationals, infinitesimal, reals, non-reals and several other subclassifications of these numbers, with specific properties. These discoveries were mostly pushed by the science of measurements, practical geometry and its numerical analysis. Since its emergence the science of the numbers obtained a very special status between all the sciences. Numbers always had a certain power of truth in them.

It was with a certain astonishment, for himself and others, that Frege felt the need of a clarification on these elements [6]. He started his inquiry by arguing that nobody, even mathematicians, knew what a number was. In fact, starting to read his book *The Foundations of Arithmetic* the reader enters in a world of bewildering astonishment, before to grasp the strength of his careful reasoning. Trying to understand a representation of imaginary numbers introduced by C. F. Gauss $(177 - 1855)$, Frege disclosed an intellectual vacuum, realizing that there was already a problem with the very familiar real ones. The work of Frege is an exercise of reason, from the initial steps of the acquisition of elemental ideas, to the process of selection, organization, reclassification and then the reach of the complete chain of arguments, revealing all the hidden steps in the method of numeration. And, even more surprising is the fact that he never acknowledged but denied that he was observing his own natural intellectual process of formation of numbers. He rejects the question of 'who' classifies classes to find out numbers. The argument was that, if we all think the same, there is no subject to occupy the 'who': numbers are themselves. This is the 'high degree' of objectivity in numbers, which is reclaimed by present-day Neo Pythagoreans [8–10].

Frege's definition of cardinal number received many criticism by other mathematicians and philosophers [11]. There are different versions, but for the purpose of our study, the one called Frege-Russell definition, described by Bertrand Russell ($1872 - 1970$), can be considered [12]. The definition describes a double classification: "The number of a class is the class of all those classes that are similar to it" .

III. FORMATION OF MATHEMATICAL CONCEPTS, THE NATURE OF THIS PROCESS.

Frege's denial of a subject of the mathematical thinking also contributed to the assumption of a metamathematics, which approaches the formation of mathematical and other concepts as a mathematical topic. The question if mathematics can describe thoughts or if it is the other way around, is a polemic one. If one considers that logic and linguistic, between others, are also sciences of thinking, we find that consciousness is able to think itself in different ways and organize its own thoughts according to different rules, unless one can show that these sciences are just one.

There are different lines of thinking on this subject. The Neo Pythagorean tendency denies a mental nature for mathematical principles. In this view, both physical and psychological existences would ultimately be in tune with these principles.

Cognitive sciences consider thinking as perception, with a physiological or physical realization. The mathematical thinking would be a refined kind of perception, performed by a special system. The existence of different levels of perception, not directly related to sensation [13] would accommodate a meta cognition. The latter is a semi-autonomous kind of process, resulting from the organization or auto-organization of the neural system. The condition of autonomy is not fully clarified, it is sometimes said to be emergent, resulting from a natural hierarchy or neural organization, between other suggestions. In this view, a mathematics of thoughts is a metamathematics, which would describe bio-physical computational systems according to its own auto-generated patterns.

Mathematical models of perception have been proposed [14–16]. And the study of mathematics and physics learning has been an important topic of study in cognitive sciences and neuro psychology [17, 18]. A mathematical treatment of psychological and social phenomena have been proposed and developed [19, 20]. The concept of shared cognition, especially for the case of shared memory, is present in the cognitive sciences [21, 22]. And the concept of collective consciousness, as distributed and shared cognition, can also be defined in this domain. But it involves learning and communication, not being a condition of an innate substratum of ideas.

The idea that the physical world *a priori* is a reality to be discovered and known by an organism, which is a subsystem of this same world, naturally separates and limits this organism from an outside. This separation is in the basis of perception, in the view of the cognitive sciences. And perception develops according to its limits, which are settled by its own formation. Therefore the 'idea that the physical world *a priori* is a reality to be discovered' is not exactly an idea but just a belief, or an illusion of this perceptual system. It is a fact that perception is also formed on common grounds and this could be an argument for this tendency to adjust itself to a common knowledge, generating processes such as meta cognition, metamathematics and rationality from perception. This adjustment would be in the basis of our sciences, as a shared product. In this interpretation, notions such as physical reality and illusion would still not be fully meaningful, due to the inherent limitations of perception.

Another idea, which would contrast with the cognitive approach on consciousness, would be to give to physical world an *a priori* fully significance. This is the view elaborated by Gottfried W. Leibniz (1646-1716). He developed logic, mathematics and physics, was a prominent defender of the existence of innate ideas and suggested the concept of unconsciousness [23]. His position radically differs from the Pythagorean and Neo Pythagorean idealism, for which the mathematical objectivity would be outside reason. He would not be in tune with cognitive sciences either. In spite of considering perception and thinking as processes, he considered principles of reason as a necessity. Leibniz also revived the topic of systems of equations and methods of solution, leading to a resurgence on the study of matrices [1, 24]. Later, a new approach on matrices emerged when physicists started to reveal and model interactions taking place beyond the experimental access of our senses.

At the end of the nineteenth century, psychoanalysis by Sigmund Freud (1856-1939) was established as a theory of consciousness and there after analytical psychology by Carl Gustav Jung (1875-1961) emerged. Both theories worked out the notions of psychic layer, psychic content and psychological composition. Analytical psychology introduced the concept of innate idea in psychology, the concept of function of consciousness and classes of conscious phenomena, between them thinking, language, perception and sensation. Jung extended the Freudian notion of unconsciousness leading to the concept of self and opening to new interpretations of objective knowledge, including archetypal patterns and innate ideas in the psychological realm [25]. Jung and collaborators had also been interested on symbols, numbers, mathematical concepts and languages, their meanings and psychic origins. Wolfgang Pauli (1900 - 1958) collaborated with Jung and gave contributions to the integration between physics and analytical psychology [26, 27]. He investigated the assumption of archetypal roots in physical theories.

Pauli's work did not elaborate on the difference between observation and theory, in the way that this topic was considered by Rudolf Carnap (1891 - 1970). Physical theories comprise more than observational concepts, they include empirical and theoretical languages and laws [28]. The question of the relation between theory and reality in fact only applies to observational sector of theories. The epistemology of the observational realism, adopted in this paper, is based on a combination of Carnap's and Jung's ideas. This epistemology has been adopted in interpretation of physical languages, especially for the case of observational descriptions and laws [29–32].

The physical languages are based on suitable mathematics which also originate from archetypal symbolism. Both sciences have a common origin in the same common self from which the collective forms of thinking emerge. According to analytical psychology, the Pythagorean encounter with ideal existences would not be possible outside the psychic domain. In this tradition of thinking, this collective or innate ground of ideas is in the basis of all human expressions and nothing beyond it could be pointed out by principle. The concept of collective unconsciousness, as emphasized by Jung, is a hypothesis to be supported by its effects, to explain a possible psychic evolution in a collective or in an individual sense. In the deepest level, this evolution would be for each and for all humans and this is the collectivity which mostly characterizes our physical world.

From this view, Frege's inquires on the formation of number could never go beyond the basis of psychological processes involved with the formation of numbers. Reconstructing the concept of number, the elements he used in it should be in the basis of all, not beyond. In other words, classification only can be possible if there are elements to be classified and not the other way around. In the domain of science we are always talking about what is explicitly conscious. In this view, it is also natural to assume that there are sciences more general than others, due to their different collective levels of approach.

Next, we will discuss an example of mathematical object which has origins in what we consider an archetypal pattern. It seems to be present since ancestral times, while concepts and a language connecting it with other elements only emerged much later. This is the case of the matrix, derived from tableaux or tabulation, which is a psychic device extensively used in intellectual activities inside and outside science.

IV. THE OBJECT OF OUR STUDY.

In the mathematical literature, the word table has been used in different senses [33, 34]. It can refer to simple associations of sets of measurements or quantities. This method was extensively adopted before the advent of the calculus, or mathematics of the continuous, generating graphics by inferences and extrapolation, inspiring the formulation of physical laws and leading to the introduction of the mathematical concept of function. But this is not the concept of table which we study here, connected with those mentioned diagrams of systems of equation. Here we will focus on the formation of a pre-numerical element and in its continuation. This is a tendency for tabulating which has been present since the early past.

Frege's number is an index of classification of classes, involving a reclassification which expels any other class which does not fall in this identification. Our element misses this reclassification by integrating class with condition. The nature of the latter opens space for an intrinsic form of systematization between elements. The idea is this basic model of association of two in one. Representation of this object, outside mathematics, can be found in scientific and non-scientific expressions such as totemism, social organization, chess game, genetics and others. And the continuation, which would compare with the natural induction of numbers, is a natural kind of multiplication or tabulation, here generated by the property of 'condition'. But the usual representation of the proliferation of elements, in form of rectangular or squared arrays, is just a matter of convenience, it does not carry in it the meaning of the concept, as we can see by the other examples outside mathematics.

As an example of tabulation, just to compare with our mathematical example, we can consider a construction connecting three different kinds of animals with three different nationalities. This would correspond to three different classes associated to three different conditions on them. Let us consider cows, horses and rabbits as classes, to table with Dutch, Brazilian and Chinese as conditions. This would produce combined elements as such: Dutch cows, Dutch horses and Dutch rabbits, in the first row; Brazilian cows, etc in the second; and similar story in the third line.

$$\begin{pmatrix} dc & dh & dr \\ bc & bh & br \\ cc & ch & cr \end{pmatrix} \tag{1}$$

Above we have all possible associations explicitely indicated. Considering the perceptual aspect of the information, because of the relative size of the picture and the capacity of the human vision to have a global view, this kind of arrangement could be of great interest. However, if one gives values to each of these elements, the information of the associations remains present in the rows and columns which were defined at the beginning, but there is also an extra gain, given by a possible comparison of the values which one had not before. The usefulness of this device would still be related to the human capacity to distinguish separations but still the mathematization of the table was of great importance as a means of storage of knowledge.

Now let us follow the steps in the formation of a matrix, according to the historical view, to highlight this relational aspect we mentioned. We now consider a case of two variables and two equations, to make it simpler. As example, let us consider these two equations below:

$$a + b = n_1 \tag{2}$$

and

$$a - b = n_2, \tag{3}$$

where a and b, are variables and n_1, n_2 are known fixed numbers. The kind of relation expressed in equations (2) and(3) can produce an infinite number of values for one and consequently for the other variable in each equation separately. In principle, there is nothing to indicate that there is a relation between these equations. This is because, considering each equation in separation, these $a's$ and $b's$, can have any value. If one a

has one specific value, the other a can still have infinite values. If one of the $a's$ is fixed, nothing says that the other is fixed too, on the contrary, all indicates that they are not at all related, and the same for the $b's$. In fact, one arithmetic statement is a very strict and closed relation.

To the reader of these equations, the message is that there is one equation and then the other, nothing else. Moreover, any possible order in the readings or thoughts, concerning each equation as a whole, and the two, is completely irrelevant. This means that the fact that there is one a in one equation and the same letter in the other equation, nothing says about their possible sameness, unless one registers this fact. So, a number or a set of equations does not make them a system. The latter asks a different condition, which is still missing. For them being a system, there must be a reason, a purpose or condition explicitely given, to relate the equations. This is clearly a very different situation, when comparing with the arithmetics involving the pairs a and b. One does not actually intend to indicate that the two equations are added or subtracted, or operated in any sense. Which would be the sense of the equations, considering them as a row? If this was the case, they could be just written in the same line, with indications to make clear what should be done. This is not the case. If there should be a purpose in a relation between equations, which purpose would this be?

So, again, history says that matrices were derived from systems of equations. But then, which was the condition that produced systematization of equations in the first place? This is not explained in the books of history. Is there some kind of hidden message or previous agreement behind the two arithmetic statements, which says that we should consider letters and variables as the one and the same thing? The answer is negative, otherwise one could take any a or b from other equations written somewhere else and consider them all the same. This would not have sense at all. Is their proximity in the paper a signal to connect them? Also not, for similar reason as the previous one. There seems to be that there is no other argument than to assume that this we call systematization exists and is a kind of relation in itself, independent from the arithmetic one. And what does this systematization mean? Is this tabulation a form of systematization? We think that yes, this is the case. The tabulation, in this case, just reflects the mentioned proliferation of elements, from an original double object. And this guides the interpretation of this set of equations, from the beginning.

In other words, this table is the representation of duality-proliferation. We then arrived to the conclusion that matrices did not result from systems of equations but are the inherent systematization of the equations. When arithmetics appeared, tables were already in the world and were tables which caught the equations, not the other way around. This kind of systematization could also have been explored and developed independently of numbers, as an independent element of expression.

V. MATRICES AND OPERATIONS.

From the view that matrices derived from systems of equations, studies and classifications of their properties developed with the sole interest to solve the systems of variables. This study had many practical purposes based on numerical calculation and analysis. The interests were to know in which conditions the equations were independent or not, to find methods to reduce the amount and time of calculations, and others which only involved the internal structural properties of the matrices. This view did not lead to any question or insight on the relational meaning of its layers and interests on a functional position of matrices outside the closed realm of their equations.

The next step was the analysis of this field and the construction of its integration with the other two, which after all were reformed as well. One was the vast field of numbers, which had new sub fields to be organized under the new theory of sets. The other comprised the geometries, combining the Euclidean with the new discovered ones. Due to the necessity of an inter-adaptation between the fields, matrices got an operational treatment specially designed for this purpose.

Addition and subtraction can be defined directly from the combination of equivalent systems of equation on the same variables, via operations on their variables, not involving the original matrix structure in any sense. This would be given by operations on elements inside the matrices. As an example we could have the addition of three matrices [A]+[B]=[C], as below:

$$[A] + [B] = [A + B], \tag{4}$$

or

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & -b_{12} \\ -b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \tag{5}$$

where

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} (a_{11} + b_{11}) & (a_{12} - b_{12}) \\ (a_{21} - b_{21}) & (-a_{22} + b_{22}) \end{pmatrix}, \tag{6}$$

where the elements a_{ij}, b_{ij} and c_{ij}, for $i, j = [1, ..n]$ are just positive numbers or quantities to which the arithmetic operations hold.

The relevant operations, which form the new algebra, are the product of matrices and their analysis in vectorial components. In this way, matrices were broken in suitable pieces, to be reconstructed again. The concept of vectorial space, as many others in mathematics, was designed to serve as mediator between mathematical fields [35, 36].

Traditionally, this has been one of the main objectives of mathematical research [37]. And the latter had been passing through reformulation, due to pressures from the natural sciences, a natural process of scientific interrelation.

For nxn matrices [A] , [B] and [D], the multiplication is defined as follows:

$$[A][B] = [D], \tag{7}$$

where, similarly, we would have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \tag{8}$$

which in fact is a very different operation as it can be seen through the results:

$$\begin{pmatrix} d_{11} & d_{12} & .. \\ d_{21} & d_{22} & .. \\ . & . & .. \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + .. & a_{11}b_{12} + a_{12}b_{22} + .. & .. \\ a_{21}b_{11} + a_{22}b_{21} + .. & a_{21}b_{12} + a_{22}b_{22} + .. & .. \\ . & . & .. \end{pmatrix}. \tag{9}$$

From this general operation, two particular cases can be defined. One is a product of vectors, given by:

$$\begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & .. \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & .. \\ a_{21}b_{11} & a_{21}b_{12} & .. \\ . & . & .. \end{pmatrix}. \tag{10}$$

The other is the so called scalar product, given by:

$$\begin{pmatrix} a_{11} & a_{12} & .. \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ . \\ . \end{pmatrix} = (n)_{1x1}, \tag{11}$$

where n in this case is a number or scalar element. Since number and matrix are different concepts, we must stress that a number n and a 1x1 matrix $(n)_{1x1}$ are not the same concept unless this is indicated, as an extra requirement.

VI. FINAL COMMENTS.

Matrices express systematization. In the physical realm, local and non-local interferences and interactions have been considered as the only expressions of physical influence on matter. Physical systematization is considered as a result of these influences under general rules of conservation only. But the matrix form of systematization can enter observational physics as an *a priori* kind of relation, not resulting from physical influence or interaction. This is an argument which deserves a careful investigation in physics interpretation, especially in the domain of the quantum physics.

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