A resource-sensitive logic of agency

Daniele Porello and **Nicolas Troquard**[1]

Abstract. We study a fragment of Intuitionistic Linear Logic combined with non-normal modal operators. Focusing on the minimal modal logic, we provide a Gentzen-style sequent calculus as well as a semantics in terms of Kripke resource models. We show that the proof theory is sound and complete with respect to the class of minimal Kripke resource models. We also show that the sequent calculus allows cut elimination. We put the logical framework to use by instantiating it as a logic of agency. In particular, we apply it to reason about the resource-sensitive use of artefacts.

1 Introduction

We propose a novel modal extension of a fragment of intuitionistic linear logic ILL [12, 30]. Linear logic is a resource-sensitive logic that allows for modeling the constructive content of deductions in logic. In particular, linear logic has been applied as a logic for representing computations [12, 1]. Moreover, intuitionistic fragments of linear logic have been used to model problems in knowledge representation and multiagent systems, for example in [19], [27], [14].

We shall extend ILL by adding non-normal modalities. Those are the modalities with a logic weaker than K, and cannot be evaluated over a Kripke semantics. The extension of ILL with *normal* modalities has already been investigated for example in [9, 16].

Neighborhood semantics were introduced independently by Scott and Montague. Early results were offered by Segerberg. Chellas built upon and gave a textbook presentation in [7]. Neighborhood semantics allow for defining non-normal modalities that are required to model a number of application domains. The significance of non-normal modal logics and their semantics in modern developments in logics of agents has been emphasized before [2]. Indeed many logics of agents are non-normal: chiefly logics of coalitional power [26], but also epistemic logics without omniscience [33, 24], or logics of agency [13], etc.

There are two main families of modal logics of action. Probably the most prominent in computer science is the one of dynamic logics. The second family has an older lineage, but the modern blooming stems from the work von Wright and others. They are the logics of agency, where action is seen as a modal notion. They do not talk about action proper but instead about what agents bring about. For instance, the formula $Does_a A \wedge Does_a \neg Does_a B$ captures the fact that agent a does that A is the case, and a refrains from doing (a does that a does not do that) B is the case. Kanger's influence led to the logics of bringing-it-about. A thorough philosophical analysis is due to Elgesem [10]. Governatori and Rotolo [13] clarified the semantics and provided a complete axiomatization. An algorithm to solve the satisfiability problem of the corresponding logic can be found in [31]. It gained much interest in the study of sociality, e.g.: parliamentary powers [17], legal reasoning [23], social influence [28], institutional agency [5], etc.

Our theoretical contributions are the following. A Kripke-like semantics allows the evaluation of connectives of linear logic. We enrich it with neighborhood functions to capture non-normal modalities. We obtain what we simply coin modal Kripke resource models. We define and study a non-normal modal logic whose propositional part is based on intuitionistic linear logic. Next, we introduce a sequent calculus, in order to investigate properties of reasoning about modal resource-bounded propositions. Moreover, we show that the sequent calculus allows cut elimination that provides a normal form for proofs. In the last sections, we motivate and discuss a number of applications of our system to represent and reasoning about artefacts.

We shall instantiate our framework with a collection of modalities E_i where the formula $\mathsf{E}_i A$ captures the fact that the acting entity i brings about the action A. Our application lies in the reasoning about artefact's function and tool use. Artefacts are special kind of entities that are characterized by the fact that they are designed by some other agent in order to achieve a purpose in a particular environment. An important aspect of the modeling of artefacts is their interaction with the environment and with the agents that *use* the artefact to achieve a specific goal [11, 4, 15, 20]. Briefly, we can view an artefact as an object that in presence of a number of preconditions $c_1, \dots, c_n$ produces the outcome o. In this work, we want to represent the function of artefacts by means of logical formulas and to view the correct behavior of an artefact by means of a form of reasoning.

Imagine we represent naïvely the behavior of a screwdriver as a classical formula that states that if there is a screw S, then we can tighten it T. We simply describe the behavior of the artefact as a material implication $S \rightarrow T$. In classical logic, we can infer that by means of a single screwdriver we can tighten two screws: $S, S, S \rightarrow T \vdash T \wedge T$. Worse, we do not even need to have two screws to begin with: $S, S \rightarrow T \vdash T \wedge T$. Thus, without specifying all the relevant constraints on the environment (e.g. that a screwdriver can handle one screw at the time) we end up with unintuitive results. Moreover, often we need to specify the relationship between the artefact and the agents: for example, there are artefacts that can be used by one agent at the time. Since a crucial point in modeling artefacts is their interaction with the environment and the users, either we carefully list all the relevant conditions, *or we need to change the logical framework that we use to represent the artefact's behavior*.

In this paper, we propose to pursue this second strategy. Our motivation is that, instead of specifying for each artefact the precondition of its application (e.g. that there is only one screw that a screw driver is supposed to operate on), the logical language that encodes the behavior of the artefact already takes care of preventing unintuitive outcomes. Thus, the formulas of ILL shall represent actions of agents and functions of artefacts, and the non-normal modality shall specify which agent or artefact brings about which process.

[1] Laboratory for Applied Ontology, ISTC-CNR, Trento. E-mail: daniele.porello@loa.istc.cnr.it, nicolas.troquard@loa.istc.cnr.it

Our decision for using an *intuitionistic* version of linear logic is that in intuitionistic sequent calculus, every sequent has a single "output" formula. This feature matches our modeling of the use of artefacts as input-output processes. Thus, we can also view the composition of a number of behaviors of artefacts as a complex input-output process.

2 A fragment of intuitionistic linear logic

The propositional language that we are going to use, $\mathcal{L}_{\mathsf{ILL}}$, is defined by the BNF

$$A ::= \mathbf{1} \mid p \mid A \otimes A \mid A \,\&\, A \mid A \multimap A$$

where $p \in Atom$. The resource-sensitive nature of linear logic is due to the lack of structural rules in the sequent calculus. ILL rejects the global validity of weakening (W), that amounts to a monotonicity of the entailment, and contraction (C), that is responsible for arbitrary duplications of formulas, e.g. $A \to A \wedge A$ is a tautology classical logic.

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}\,(\mathrm{W}) \qquad \frac{\Gamma, B, B \vdash A}{\Gamma, B \vdash A}\,(\mathrm{C})$$

Exchange still holds, thus contexts of formulas Γ in sequent calculus are multisets. By dropping weakening and contraction, we are led to define two non-equivalent conjunctions with different behavior: the multiplicative conjunction $\otimes$ (tensor) and the additive conjunction $\&$ (with). The intuitive meaning of $\otimes$ is that an action of type $A \otimes B$ can be performed by summing the resources that are relevant to perform A and to perform B. The lack of weakening entails that $A \otimes B$ no longer implies A, namely the resources that are relevant to perform $A \otimes B$ may not be relevant to perform just A. The lack of contraction entails that $A \multimap A \otimes A$ is no longer valid. The additive conjunction $A \,\&\, B$ expresses an option, the choice to perform A or B. Accordingly $A \,\&\, B \multimap A$ holds in linear logic, the resources that enable the choice between A and B are relevant also to make A or to make B. The linear implication $A \multimap B$ expresses a form of causality, for example "If I strike a match, I can light the room" the action of striking that match is consumed, in the sense that it is no longer available after the room is lighted. The unit $\mathbf{1}$ is the neutral element for $\otimes$ and can represent model a null action.

For better readability, we have labeled our fragment intuitionistic linear logic ILL although the full fragment of intuitionistic linear logic includes also an additive disjunction $\oplus$ and the exponential $!A$. Exponentials in linear logic allow for defining a local form of structural rules. However, exponentials introduce very high complexity in the calculus, for instance even the multiplicative exponential fragment ($!$, $\otimes$, $\multimap$) is at least EXPSPACE-hard and an upper bound is not known [21]. We will not introduce an additive disjunction. One reason for that is that the Kripke resource model cannot handle the additive fragment faithfully. In particular, the $\&$, $\oplus$ fragment of linear logic is not complete for the Kripke semantics:[2] We have decided to exploit Kripke resource frames as far as possible, as they allow for a simple semantics for an interesting fragment of linear logic and for a *natural modal extension*. Moreover, Kripke resource frames provides model for the logic of bunched implication (BI)[25, 29], for that reason, our modal extension can be adapted to BI. An intuitionistic negation can be added to our language. We simply define a designated atom $\perp \in Atom$ and define negation by means of implication $\sim A \equiv A \multimap \perp$ [18]. There is no specific rule for negation.

[2] A more crude way to put is that the Kripke semantics is inadequate.

Given a multiset of formulas, it will be useful to combine them into a unique formula. We adopt the following notation: $\emptyset^* = \mathbf{1}$, and $\Delta^* = A_1 \otimes \ldots \otimes A_k$ when $\Delta = \{A_1, \ldots, A_k\}$.

3 Models of ILL

We introduce a Kripke-like class of models for ILL that is basically due to Urquhart [32]. A *Kripke resource frame* is a structure $\mathcal{M} = (M, e, \circ, \geq)$, where $(M, e, \circ)$ is a commutative monoid with neutral element e, and $\geq$ is a pre-order on M. The frame has to satisfy the condition of *bifunctoriality*: if $m \geq n$, and $m' \geq n'$, then $m \circ m' \geq n \circ n'$. To obtain a *Kripke resource model*, a valuation on atoms $V : Atom \to \mathcal{P}(M)$ is added. It has to satisfy the *heredity* condition: if $m \in V(p)$ and $n \geq m$ then $n \in V(p)$. The truth conditions of the formulas of $\mathcal{L}_{\mathsf{ILL}}$ in the Kripke resource model $\mathcal{M} = (M, e, \circ, \geq, V)$ are the following:

$m \models_{\mathcal{M}} p$ iff $m \in V(p)$.
$m \models_{\mathcal{M}} \mathbf{1}$ iff $m \geq e$.
$m \models_{\mathcal{M}} A \otimes B$ iff there exist m_1 and m_2 such that $m \geq m_1 \circ m_2$ and $m_1 \models_{\mathcal{M}} A$ and $m_2 \models_{\mathcal{M}} B$.
$m \models_{\mathcal{M}} A \,\&\, B$ iff $m \models_{\mathcal{M}} A$ and $m \models_{\mathcal{M}} B$.
$m \models_{\mathcal{M}} A \multimap B$ iff for all $n \in M$, if $n \models_{\mathcal{M}} A$, then $n \circ m \models_{\mathcal{M}} B$.

Denote $||A||^{\mathcal{M}}$ the extension of A in $\mathcal{M}$, i.e. the set of worlds of $\mathcal{M}$ in which A holds. A formula A is *true* in a model $\mathcal{M}$ if $e \models_{\mathcal{M}} A$.[3] A formula A is *valid* in Kripke resource frames, noted $\models A$, iff it is true in every model.

With $\models_{\mathcal{M}}$ now defined, observe that heredity can be shown to extend naturally to every formula, in the sense that:

Proposition 1. *For every formula A, if $m \models A$ and $m' \geq m$, then $m' \models A$.*

4 Modal Kripke resource models

We now design a version of ILL with a minimal modality $\Box$ and obtain MILL. The language of MILL, $\mathcal{L}_{\mathsf{MILL}}$, then becomes

$$A ::= \mathbf{1} \mid p \mid A \otimes A \mid A \,\&\, A \mid A \multimap A \mid \Box A$$

where $p \in Atom$.[4]

To give a meaning to the new modality, we define a neighborhood semantics on top of the Kripke resource frame. A neighborhood function is a mapping $N : M \to \mathcal{P}(\mathcal{P}(M))$ that associates a world m with a set of sets of worlds. (See [7].) We define:

$m \models \Box A$ iff $||A|| \in N(m)$

This is not enough, though. It is possible that $m \models \Box A$, yet $m' \not\models \Box A$ for some $m' \geq m$. That is, Proposition 1 does not hold with the simple extension of $\models$ for $\mathcal{L}_{\mathsf{MILL}}$. (One disastrous consequence is that the resulting logic does not satisfy the *modus ponens* or the cut rule.) We could define the clause concerning the modality alternatively as: $m \models \Box A$ iff there is $n \in M$, such that $m \geq n$ and $||A|| \in N(n)$. However, this is bothersome because this is not how a non-normal modality is traditionally defined [7].

[3] When the context is clear we will write $||A||$ instead of $||A||^{\mathcal{M}}$, and $m \models A$ instead of $m \models_{\mathcal{M}} A$.

[4] Note that we are working with a 'necessity' modality only. We do not deal with a 'possibility' operator. In intuitionistic logics, they are not dual, therefore they are not interdefinable. We leave a discussion of their logical relations for future work. For what is worth, we will not feel the need of it for our application domain of agency.

Instead, we will require our neighborhood function to satisfy the condition that if some set $X \subseteq M$ is in the neighborhood of a world, then X is also in the neighborhood of all "greater" worlds.[5] Formally, our modal linear logic is evaluated over the following models:

Definition 1. *A* modal Kripke resource model *is a structure* $\mathcal{M} = (M, e, \circ, \geq, N, V)$ *such that:*

- $(M, e, \circ, \geq)$ *is a Kripke resource frame;*
- N *is a neighborhood function such that:*

$$\textit{if } X \in N(m) \textit{ and } n \geq m \textit{ then } X \in N(n) \qquad (1)$$

It is readily checked that Proposition 1 is true as well for $\mathcal{L}_{\mathsf{MILL}}$ over modal Kripke resource models for modal formulas.

5 Sequent calculus MILL and completeness

In this section, we introduce the sequent calculus for our logic. A *sequent* is a statement $\Gamma \vdash A$ where Γ is a finite multiset of occurrences of formulas of ILL and A is a formula. The fact that we allow for a single formula in the conclusions of the sequent corresponds to the fact that we are working with the intuitionistic version of the calculus [12].

$$\dfrac{}{A \vdash A}\,\text{ax} \qquad \dfrac{\Gamma, A \vdash C \qquad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash C}\,\text{cut}$$

$$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,\otimes\text{L} \qquad \dfrac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\,\otimes\text{R}$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma', B \vdash C}{\Gamma', \Gamma, A \multimap B \vdash C}\,\multimap\text{L} \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,\multimap\text{R}$$

$$\dfrac{\Gamma, A, \Gamma' \vdash C}{\Gamma, A \,\&\, B, \Gamma' \vdash C}\,\&\text{L} \qquad \dfrac{\Gamma, B, \Gamma' \vdash C}{\Gamma, A \,\&\, B, \Gamma' \vdash C}\,\&\text{L}$$

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,\&\text{R} \qquad \dfrac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}\,\mathbf{1}\text{L} \qquad \dfrac{}{\vdash \mathbf{1}}\,\mathbf{1}\text{R}$$

Table 1. Sequent calculus MILL

Since in a sequent $\Gamma \vdash A$ we identify Γ to a multiset of formulas, the exchange rule—the reshuffling of Γ—is implicit.

A sequent $\Gamma \vdash A$ where $\Gamma = A_1, \ldots, A_n$ is *valid* in a modal Kripke resource frame iff the formula $A_1 \otimes \ldots \otimes A_n \multimap A$ is valid, namely $\models \Gamma^* \multimap A$. The calculus of ILL presented above is sound and complete wrt. the class of Kripke resource models [32].

We obtain the sequent calculus for our minimal modal logic MILL by extending the language of ILL with modal formulas and by adding a new rule $\Box$(re).

$$\dfrac{A \vdash B \qquad B \vdash A}{\Box A \vdash \Box B}\,\Box(\text{re})$$

Crucially, the modal extension does not affect cut elimination.

Theorem 2. *Cut elimination holds for* MILL*.*

Proof. (Sketch) Cut elimination holds for linear logic [12]. The proof for MILL largely adapts the proof for linear logic [30]. By reasoning by induction on the length of the proof, we need to show that we can reduce the depth of cuts and we need to show that we can reduce cuts on complex formulas to cuts on sub formulas and then eliminate them by replacing them with axioms.
For example, take the case in which $\Box A$ is the cut formula and is principal in both premises (i.e. it has been introduced by $\Box$(re)):

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash B}{\Box B \vdash \Box C}\,\Box(\text{re}) \qquad \dfrac{C \vdash D \qquad D \vdash C}{\Box C \vdash \Box D}\,\Box(\text{re})}{\Box B \vdash \Box D}\,\text{cut}$$

It is reduced by replacing the cut on $\Box C$ by less complex cuts on C.

$$\dfrac{\dfrac{B \vdash C \qquad C \vdash D}{B \vdash D}\,\text{cut} \qquad \dfrac{D \vdash C \qquad C \vdash B}{D \vdash B}\,\text{cut}}{\Box B \vdash \Box D}\,\Box(\text{re})$$

This reduction extends to the case where $\Box A$ is the non-principal cut formula. $\square$

By inspecting the rules others than cut, it is easy to see that cut elimination entails the subformula property, namely if $\Gamma \vdash A$ is derivable, then there is a derivation containing subformulas of Γ and A only. The decidability remains to be established. We can show that the proof-search for MILL is no more costly than the proof-search for propositional intuitionistic multiplicative additive linear logic [22].

Theorem 3. *Proof search complexity for* MILL *is in PSPACE.*

Proof. (Sketch) The proof adapts the argument in [22]. By cut elimination, Theorem 2, for every provable sequent in MILL there is a cut-free proof with same conclusion. For every rule in MILL other than (cut), the premises have a strictly lower complexity wrt. the conclusion. Hence, for every provable sequent, there is a proof whose branches have a depth at most linear in the size of the sequent. The size of a branch is at most quadratic in the size of the conclusion. And it contains only subformulas of the conclusion sequent because of the subformula property. This means that one can non-deterministically guess such a proof, and check each branch one by one using only a polynomial space. Proof search is then in NPSPACE = PSPACE. $\square$

We sketch the proof of soundness and completeness of MILL wrt. the class of modal Kripke resource frames.

Theorem 4. $\models \Gamma^* \multimap A$ *iff* $\Gamma \vdash A$.

The direction of soundness is established by proving by induction that sequents rules preserve validity. We only give the proof for the cases that differ from the proof of soundness for ILL, since soundness for ILL wrt Kripke resource frames has been established in [16].
Soundness of $\Box$*(re).* We show that $\Box$ preserves validity, namely, if premises are valid, then the conclusion is valid: If $e \models A \multimap B$ and $e \models B \multimap A$, then $e \models \Box A \multimap \Box B$. Our assumptions imply that, for all x, if $x \models A$, then $x \models B$, and if $x \models B$ then $x \models A$. Thus, $||A|| = ||B||$. We need to show that for all x, if $x \models \Box A$, then $x \models \Box B$. By definition, $x \models \Box A$ iff $||A|| \in N(x)$. Thus, since $||A|| = ||B||$, we have that $||B|| \in N(x)$, that means $x \models \Box B$.

The proof of completeness can be summarized as follow. We build a canonical model $\mathcal{M}^c$ (Definition 2). In particular, the set M^c of states consists in the set of finite multisets of formulas, and the neutral element e^c is the empty multiset. We first need to show that it is indeed a modal Kripke resource model (Lemma 5). Second we need to show a correspondence, the "Truth Lemma", between $\vdash$ and truth in $\mathcal{M}^c$. Precisely we show that for a formula A and a multiset of formulas $\Gamma \in M^c$, it is the case that Γ satisfies A iff $\Gamma \vdash A$ is provable in the calculus (Lemma 6). Finally, to show completeness, assume

[5] An analogous yet less transparent condition was used in [9] for a normal modality.

that it is not the case that $\vdash \Gamma^* \multimap A$. By the Truth Lemma, it means that in the canonical model $\Gamma^* \multimap A$ is not satisfied at e^c. So $\mathcal{M}^c$ does not satisfy $\Gamma^* \multimap A$. So it is not the case that $\models \Gamma^* \multimap A$.

In the following, $\sqcup$ is the multiset union. Also, $\mid A \mid^c = \{\Gamma \mid \Gamma \vdash A\}$.

Definition 2. *Let $\mathcal{M}^c = (M^c, e^c, o^c, \geq^c, N^c, V^c)$ such that:*

- $M^c = \{\Gamma \mid \Gamma$ *is a finite multiset of formulas*$\}$;
- $\Gamma \circ^c \Delta = \Gamma \sqcup \Delta$;
- $e^c = \emptyset$;
- $\Gamma \geq^c \Delta$ *iff* $\Gamma \vdash \Delta^*$;
- $\Gamma \in V^c(p)$ *iff* $\Gamma \vdash p$;
- $N^c(\Gamma) = \{\mid A \mid^c \mid \Gamma \vdash \Box A\}$.

Lemma 5. *$\mathcal{M}^c$ is a modal Kripke resource model.*

Proof. 1. $(M^c, e^c, o^c, \geq^c)$ is the "right type" of ordered monoid: (i) $(M^c, e^c, \circ^c)$ is a commutative monoid with neutral element e^c, and (ii) $\geq^c$ is a pre-order on M^c. Finally, (iii) if $\Gamma \geq^c \Gamma'$ and $\Delta \geq^c \Delta'$ then $\Gamma \circ^c \Delta \geq^c \Gamma' \circ^c \Delta'$.

For (i), commutativity and neutrality follows from the definition of $\circ^c$ as the multiset union, and the neutrality of e^c follows from it being the empty multiset.

For (ii), $\geq^c$ is reflexive because $\{A_1, \ldots, A_n\} \vdash \{A_1, \ldots, A_n\}^*$ can be proved from the axioms (ax) $A_k \vdash A_k$, $1 \leq k \leq n$, and by applying $\otimes$R $n-1$ times. The key rule to establish that $\geq^c$ is transitive is cut.

For (iii), assume $\Gamma \geq^c \Gamma'$ and $\Delta \geq^c \Delta'$, that is, $\Gamma \vdash (\Gamma')^*$ and $\Delta \vdash (\Delta')^*$. By $\otimes$R we have $\Gamma, \Delta \vdash (\Gamma')^* \otimes (\Delta')^*$. By applying the definitions we end up with $\Gamma \sqcup \Delta \vdash (\Gamma' \sqcup \Delta')^*$ and the expected result follows.

2. V^c is a valuation function and satisfies heredity: if $\Gamma \in V(p)$ and $\Delta \geq^c \Gamma$ then $\Delta \in V(p)$. To see this, suppose $\Gamma \vdash p$ and $\Delta \vdash \Gamma^*$. By applying $\otimes$L enough times, we have $\Gamma^* \vdash p$. By cut, we obtain $\Delta \vdash p$.

3. N^c is well-defined: Suppose that $\mid A \mid^c = \mid B \mid^c$. We need to show that $\mid A \mid^c \in N^c(\Gamma)$ iff $\mid B \mid^c \in N^c(\Gamma)$.

From $\mid A \mid^c = \mid B \mid^c$, we have $\Gamma \vdash A \Rightarrow \Gamma \vdash B$. In particular, we have $A \vdash A \Rightarrow A \vdash B$. Hence, $A \vdash B$ is provable (by rule (ax)). We show symmetrically that $B \vdash A$ is provable.

From $A \vdash B$ and $B \vdash A$, we have by rule $\Box$(re) that $\Box A \vdash \Box B$ is provable, and also that $\Box B \vdash \Box A$ is provable.

Now suppose that $\Gamma \vdash \Box A$. Since $\Box A \vdash \Box B$ is provable, we obtain by cut that $\Gamma \vdash \Box B$ is provable. Symmetrically, suppose that $\Gamma \vdash \Box B$. Since $\Box B \vdash \Box A$ is provable, we obtain by cut that $\Gamma \vdash \Box A$ is provable.

Hence, we have that $\Gamma \vdash \Box A$ iff $\Gamma \vdash \Box B$. By definition of N^c, it means that $\mid A \mid^c \in N^c(\Gamma)$ iff $\mid B \mid^c \in N^c(\Gamma)$.

4. If $X \in N^c(\Gamma)$ and $\Delta \geq^c \Gamma$ then $X \in N^c(\Delta)$. To see that this is the case, the hypotheses are equivalent to $\Gamma \vdash \Box A$ for some A such that $\mid A \mid^c = X$, and $\Delta \vdash \Gamma^*$. By repeatedly applying $\otimes$L to obtain $\Gamma^* \vdash \Box A$ and by using cut, we infer that $\Delta \vdash \Box A$. Which is equivalent to the statement that $X \in N^c(\Delta)$. ☐

The following can be proved with a routine induction on the complexity of A.

Lemma 6. *Let us then note $\models_c$ the truth relation in $\mathcal{M}^c$. We have $\Gamma \models_c A$ iff $\Gamma \vdash A$.*

6 A resource-sensitive logic of agency

We present the (non-normal modal) logic of agency of bringing-it-about [10, 13], and propose a version of it in linear logic coined Linear BIAT. Then, we illustrate the logic by representing a few actions of agents, functions of artefacts, and their interaction. We shall emphasize how these interactions depend on resources by means of proof search in Linear BIAT. We specialize our minimal modality to a *bringing-it-about* modality [10, 13]. For each agent a in a set $\mathcal{A}$, we define a modality E_a, and $\mathsf{E}_a A$ specifies that agent $a \in \mathcal{A}$ brings about A. As previously, to interpret them in a modal Kripke resource frame, we take one neighborhood function N_a for each agent a that obeys Condition (1) in Definition 1. We have $m \models \mathsf{E}_a A$ iff $||A|| \in N_a(m)$.

The four following principles typically constitute the core of logics of agency [28, 10, 3]:

1. If something is brought about, then this something holds.
2. It is not possible to bring about a tautology.
3. If an agent brings about two things concomitantly then the agent also brings about the conjunction of these two things.
4. If two statements are equivalent, then bringing about one is equivalent to bringing about the other.

Item 1 is a principle of success. It corresponds to the axiom T: $\mathsf{E}_i A \rightarrow A$. Item 2 has been open to some debate, although Chellas is essentially the only antagonist. (See [6] and [8].) It corresponds to the axiom $\neg \mathsf{E}_i \top$ (notaut). Item 3 corresponds to the axiom: $\mathsf{E}_i A \wedge \mathsf{E}_i B \rightarrow \mathsf{E}_i(A \wedge B)$. That is, co-temporality is tacitly presupposed. Item 4 confers to the concept of bringing about the quality of being a modality, effectively obeying the rule of equivalents: if $\vdash A \leftrightarrow B$ then $\vdash \mathsf{E}_i A \leftrightarrow \mathsf{E}_i B$. We capture the four principles, adapted to the resource-sensitive framework, by means of rules in the sequent calculus, cf. Table 2

We already know that the logic MILL satisfies the rule of equivalents for E_a: from $A \vdash B$ and $B \vdash A$ we infer $\mathsf{E}_a A \vdash \mathsf{E}_a B$, so principle 4 is fine.

Because of the difference between the unities in LL and in classical logic (i.e. in LL all the tautologies are not equally provable), principle 2 must be changed into an inference rule ($\sim$ nec): if $\vdash A$, then $E_a A \vdash \bot$. So, if a formula is a theorem, an agent that brings it about implies the contradiction[6].

Principle 1 is captured by E_a(refl) that entails the linear version of T: $\mathsf{E}_a A \multimap A$. In our interpretation, it means that if an agent brings about A, then A affects the environment.

The principle of BIAT for combining actions (Item 3 in the list) is the interesting bit here: it can be interpreted in linear logic in two ways, namely, in a multiplicative and in an additive way. Both version can be easily handled from a technical point of view, however we focus here on the multiplicative interpretation of principle 3, and we leave a discussion of the additive for future work. The additive combination would mean that if there is a choice for agent a between bringing about A and bringing about B, then agent a can bring about a choice between A and B. $\mathsf{E}_a \otimes$ means that if an agent a brings about action A and brings about action B then a brings about both actions $A \otimes B$. Moreover, in order to bring about $A \otimes B$, the sum of the resources for A and the resources for B is required. The following conditions on modal Kripke resource frames are now required.

($\sim$ nec) requires:

$$\text{if } (X \in N_a(w)) \text{ and } (e \in X) \text{ then } (w \in V(\bot)) \tag{2}$$

(E_a(refl)) requires:

$$\text{if } X \in N_a(w) \text{ then } w \in X \tag{3}$$

[6] This amounts to negating $E_a A$, according to intuitionitstic negation.

Let $X \circ Y = \{x \circ y \mid x \in X \text{ and } y \in Y\}$, the condition corresponding to the multiplicative version of action combination ($\mathsf{E}_a\otimes$) requires that the upper closure of $X \circ Y$, denote it by $(X \circ Y)^{\uparrow}$, is in $N_a(x \circ y)$:

$$\text{if } X \in N_a(x) \text{ and } Y \in N_a(y) \text{ , then } (X \circ Y)^{\uparrow} \in N_a(x \circ y) \quad (4)$$

Theorem 7. *Linear BIAT is sound and complete wrt. the class of Kripke modal frames that satisfy (2), (3), and (4).*

Proof. (Sketch) We only consider the case of Condition (2) and rule ($\sim$nec). ($\sim$nec) is sound. Assume that for every model, $e \models A$. We need to show that $e \models \mathsf{E}_a A \multimap \bot$. That is. for every x, if $x \models \mathsf{E}_a A$, then x models $\bot$. If $x \models \mathsf{E}_a A$, then by definition, $||A|| \in N_a(x)$. Since A is a theorem, $e \in ||A||$, thus by Condition 2, $x \in V(\bot)$, so $x \models \bot$. For completeness, it suffices to adapt our canonical model construction. Build the canonical model for Linear BIAT as in Def 2 (we have now more valid sequents). Now suppose (1) $X \in N^c(\Gamma)$, and (2) $e^c \in X$. By definition of N^c and of $|.|^c$, there is A, s.t. $|A|^c = X$, (1) $\Gamma \vdash \mathsf{E}_a A$ and (2) $\vdash A$. From (2), and ($\sim$nec): $\mathsf{E}_a A \vdash \bot$. From (1), and previous, we obtain $\Gamma \vdash \bot$ using (cut). By definition of V^c, $\Gamma \in V^c(\bot)$. □

$$\frac{A \vdash B \qquad B \vdash A}{\mathsf{E}_a A \vdash \mathsf{E}_a B}\ \mathsf{E}_a(\text{re}) \qquad\qquad \frac{\vdash A}{\mathsf{E}_a A \vdash \bot}\ \sim\text{nec}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, \mathsf{E}_a A \vdash B}\ \mathsf{E}_a(\text{refl}) \qquad\qquad \frac{\Gamma \vdash \mathsf{E}_a A \qquad \Delta \vdash \mathsf{E}_a B}{\Gamma, \Delta \vdash \mathsf{E}_a(A \otimes B)}\ \mathsf{E}_a\otimes$$

Table 2. Linear BIAT

Behavior of artefacts There is a striking similarity between functions of artefacts as displayed behavior and the meaning of the propositions A that are purposefully brought about. Artefacts are not living things but have a purpose, attributed by a designer or a user [4, 20]. If A is a function of an artefact t, then one can represent t's behavior as $\mathsf{E}_t A$ in a conceptually consistent manner. With linear logic, we are equipped with a formalism to represent and reason about processes and resources. We represent the function of an artefact as a formula of Linear BIAT. With a resource consumption and production reading of linear formulas, this view of artefact functions has an immediate appeal.

At an abstract level, an artefact can be seen as an agent. It takes actions in a reactive manner. When t is an artefact, and $\mathsf{E}_t A$ is deemed true, the formula A is a realized function of t. Thus, the formula A describes a behavior of t. Clearly, functions do not have a unique formulation. The functions $(A \otimes B) \multimap C$, and $A \multimap (B \multimap C)$ are provably equivalent. However, the rule $\mathsf{E}_a(\text{re})$ ensures that an agent bringing about a function is provably equivalent to this agent bringing about any of its equivalent forms.

Take a very simple example. We can represent the behavior of a screwdriver s as an implication that states that if there is a screw (formula S) and some agent brings about the right force (F), then the screw gets tighten (T): $\mathsf{E}_s(S \otimes F \multimap T)$. Suppose the environment provides S and an agent i is providing the right force $\mathsf{E}_i F$, we can show that the goal T can be achieved by means of the following proof in Linear BIAT.

$$\cfrac{\cfrac{\cfrac{S \vdash S \qquad \cfrac{F \vdash F}{\mathsf{E}_i F \vdash F}\ \mathsf{E}_i(\text{refl})}{S, \mathsf{E}_i F \vdash S \otimes F}\ \otimes\text{R} \qquad T \vdash T}{S, \mathsf{E}_i F, S \otimes F \multimap T \vdash T}\ \multimap\text{L}}{S, \mathsf{E}_i F, \mathsf{E}_s(S \otimes F \multimap T) \vdash T}\ \mathsf{E}_s(\text{refl})$$

Our calculus is resource sensitive, thus, as expected, we cannot infer for example that two agents can use the same screwdriver at the same time to tighten two screws:

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \nvdash T \otimes T$$

Linear BIAT allows for expressing much more. For instance, it can capture functions that are user-specific:

$$\mathsf{E}_t((\mathsf{E}_{a_1} A \multimap O) \,\&\, (\mathsf{E}_{a_2} A \multimap O) \,\&\, \cdots \,\&\, (\mathsf{E}_{a_m} A \multimap O)) \quad (5)$$

where $a_i \in B \subseteq \mathcal{A}$. The meaning of (6) is that implications $\mathsf{E}_{a_i} A \multimap O$ specify which agents among those in $\mathcal{A}$ are entitled of using the artefact t to obtain O. Accordingly, if one of those agents brings about A in a context of manipulating the tool t, then the outcome O is provable in Linear BIAT. Formula (5) can be shortened as $\mathsf{E}_t(\&_{x \in B \subseteq \mathcal{A}} \mathsf{E}_x(A) \multimap O)$. Let $i \in B \subseteq \mathcal{A}$, we have the following proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{\dots}{\mathsf{E}_i A, \mathsf{E}_i A \multimap O \vdash O}}{\dots}}{\mathsf{E}_i A, \&_{x \in B}(\mathsf{E}_x A \multimap O) \vdash O}\ \&\text{ L (enough times)}}{\mathsf{E}_i A, \mathsf{E}_t(\&_{x \in B}(\mathsf{E}_x A \multimap O)) \vdash O}\ \mathsf{E}_t(\text{refl})$$

Moreover, the behavior represented by $\mathsf{E}_t(\&_{x \in \mathcal{A}} \mathsf{E}_x(A \otimes B) \multimap O)$ requires the same agent x to perform both actions A and B in order to get O (e.g. in order to access my email I have to insert my login and my password). This is due to our $\mathsf{E}_i\otimes$ rule.

$$\cfrac{\cfrac{\cfrac{\mathsf{E}_i A \vdash \mathsf{E}_i A \qquad \mathsf{E}_i B \vdash \mathsf{E}_i B}{\mathsf{E}_i A, \mathsf{E}_i B \vdash \mathsf{E}_i(A \otimes B)}\ \mathsf{E}_i\otimes \qquad O \vdash O}{\mathsf{E}_i A, \mathsf{E}_i B, \mathsf{E}_i(A \otimes B) \multimap O \vdash O}\ \multimap\text{L}}{\mathsf{E}_i A, \mathsf{E}_i B, \mathsf{E}_t(\mathsf{E}_i(A \otimes B) \multimap O) \vdash O}\ \mathsf{E}_t(\text{refl})$$

On the other hand, the behavior $\mathsf{E}_t(\&_{x,y \in \mathcal{A}, x \neq y}(\mathsf{E}_x A \otimes \mathsf{E}_y B) \multimap O)$, forces the agents who operate tool t to be different (e.g. a crosscut saw). In a similar way, we can represent in a purely logical manner, tools that require any number of agents to operate (of course, if we want to express that any subsets of $\mathcal{A}$ can operate the tool, then we need an exponentially long formula).

Linear BIAT is resource-sensitive as the previous non-provable sequent in our screwdriver example illustrates: the screwdriver *cannot be reused*, despite the fact that an additional screw is available and an appropriate force is brought about. This is perfectly fine as long as our interpretation of resource consumption is *concurrent*: all resources are consumed at once. Abandoning a concurrent interpretation of resource consumption, we may specialize the modality E_a when a is an artefactual agent in such a way that the function of an artefact can be used at will. After all, using a screwdriver once does not destroy the screwdriver. Its function is still present after. We are after a property of *contraction* for our operator E_s.

$$\frac{\Gamma, \mathsf{E}_s A, \mathsf{E}_s A \vdash B}{\Gamma, \mathsf{E}_s A \vdash B}\ \text{c}(\mathsf{E}_s)$$

Now, if we adopt the rule, c(E_s) we can easily see that indeed

$$S, S, \mathsf{E}_i F, \mathsf{E}_j F, \mathsf{E}_s(S \otimes F \multimap T) \vdash T \otimes T$$

is provable.There are several issues with this solution to 'reuse' as a duplication of assumptions. Some technical, some conceptual. The main technical issue is that we lose a lot of control on the proof search, as contraction is the main source of non-termination (of bottom-up proof search). Another technical (or theoretical) issue is that trying to give a natural condition on our frames that would be canonical for contraction is out of question. The conceptual issue is the same as the one posed by Girard in creating linear logic: duplication of assumptions should not be automatic. Similarly, *ad lib* reuse of an artefact does not reflect a commonsensical experience. In general, although they don't consume after the first use, tools will nonetheless eventually become so worn out that they will not realize their original function. We can capitalize on the 'additive' feature of linear logic language: employing the 'with' operator $\&$, we can specify a sort of warranty of artefact functions.

$$A^{\leq n} = \mathbf{1}\&(A)\&(A \otimes A)\& \ldots \&(\overbrace{A \otimes \ldots \otimes A}^{n \text{ times}})$$

The formula $A^{\leq n}$ can be read as "it is guaranteed that A can be used n times". We can apply this concept of warranty to any artefact that we have discussed, therefore characterizing a set of behavior and their warranty. For example, with three screws, and by applying three times the appropriate force, then using a decently robust screwdriver, one can obtain three tighten screws:

$$F, F, F, S, S, S, \mathsf{E}_s(S \otimes F \multimap T)^{\leq 10000} \vdash T \otimes T \otimes T$$

7 Conclusions

We have studied a non-normal modal logic based on intuitionistic linear logic and we have provided the main logical results in order to show that MILL is a well-behaved system. We have extended MILL to Linear BIAT and presented a number of applications to artefacts. We conjecture that cut elimination holds also for Linear BIAT and that would provide us with a PSPACE upper bound for proof-search complexity for Linear BIAT. The proof theory of sequent calculus can be viewed as the theoretical background for developing a logic programming approach to modal linear logics. For this purpose we plan to implement an automated prover for Linear BIAT. Moreover, by exploiting our methodology for modeling functions and the intrinsic power of logic representations at modularity, we can represent complex combinations of artefacts. Therefore, more complex artefacts, as well as more complete industrial and business processes, can all in principle be modeled within a generalization of our framework. To that end, Linear BIAT can later be extended to coalitional agency where sets of artefacts and agents can combine their functions and actions. A resource-sensitive framework for strategic ability in multi-agent settings will then be a prime objective.

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