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APPROXIMATION OF B-CONTINUOUS AND B-DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS DEFINED BY FINITE SUM

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Abstract. In this paper we start from a class of linear and positive operators defined by finite sum. We consider the associated GBS operators and we give an approximation of B-continuous and B-differentiable functions by these operators. Through particular cases, we obtain statement verified by the GBS operators of Bernstein, Stancu, Schurer and Schurer-Stancu type.

1. Introduction

In this section, we recall some notions and results which we will use in this article.

In the following, let X and Y be real intervals. A function $f : X \times Y$ is called B-continuous function in $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x,y),(x_0,y_0)] = 0.$$

Here $\Delta f[(x,y),(x_0,y_0)] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ denotes a so-called mixed difference of f.

The definition of B-continuity was introduced by K. Bögel in the paper [9] and [10].

A function $f : X \times Y \to \mathbb{R}$ is called B-differentiable function in $(x_0, y_0) \in X \times Y$ if it exists and if the limit is finite

$$\lim_{(x,y)\to(x_0,y_0)} = \frac{\Delta f[(x,y),(x_0,y_0)]}{(x-x_0)(y-y_0)}.$$

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This limit is named the B-differential of f in the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if there exists $K > 0$ such that

$$|\Delta f[(x, y), (s, t)]| \le K$$

for any $(x, y), (s, t) \in X \times Y$.

We shall use the function sets

$$B(X \times Y) = \{f \mid f : X \times Y \to \mathbb{R},\ f \text{ bounded on } X \times Y\}$$

with the usual sup-norm $\|\cdot\|_\infty$,

$$B_b(X \times Y) = \{f \mid f : X \times Y \to \mathbb{R},\ f\ B\text{-bounded on } X \times Y\}$$

and we set $\|f\|_B = \sup\limits_{(x,y),(s,t)\in X\times Y} |\Delta f[(x, y), (s, t)]|$ where $f \in B_b(X \times Y)$,

$$C_b(X \times Y) = \{f \mid f : X \times Y \to \mathbb{R},\ f\ B\text{-continuous on } X \times Y\}$$

and

$$D_b(X \times Y) = \{f \mid f : X \times Y \to \mathbb{R},\ f\ B\text{-differentiable on } X \times Y\}.$$

Let $f \in B_b(X \times Y)$. The function $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\{|\Delta f[(x, y), (s, t)]| \mid |x - s| \le \delta_1, |y - t| \le \delta_2\} \tag{1.1}$$

for any $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$ is called the mixed modulus of smoothness.

For other information, see the following papers: [1], [3], [15] and [19].

Let the functions test $e_{ij} : X \times Y \to \mathbb{R}$, $e_{ij}(x, y) = x^i y^j$ for any $(x, y) \in X \times Y$, where $i, j \in \mathbb{N}$.

The inequality of Corollary 5 from [4], in the condition of (1.2), becomes (1.3) inequality. The (1.4) inequality is demonstrated in [17].

Theorem 1.1. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Supposing that the operator L has the property*

$$\left(L(\cdot - x)^{2i}(* - y)^{2j}\right)(x, y) = \left(L(\cdot - x)^{2i}\right)(x, y)\left(L(* - y)^{2j}\right)(x, y) \tag{1.2}$$

for any $(x,y) \in X \times Y$ *and any* $i,j \in \{1,2\}$, *where "$\cdot$" and "$*$" stand for the first and second variable.*

(i) *For any* $f \in C_b(X \times Y)$, *any* $(x,y) \in X \times Y$ *and any* $\delta_1, \delta_2 > 0$, *we have*

$$(1.3)\quad |f(x,y) - (ULf)(x,y)| \le |f(x,y)||1 - (Le_{00})(x,y)| + \Big[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)} + \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}\Big]\omega_{\text{mixed}}(f;\delta_1,\delta_2).$$

(ii) *For any* $f \in D_b(X \times Y)$ *with* $D_B f \in B(X \times Y)$, *any* $(x,y) \in X \times Y$ *and any* $\delta_1, \delta_2 > 0$, *we have*

$$(1.4)\quad |f(x,y) - (ULf)(x,y)| \le |f(x,y)||1 - (Le_{00})(x,y)| + 3\|D_B f\|_\infty \sqrt{(L(\cdot - x)^2(x,y)(L(* - y)^2)(x,y)} + \Big[\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)} + \delta_1^{-1}\sqrt{(L(\cdot - x)^4)(x,y)(L(* - y)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^4)(x,y)} + \delta_1^{-1}\delta_2^{-1}(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)\Big]\omega_{\text{mixed}}(D_B f;\delta_1,\delta_2).$$

The following Korovkin type theorem for convergence of B-continuous functions is due to C. Badea, I. Badea and H. H. Gonska (see [2]).

Theorem 1.2. *Let* $(L_{m,n})_{m,n\ge 1}$ *be a sequence of linear positive bivariate operators,* $L_{m,n} : C_b([a,b] \times [c,d]) \to B([a,b] \times [c,d])$, $m,n \in \mathbb{N}$, $m \ne 0$ *and* $n \ne 0$. *If*

(i) $(L_{m,n}e_{00})(x,y) = 1$,

(ii) $(L_{m,n}e_{10})(x,y) = x + u_{m,n}(x,y)$,

(iii) $(L_{m,n}e_{01})(x,y) = y + v_{m,n}(x,y)$,

(iv) $(L_{m,n}(e_{20} + e_{02}))(x,y) = x^2 + y^2 + w_{m,n}(x,y)$

for any $(x,y) \in [a,b] \times [c,d]$, *any non zero natural number* m, n *and*

(v) $\lim\limits_{m,n\to\infty} u_{m,n}(x,y) = \lim\limits_{m,n\to\infty} v_{m,n}(x,y) = \lim\limits_{m,n\to\infty} w_{m,n}(x,y) = 0$

uniformly on $[a,b] \times [c,d]$, *then the sequence* $(UL_{m,n})_{m,n\ge 1}$ *converge to* f, *uniformly on* $[a,b] \times [c,d]$ *for any function* $f \in C_b([a,b] \times [c,d])$.

2. Preliminaries

Let $I, J, K \subset \mathbb{R}$ intervals, $J \subset K$ and $I \cap J \neq \varnothing$. For any non zero natural number m, we consider the sequence of nodes $\left((x_{m,k})_{k=\overline{0,m}}\right)_{m\geq 1}$ such that $x_{m,k} \in I \cap J$, $k \in \{0, 1, \dots, m\}$ and the functions $p^*_{m,k} : K \to \mathbb{R}$ with the property that $p^*_{m,k}(x) \geq 0$ for any $x \in J$ and $k \in \{0, 1, \dots, m\}$.

Definition 2.1. Let m be a non zero natural number. Define the operator $L^*_m : E(I) \to F(K)$ by

$$\left(L^*_m f\right)(x) = \sum_{k=0}^{m} p^*_{m,k}(x) f(x_{m,k}) \tag{2.1}$$

for any function $f \in E(I)$ and any $x \in K$, where $E(I)$ and $F(K)$ are subsets of the set of real function defined on I, respectively on K.

Proposition 2.1. *The operator* $(L^*_m)_{m\geq 1}$ *are linear and positive on* $E(I \cap J)$.

Proof. The proof follows immediately. □

In the following, we suppose that for any function $f \in C(I)$, we have

$$\lim_{m\to\infty} \left(L^*_m f\right)(x) = f(x) \tag{2.2}$$

uniformly on $I \cap J$ and

$$\left(L^*_m e_0\right)(x) = 1 \tag{2.3}$$

for any $x \in K$ and any non zero natural number m.

Definition 2.2. Let m and n be non zero natural numbers. The operator $L^*_{m,n} : E(I \times I) \to F(K \times K)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in K \times K$ by

$$\left(L^*_{m,n} f\right)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p^*_{m,k}(x) p^*_{n,j}(y) f(x_{m,k}, y_{n,j}) \tag{2.4}$$

is named the bivariate operator of L^* type.

Proposition 2.2. *The operator* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *are linear and positive on* $E\left[(I \times I) \cap (J \times J)\right]$.

Proof. The proof follows immediately. □

Definition 2.3. Let m and n be a non zero natural numbers. The operator $UL^*_{m,n} : E(I \times I) \to F(K \times K)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in K \times K$ by

$$(2.5)\quad \left(UL^*_{m,n} f\right)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p^*_{m,k}(x) p^*_{n,j}(y) [f(x_{m,k}, y) + f(x, x_{n,j}) - f(x_{m,k}, x_{n,j})]$$

is named GBS operator of L^* type.

3. Main Results

Lemma 3.1. *For any non zero natural numbers m, n and any $(x, y) \in K \times K$*

$$(3.1)\quad \left(L^*_{m,n}(\cdot - x)^{2i}(* - y)^{2j}\right)(x, y) = \left(L^*_m(\cdot - x)^{2i}\right)(x) \left(L^*_n(* - y)^{2j}\right)(y)$$

takes place.

Proof. We have

$$\begin{aligned}
\left(L^*_{m,n}(\cdot - x)^{2i}(* - y)^{2j}\right)(x, y) &= \sum_{k=0}^{m} \sum_{j=0}^{n} p^*_{m,k}(x) p^*_{n,j}(y) (x_{m,k} - x)^{2i} (x_{n,j} - y)^{2j} \\
&= \sum_{k=0}^{m} p^*_{m,k}(x) (x_{m,k} - x)^{2i} \sum_{j=0}^{n} p^*_{n,j}(y) (x_{n,j} - y)^{2j} \\
&= 1 \left(L^*_m(\cdot - x)^{2i}\right)(x) \left(L^*_n(* - y)^{2j}\right)(y),
\end{aligned}$$

so (3.1) takes place. □

Lemma 3.2. *The operators $\left(L^*_{m,n}\right)_{m,n \geq 1}$ verify*

$$(3.2)\quad \left(L^*_{m,n} e_{00}\right)(x, y) = 1,$$

$$(3.3)\quad \left(L^*_{m,n} e_{10}\right)(x, y) = x + u_{m,n}(x, y),$$

$$(3.4)\quad \left(L^*_{m,n} e_{01}\right)(x, y) = y + v_{m,n}(x, y),$$

$$(3.5)\quad \left(L^*_{m,n}(e_{20} + e_{02})\right)(x, y) = x^2 + y^2 + w_{m,n}(x, y)$$

for any $(x, y) \in (I \times I) \cap (J \times J)$, *any non zero natural numbers* m, n *and*

$$\lim_{m,n\to\infty} u_{m,n}(x, y) = \lim_{m,n\to\infty} v_{m,n}(x, y) = \lim_{m,n\to\infty} w_{m,n}(x, y) = 0 \quad (3.6)$$

uniformly on $(I \times I) \cap (J \times J)$.

Proof. Applying Lemma 3.1, we have

$$\left(L^*_{m,n}e_{00}\right)(x, y) = \left(L^*_m e_0\right)(x) \left(L^*_n e_0\right)(y)$$

and taking (2.1) into account, it results (3.2). From (2.2), by Bohman-Korovkin theorem, it results that the functions $u_m, w_m : I \cap J \to \mathbb{R}$ exist such that

$$\left(L^*_m e_1\right)(x) = x + u_m(x), \quad (3.7)$$

$$\left(L^*_m e_2\right)(x) = x^2 + w_m(x) \quad (3.8)$$

for any $x \in I \cap J$, any non zero natural number m and

$$\lim_{m\to\infty} u_m(x) = \lim_{m\to\infty} w_m(x) = 0 \quad (3.9)$$

uniform on $I \cap J$.

For $(x, y) \in (I \times I) \cap (J \times J)$, $m, n \in \mathbb{N}$, $m \neq 0$, $n \neq 0$ and taking Lemma 3.1 and (2.3) into account, we have

$$\left(L^*_{m,n}e_{10}\right)(x, y) = \left(L^*_m e_1\right)(x) \left(L^*_n e_0\right)(y) = \left(L^*_m e_1\right)(x).$$

¿From (3.7) considering $u_{m,n}(x, y) = u_m(x)$, we obtain (3.3). Similarly follows (3.4). We have

$$\begin{aligned} \left(L^*_{m,n}(e_{20} + e_{02})\right)(x, y) &= \left(L^*_{m,n}e_{20}\right)(x, y) + \left(L^*_{m,n}e_{02}\right)(x, y) \\ &= \left(L^*_m e_2\right)(x) \left(L^*_n e_0\right)(y) + \left(L^*_m e_0\right)(x) \left(L^*_n e_2\right)(y) \\ &= x^2 + y^2 + w_{m,n}(x, y), \end{aligned}$$

when, taking (3.8) into account and $w_{m,n}(x, y) = w_m(x) + w_n(y)$.

Thus, the relations (3.2)–(3.5) take place and from the definition of the functions $u_{m,n}$, $v_{m,n}$ and $w_{m,n}$, it results that the relation (3.6) holds. □

Theorem 3.1. *The sequence* $\left(UL^*_{m,n}f\right)_{m,n\geq 1}$ *converges uniformly to the function* f *on* $(I \times I) \cap (J \times J)$, *for any* $f \in C_b[(I \times I) \cap (J \times J)]$.

Proof. It results from Lemma 3.2 and Theorem 1.2. □

For the operators constructed in this sections, we note

$$\delta_m(x) = \sqrt{\left(L_m^* \varphi_x^2\right)(x)}, \quad \delta_{m,x} = \sqrt{\left(L_m^* \varphi_x^4\right)(x)},$$

where $x \in I \cap J$, $m \in \mathbb{N}$, $m \neq 0$ and $\varphi_x : I \to \mathbb{R}$, $\varphi_x(t) = |t - x|$, for any $t \in I$. Then, taking Lemma 3.1 into account, the Theorem 1.1 becomes:

Theorem 3.2. *(i) For any function $f \in C_b(I \times I)$, any $(x,y) \in (I \times I) \cap (J \times J)$, any non zero natural number m, n, we have*

$$\left|f(x,y) - \left(UL_{m,n}^* f\right)(x,y)\right| \leq (1 + \delta_1^{-1}\delta_m(x) + \delta_2^{-1}\delta_n(y) + \delta_1^{-1}\delta_2^{-1}\delta_m(x)\delta_n(y))\omega_{\text{mixed}}(f;\delta_1,\delta_2) \tag{3.10}$$

for any $\delta_1, \delta_2 > 0$ and

$$\left|f(x,y) - \left(UL_{m,n}^* f\right)(x,y)\right| \leq 4\omega_{\text{mixed}}(f;\delta_m(x),\delta_n(y)). \tag{3.11}$$

(ii) For any function $f \in D_b \in (I \times I)$ with $D_B f \in B(I \times I)$, any $(x,y) \in (I \times I) \cap (J \times J)$, any non zero natural number m, n, any $\delta_1, \delta_2 > 0$, we have

$$|f(x,y) - (UL^* f)(x,y)| \leq 3\|D_B f\|_\infty \delta_m(x)\delta_n(y) + [\delta_m(x)\delta_n(y) + \delta_1^{-1}\delta_{m,x}\delta_n(y) + \delta_2^{-1}\delta_m(x)\delta_{n,y} + \delta_1^{-1}\delta_2^{-1}\delta_m^2(x)\delta_n^2(y)]\omega_{\text{mixed}}(D_B f;\delta_1,\delta_2). \tag{3.12}$$

In the following, we give examples of GBS operators associated, which verify Theorem 3.1 and Theorem 3.2. In these applications, we consider $p_{m,k}^* = p_{m,k}$, where $p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$, $m \in \mathbb{N}$, $m \neq 0$, $k \in \{0, 1, \ldots, m\}$, $x \in [0,1]$ and $E(I) = C(I)$, $F(K) = C(K)$.

Application 1. If $I = J = K = [0,1]$, $x_{m,k} = \dfrac{k}{m}$ for $m \in \mathbb{N}$, $m \neq 0$, $k \in \{0, 1, \ldots, m\}$, then we obtain the Bernstein operators $(B_m)_{m \geq 1}$.

Application 2. Let $\alpha \geq 0$ and $\beta \in \mathbb{R}$. If $I = \left[0, \mu^{(\alpha,\beta)}\right]$, $J = K = [0,1]$, $x_{m,k} = \dfrac{k+\alpha}{m+\beta}$, $m \in \mathbb{N}$, $m \geq m_0$, $k \in \{0, 1, \ldots, m\}$, then we obtain the Stancu operators $\left(P_m^{(\alpha,\beta)}\right)_{m \geq m_0}$ (see [16] or [17]).

Application 3. Let p be a natural number. If $I=[0,1+p]$, $J=K=[0,1]$, $p^*_{m,k}=\widetilde{p}_{m,k}=p_{m+p,k}$, $x_{m,k}=\dfrac{k}{m}$, $m\in\mathbb{N}$, $m\neq 0$, $k\in\{0,1,\dots,m+p\}$, then we obtain the Schurer operators $\left(\widetilde{B}_{m,p}\right)_{m\geq 1}$ (see [7]).

Application 4. Let p be a natural number and $0\leq\alpha\leq\beta$. If $I=[0,1+p]$, $J=K=[0,1]$, $p^*_{m,k}=\widetilde{p}_{m,k}$, $x_{m,k}=\dfrac{k+\alpha}{m+\beta}$, $m\in\mathbb{N}$, $m\neq 0$, $k\in\{0,1,\dots,m+p\}$, then we obtain the Schurer-Stancu operators $\left(S^{(\alpha,\beta)}_{m,p}\right)_{m\geq 1}$ (see [5]).

Application 5. In this application we consider $I=J=K=[0,\infty)$, $E(I)=F(K)=C_B([0,\infty))$, $p^*_{m,k}(x)=(1+x)^{-m}\binom{m}{k}x^k$, $x\in[0,\infty)$ and $x_{m,k}=\dfrac{k}{m+1-k}$, $m\in\mathbb{N}$, $m\neq 0$, $k\in\{0,1,\dots,m\}$. Then we obtain the Bleimann, Butzer and Hahn operators $(L_m)_{m\geq 1}$ (see [8]).

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